The Geometry of Quantum Field Theory

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Some Ideology

TRUTH AND BEAUTY
Seal of the Institute for Advanced Study
“My work always tried to unite the true with the beautiful, but when I had to choose one or the other, I usually chose the beautiful.”

Hermann Weyl
“It is more important for our equations to be beautiful than to have them fit experiment.”

Paul Dirac
“Every law of physics, pushed to the extreme, will be found to be statistical and approximate, not mathematical perfect and precise.”

John Wheeler
“Any theory that can account for all of the facts is wrong, because some of the facts are always wrong.”

Francis Crick
Garbage

Reduction

Beauty

Standard Model

macro

micro
Emergence

Beauty
- hydrodynamics
- thermodynamics

Garbage
- statistical mechanics

macro

micro
Infrared fixed points
\[ \Lambda = 0 \]
Large distances
Low energy
macro

Renormalization Group

Ultraviolet fixed points
\[ \Lambda = \infty \]
Short distances
High energy
micro
Geometric object $K$ is quantized to a quantum invariant $Z(K) \in C$. The process of quantization converts geometric properties into quantum properties.
Emergent Geometry

Geometry

effective geometry

Algebra

quantum system
Quantum Geometry

String Theory
Classical Mechanics

calculus, geometry, dynamical systems, chaos,…
Quantum Mechanics

Sum over histories \[ \sum e^{-\frac{\text{Action}}{\hbar}} \]

functional analysis, operator algebra, differential topology, …
Quantum Field Theory

creation/annihilation of particles

quantum topology: knots, 3- & 4-manifolds, twistors, Grassmannians & amplitudology
String Theory

conformal field theory, algebraic curves, moduli spaces, mirror symmetry, quantum cohomology
Quantum Gravity

non-commutative geometry, emergent geometry, automorphic forms,…
What Is A Quantum Field Theory?
Classical Field Theory

Space-time manifold $M$, field space $F$

$$\phi : M \rightarrow F$$

Action, critical points

$$S[\phi], \quad dS = 0$$

Classical field equations

$$\nabla^2 \phi + \ldots = 0$$

Moduli space of classical solutions

$$\{dS = 0\} / \text{Sym} = \bigsqcup_n \text{Mod}_n$$
Quantum Field Theory

Euclidean path integral

\[ Z(M) = \int D\phi \cdot e^{-S(\phi)/\hbar} \]

Semi-classical localisation to critical points

\[ Z(M) \sim \sum_n e^{-S_n/\hbar} \cdot \int_{\text{Mod}_n} \text{vol}_n \]

Typical form

\[ Z(M) \sim \sum_n d_n q^n, \quad q \sim e^{-1/\hbar} \]

Classical limit

\[ \hbar \to \infty, \quad q \to 0 \]
Dualities

Quantum equivalences

\[ Z(M, F) = Z(M, F^*) \]

e.g. D=2 T-dualities, \( M = \text{Riemann surface} \), \( F = n\)-torus

where

\[ F = T^n = R^n / \Lambda, \quad F^* = T^* = R^n / \Lambda^* \]

\[ d\phi^* = *d\phi \]

Partition function = theta function

\[ Z = \theta_{\Lambda} = \sum_{p \in \Lambda, w \in \Lambda^*} e^{\pi i (p + w) \tau (p + w) - \pi i (p - w) \bar{\tau} (p - w)} \]

Extended symmetry

\[ SL(n, Z) \subset SO(n, n, Z) = Aut(\Lambda \oplus \Lambda^*) \]
Special automorphisms of self-dual QFTs lead to dualities.

\[ \text{Space of Quantum Field Theories} \]

Classical field theory $F$ maps to the dual classical limit $\frac{\hbar}{0}$.

Dual classical field theory $F^*$ maps to the dual classical limit $\frac{\hbar^*}{0}$.

Dualities connect these limits, leading to special automorphisms of self-dual QFTs.
Dualities of Effective QFT

Renormalization group

\[ Z(M, F_\varepsilon) \]

Depends on cut-off distance \( \varepsilon \) or energy scale \( \Lambda = 1/\varepsilon \)

RG flow on space of regularized quantum field theories

\[ F = F_\infty, \quad -\varepsilon \partial_\varepsilon, \quad F^* = F_0 \]

\( \varepsilon = \infty \rightarrow \varepsilon = 0 \)

Sometimes \( Z \) independent of \( \varepsilon \)

\[ Z(M, F_0) = Z(M, F_\infty) \]
Defining Quantum Field Theory: Algebra

Operator products of local operators

\[ \langle O_1(x_1) \ldots O_n(x_n) \rangle \]

