Geometry and Algebra -
From Local to Global

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What is a local-global principle?

Question:

Can properties of an (algebraic) object be detected “locally”? 
Example

Smooth
Example

Simply connected
What is a local-global principle?

Question:
Can properties of an (algebraic) object be detected “locally”?

Answer: Sometimes.

More questions:
Why (not)? Can we describe the obstruction?
Local-Global Principles in Mathematics

Extreme Value Theorem:

A continuous function on a closed and bounded interval \([a,b]\) must attain its minimum and maximum.

In particular, this asserts that local boundedness of a function (continuity) implies global boundedness.
Local-Global Principles in Mathematics

Euler’s theorem:

A connected multigraph has a Eulerian circuit (a loop which uses every edge exactly once) if and only if every vertex touches an even number of edges.

In particular, local information (degree of vertices) implies global information.
Local-Global Principles in Mathematics

Gauss-Bonnet theorem:

On a compact 2-dimensional Riemannian manifold, the curvature determines the Euler characteristic.

In particular, \textit{local} information (curvature) determines \textit{global} information (Euler characteristic).
Local-Global Principles in Mathematics

• Extreme Value Theorem

• Euler’s Theorem

• Gauss-Bonnet Theorem

• Analytic Continuation

• Integrability of differentiable complex functions

• …
What does “local” mean algebraically?

Given a prime number $p$ and a number $n$, we can write $n$ as a combination of powers of $p$.

Example: $47 = 5 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0$

This called the $p$-adic expansion of $n$.

If we allow these expansions to be of infinite length and we also allow fractions, we obtain the $p$-adic numbers $\mathbb{Q}_p$. 
Quadratic Forms

A quadratic form of dimension $n$ is a polynomial in $n$ variables in which every term has degree 2 (homogeneous).

Example: $3x^2 + 4y^2 + 5xy + xz$

Quadratic forms play an important role in many areas of algebra and mathematics in general.
Quadratic Forms

A quadratic form is called *isotropic* if it has a solution with not all entries identical to zero. It is called *anisotropic* otherwise.

Example: \[3x^2 + 4y^2 + 5xy + xz\]

This example has the solution (1,0,-3) and hence is isotropic.
Quadratic Forms

Question: Is there an upper bound on the dimension of anisotropic quadratic forms?
Quadratic Forms

Question: Is there an upper bound on the dimension of anisotropic quadratic forms?

Example: Every quadratic form with at least two variables has a nontrivial solution in the complex numbers.
Quadratic Forms

Question: Is there an upper bound on the dimension of anisotropic quadratic forms?

Example: In a finite field, exactly half the nonzero elements are squares (and zero is a square). A simple counting argument then shows that any form in more than two variables is isotropic.
Quadratic Forms

Question: Is there an upper bound on the dimension of anisotropic quadratic forms?

The $u$-invariant of a field is the maximal dimension of anisotropic quadratic forms (with coefficients in that field).

We have seen that the $u$-invariant of the complex numbers is 1, and the $u$-invariant of a finite field is 2.
# Quadratic Forms

<table>
<thead>
<tr>
<th>$\mathbb{F}_p$</th>
<th>$\mathbb{F}_p((x))$</th>
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<th>$\mathbb{Q}_p$</th>
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<tr>
<td>2</td>
<td>4</td>
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Quadratic Forms

\[ u(\mathbb{Q}_p(x)) = ? \]
Quadratic Forms over the rational numbers
Quadratic Forms over the rational numbers

\[ u(\mathbb{Q}) = ? \]
Quadratic Forms over the rational numbers

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Better question: What is the maximal dimension of indefinite anisotropic quadratic forms over the rational numbers?
Quadratic Forms over the rational numbers

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Better question: What is the maximal dimension of indefinite anisotropic quadratic forms over the rational numbers?

**Theorem (Meyer, 1884):** The maximum dimension of indefinite anisotropic quadratic forms over the rational numbers is 4.

Modern proofs of this rely on

**Theorem (Hasse-Minkowski, 1921):** A quadratic form over the rational numbers is isotropic if and only if it is isotropic over the real numbers and over \( \mathbb{Q}_p \) for all primes \( p \).
Quadratic Forms over the rational numbers

The Hasse-Minkowski theorem is the first example of an algebraic local-global principle (Hasse principle).

It fails for forms of higher degree, e.g. for the cubic form

\[3x^3 + 4y^3 + 5z^3\]
What about $\mathbb{Q}_p(x)$?

Fields of this form are called *semi-local*.

Nice: These fields have some “geometry”.
In particular, we have *patching*.
Patching

Algebraic Problem → Geometric Problem → (Finite) set of subproblems

Solution to AP ← Solution to GP ← Solution to the subproblems
Algebraic Problem → (Finite) set of subproblems
Solution to AP ← Solution to the subproblems

Patching
A simple patching setup

Geometrically:

Algebraically:
A simple patching setup

Algebraically:

Patching asserts that objects defined over $F_1$ and $F_2$ can be glued to an object over $F$ provided that they agree as objects over $F_0$.

There is a rather simple criterion on the fields to check whether they satisfy patching.
A simple patching setup

Algebraically:

Patching asserts that objects defined over $F_1$ and $F_2$ can be glued to an object over $F$ provided that they agree as objects over $F_0$.

There are lots of suitable choices for the overfields when the base field is $\mathbb{Q}_p(x)$. 
Patching and Local-Global Principles

**Local-Global Principles:** Consider a field and an infinite collection of overfields which are completions w.r.t. absolute values.

**Patching:** Consider a field and a finite collection of overfields. Indeed: Uniqueness assertion in patching gives “finite” local-global principle.
Obstructions

Back to quadratic forms:

• The orthogonal group of a quadratic form acts transitively on the set of solutions of that form (“principal homogeneous space”).

• The principal homogeneous space is “trivial” if and only if there is a solution.

• The obstruction to a local-global principle is the set of all “locally” trivial principal homogeneous spaces.

• The obstruction to patching is the set of all “patching-locally” trivial principal homogeneous spaces.
How patching helps

We can describe the obstruction w.r.t. the patching setup very explicitly:

$$\text{Hom}(\pi_1(\Gamma), \mathbb{Z}/2\mathbb{Z})/\sim$$
How patching helps

In general, the obstruction obtained from the patching setup is a subset of the obstruction to a local-global principle.

For quadratic forms, one can show that these obstruction sets are indeed the same.

Hence we understand the obstruction to the local-global principle for quadratic forms over $\mathbb{Q}_p(x)$. In particular, one can show that there is no obstruction if the form has at least 3 variables.

As a corollary, the u-invariant of $\mathbb{Q}_p(x)$ is indeed 8 - the expected value. This was first shown by Parimala and Suresh in a 2010 article.
How patching helps

The same strategy works for other types of algebraic objects (corresponding to other transformation groups). One has to check in each case that the obstruction sets coincide.

This also gives local-global principles e.g. for central simple algebras.
What’s next?

• There are still many questions around both types of obstruction sets, e.g. finiteness.

• There are potential generalizations to other kinds of homogeneous spaces.

• Applications to algebraic structures, e.g. field invariants.

• More local-global: Passage to $\mathbb{Q}(x)$?