Scattering amplitudes
D=2 Conformal Field Theory

Operator product expansion

$$\lim_{x \to y} O_i(x) \cdot O_j(y)$$

$$\sum_k O_k(y)$$

Vertex algebra

$${\mathcal H} \otimes {\mathcal H} \to {\mathcal H}$$

$\infty$-dim vertex algebras classification reps Virasoro

Associative algebra
Defining Quantum Field Theory: Geometry

cut & paste
topological indices
defect operators
Category Theory

$M$ codim 0: number, partition function

$S$ codim 1: vector space, Hilbert space of states

Gluing laws

$\Phi(M_1) \circ \Phi(M_2) = \Phi(M)$
Category Theory

codim 2: category of boundary conditions

Composition law of morphisms

\[ H : \alpha \to \beta \]

\[ H : \alpha \to \beta \]

\[ H_1 : \alpha \to \beta \]
\[ H_2 : \beta \to \gamma \]
\[ H_2 \circ H_1 : \alpha \to \gamma \]
D=1 Supersymmetric QM

Differential geometry & topology

\[ Hilbert \ space = \Omega^*(X) \]
\[ H = -\Delta = -(dd^* + d^*d) \]

Ground states = harmonic forms

\[ \text{Harm}^*(X) \cong H^*(X) \]

Partition function = Witten index

\[ \text{Tr}(((-1)^{\text{deg}} e^{-iH})) = \text{Euler}(X) \]
D=2 Conformal Field Theory

Quantize loop space

\[ \mathcal{H} = L^2(X, \mathcal{F} \otimes \mathcal{F}) \]

Fock space created by string oscillators

Infinite-dimensional analysis

Loop space thickening \( X \subset LX \)
String Product

Physical fields = cohomology classes \( a, b \in H^* (X) \)
Topological String Theory

\[ F_0(t) = \sum_{d \geq 0} GW_{d,0} e^{-dt} \]

\( d = \text{deg}, \quad t = \frac{\text{Area}}{\ell_{\text{string}}^2} \)
Classical Intersection Product

\[ \ell_s = 0 \]

\[ a \wedge b \wedge c = \int_X (a \wedge b \wedge c) \]

dual cycles
Quantum Cohomology

\[ \ell_s > 0 \]

\[ = \sum_{\text{rat curves degree } d} e^{-dt/\ell_s^2} \]

for \( \mathbb{P}^n \quad x^{n+1} = 0 \quad \Rightarrow \quad x^{n+1} = e^{-t} \)

smooth under flops \( t \leftrightarrow -t \)
\[ \ell_s = 0 \]  

local \( \mathbb{P}^1 \)

\[ t < 0 \quad t > 0 \]

moduli space

\[ \ell_s > 0 \]

singularity at zero size

\[ t = 0 \]
Open Strings & Branes

fixed time picture

brane

open strings (brane)

closed strings (bulk)

X
D-branes: Relative CFT

Space-time picture
D-Branes

4D Space-time

multiplicity $N$

Internal space
$U(N)$ Yang-Mills Theory

$N \times N$ matrix of strings $A_{ij}$
U(N) Yang-Mills Theory

Matrix multiplication: $\sum_k A_{ik} A_{kj}$
Closed/Open Dualities

closed strings

\[ \text{inf-dim Lie algebras} \]
\[ \text{loop spaces} \]
\[ \text{Virasoro algebra} \]
\[ \text{genus expansion} \]

\[ \Leftrightarrow \]

open strings

\[ \text{vector bundles} \]
\[ \text{K-theory} \]
\[ \text{moduli spaces} \]
\[ \text{non-commutative} \]
Atiyah-Singer Index Theorem

analysis $\Leftrightarrow$ geometry

$$\text{Index } \mathcal{D}_{E_1 \otimes E_2} = \int ch(E_1 \otimes E_2^*) \hat{A}(X)$$

$$\text{Tr}(-1)^F = \text{Index } \mathcal{D}_{E_1 \otimes E_2}$$

$$\langle E_1, E_2 \rangle = \int ch(E_1 \otimes E_2^*) \hat{A}(X)$$
String Theory

Two fundamental parameters

String length \((\text{Planck’s constant on world sheet})\)

\[
\ell_s
\]

String coupling \((\text{Planck’s constant in space time})\)

\[
\ell_{\text{Planck}} = g_s \ell_s
\]
String Partition Function

\[ Z_{\text{string}} = \exp \sum_{g \geq 0} g_s^{2g-2} F_g(t; \ell_s) \]
Phase Diagram

- **Branes**
- **Strings**
- **Particles**
- **Fields**

Parameters:
- \( \ell_s \)
- \( g_s \)
Quantum Geometry (emergent)

Stringy Geometry (deformed)

Classical Geometry

\[ \ell_{\text{Planck}} = g_s \ell_s \]

\[ \ell_s \]

smooth
Simplest Calabi-Yau 3-fold

$$\mathbb{C}^3$$

$$(|z_1|, |z_2|, |z_3|) \in \mathbb{R}^3$$
\[ N_{g,0} = \int_{\overline{M}_g} \lambda^3_{g-1} = \frac{B_{2g} B_{2g-2}}{2g(2g - 2)(2g - 2)!} \]
$Z = \exp \sum_{g \geq 0} N_g g_s^{2g-g} \geq 0 \sum_{q^n(\ )} -n \prod_{n>0} (1-q^n)^{-n} = \sum_{\text{3d partitions of } N} q^N$ 

$q = e^{-g_s}$
\[ Z_{top} = \prod_{n > 0} \left(1 - q^n \right)^{-n} = 1 + q + 3q^2 + 6q^3 + \ldots \]
Melting Crystals

\[ Z_{\text{top}} = \sum q^{\# \text{atoms}} = \prod_{n>0} \left(1 - q^n\right)^{-n} \]
Limit Shape = Mirror Manifold
$\ell_{\text{Planck}}$  $\ell_{\text{string}}$  smooth
D=4 Gauge Theories

Instantons: self-dual connection $F = \ast F$

4-manifold $M^4$

$\text{ch}_2 = n$

moduli space $\text{Mod}_{N,n}(M)$

Action

$$S = \frac{4\pi}{g^2} \int Tr F \wedge \ast F + \frac{\theta}{8\pi^2} \int Tr F \wedge F = -n \cdot 2\pi i \tau$$

Gauge coupling

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \in H$$
**Dualities of D=4 Gauge Theories**

Electric-magnetic dualities

\[ F(A^*) = {}^*F(A) \]

Langlands dual gauge group

\[ G \leftrightarrow {}^L G \]

Dual coupling

\[ \tau \leftrightarrow -1/ \tau \]

Extended symmetry \( SL(2, Z) \)

\[ \tau \rightarrow \frac{a \tau + b}{c \tau + d} \]
$N=4$ SUSY Gauge Theory

Vafa-Witten: partition function is a modular form

$$Z(M; q) = \sum_{n \geq 0} d(n) q^n, \quad d(n) = Euler\left(\text{Mod}_{N,n}\right)$$

- $SL(2,\mathbb{Z})$ S-duality in $N=4$ gauge theories $\leftrightarrow$ modular invariance of a quantum CFT on 2-torus
  $$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

- Related to 2d CFT characters
  $$Z(q) = Tr_v q^{L_0} = \chi(q)$$
$N = 4$ Gauge Theories on ALE spaces

- Resolved singularity $M_{\Gamma} \rightarrow C^2 / \Gamma$
- Finite subgroup $\Gamma \subset SU(2)$
- Boundary condition: flat connection
  \[ \rho \in Hom(\Gamma, U(N)), \quad \rho = \oplus N_i \rho_i \]
- MacKay correspondence: finite groups $\Gamma \leftrightarrow \hat{g}$ affine KM algebras
N = 4 Gauge Theories on ALE spaces

• $\rho$ rep of dim $N$ of $\Gamma \leftrightarrow V_\rho$ integrable rep at level $N$ of $\hat{g}$

• $A_{k-1} : C^2 / Z_k$ characters of $SU(k)_N$ WZW model

$$Z_{\text{gauge}}(q)_\rho = Tr_{V_\rho} q^{L_0} = \chi_\rho(q)$$

• Level-rank duality

$$SU(N)_k \leftrightarrow SU(k)_N$$
**N=2 SUSY Gauge Theory**

Equivariant action of $SO(4) \times U(N)$ on $\text{Mod}_{N,n}(R^4)$

$$(\varepsilon_1, \varepsilon_2; a) \in T^2 \times T^N$$

Nekrasov partition function: equivariant fundamental class

$$Z_{\text{gauge}}(q; \varepsilon_1, \varepsilon_2, a) = \sum_{n \geq 0} q^n \int_{M_{N,n}} 1(\varepsilon_1, \varepsilon_2, a)$$

Traditional (supersymmetric) case $\varepsilon_1 = -\varepsilon_2 = \varepsilon = g_s$
Seiberg-Witten solution

\[ Z_{\text{gauge}} = e^F, \quad F = \sum_{g} \varepsilon^{2g-2} F_g \]

Quantum (symplectic) invariants of the spectral curve

\[(C, \omega), \quad \omega = \text{meromorphic one-form}\]

\[ F_0 : \mu_i = \int_{A_i} \omega \quad \text{(moduli)}, \quad \frac{\partial F_0}{\partial \mu_i} = \int_{B_i} \omega \]

\[ F_1 = \frac{1}{2} \log \det \Delta \]

\[ F_g : \text{genus } g \text{ graphs} \]
Gauge Theories

Emergent Geometry

$M^4$

self-dual connections on 4-manifolds

Conformal Field Theory

Emergent Geometry

$C^2$

2d QFT on algebraic curves reps of infinite dim algebras
D=6 Tensor (2-form) Gauge Theory

D=2 CFT

D=4 Gauge Theory
Geometry/Algebra Duality

Gauge Theories

Geometry

$M^4$

4-manifold

Algebra

$G$

gauge group (vector bundle)

Conformal Field Theory

Algebra

$\hat{g}$

affine symmetries

Geometry

$C^2$

algebraic curve

Riemann surface
Wigner’s Random Matrix Model

$$\lim_{N \to \infty} Z_N, \quad Z_N = \int_{N \times N} d\Phi \cdot e^{-Tr\Phi^2 / g_s}$$

Eigenvalue distribution in ’t Hooft limit

$$N \to \infty, \quad g_s \to 0, \quad Ng_s = \mu = \text{fixed}$$
Eigenvalue Dynamics

\[
Z_{\text{matrix}} = \int d^N \lambda \cdot \prod (\lambda_I - \lambda_J)^2 \cdot e^{-\sum \frac{\lambda_i^2}{g_s}}
\]

Effective action (repulsive Coulomb force)

\[
S_{\text{eff}} = \sum_I \lambda_I^2 - 2g_s \sum_{I < J} \log(\lambda_I - \lambda_J)
\]

\[
W(x) = x^2
\]
General Matrix Model

\[ Z_{\text{matrix}} = \int d\Phi \cdot e^{TrW(\Phi)/g_s} \]

‘t Hooft limit

\[ N_I \to \infty, \; g_s \to 0, \; N_I g_s = \mu_I = \text{fixed} \]

Filling fractions

\[ W(\Phi) \]
Hyperelliptic curve

Spectral Curve

\[ C : y^2 = W'(x)^2 + f(x) \]

Quantum invariants of \( C, \quad \omega = ydx \)

\[ \mu_i = \oint_{A_i} ydx \]

\[ \frac{\partial F_0}{\partial \mu_i} = \oint_{B_i} ydx \]
Wigner’s Semi-Circle

Rational spectral curve

\[ C : y^2 = x^2 + \mu \]

\[
\frac{\partial F_0}{\partial \mu} = \mu \log \mu, \quad F_0 = \frac{1}{2} \mu^2 \log \mu
\]
Quantum Curves

$N \rightarrow \infty$

$O(1/N)$

$N \text{ finite}$

$F_0$

$\sum_{g \geq 0} N^{2-2g} F_g$
Categorification

“Miraculous” integrality

\[ Z(M) = \sum_{n} d_n q^n \]

\[ d_n \in \mathbb{Z} \]

Suggests higher dimensional QFT

\[ d_n = \dim V_n(M) \]

The vector spaces \( V_n \) can carry representations of algebraic structures
Knot Polynomials

Knot $K$ in $S^3$ in Chern-Simons theory

$$Z(K, G) = \int DA \, e^{-kCS(A)} \cdot Tr_{RHol}(K)$$

Knot polynomials (Jones,...)

$$Z(K, G) = \sum_{n} d_n q^n, \ d_n \in \mathbb{Z}$$

$$q = e^{2\pi i/(k+h)} = e^{\hbar}$$

Khovanov cohomology
Global Definition?
Mathematics of QFT

- Constructive and algebraic QFT, asymptotically free theories (Clay prize).
- Surprisingly rich structure in Feynman diagrams (Hopf algebra of Connes-Kreimer, multiple zeta-functions, number theory).
- Twistor reformulations, MHV calculus, amplituhedron,...
- Dualities: relating different gauge groups & dynamical variables, not always a semi-classical expansion.
- Is the path-integral fundamental?
What Kind of Mathematical Beauty?

universal

*calculus,*

*Hilbert spaces*

exceptional

*E₈, Monster Group,*

*Calabi-Yau manifolds*
Plato’s Cave

Mathematical Dream

Physical Reality
Quantum Cave

Physical Dream

Mathematical Reality