Interplays between stochastic calculus and geometric inequalities and some new bounds related to the Gaussian convolution operator

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Our setting is $\mathbb{R}^n$ equipped with the standard Gaussian measure $\gamma$.

We consider the convolution operator

$$P_t[f](x) := \mathbb{E} \left[ f \left( \sqrt{tx} + \sqrt{1 - t} \right) \right], \quad 0 \leq t \leq 1$$

where $\Gamma$ is a standard Gaussian random vector.

The renormalization is chosen such that $\int f d\gamma = \int P_t f d\gamma$ for all $t, f$.

We will also consider the associated quadratic form

$$Q_t(f) = \langle f, P_t f \rangle_\gamma = \int f(x) P_t[f](x) d\gamma(x).$$
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The Ornstein-Uhlenbeck convolution operator

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An analytic vs. a probabilistic point of view

For a function $f \in L_2(\gamma)$, consider its Fourier-Hermite representation

$$f(x) = \sum_{\ell \in \mathbb{N}^n} C_\ell(f) H_\ell(x).$$

Then $Q_t(f)$ is diagonalizable over the above basis, and we have

$$Q_t(f) = \sum_{\ell \in \mathbb{N}^n} t^{\ell} C_\ell(f)^2.$$

On the other hand, we have

$$Q_t(f) = \mathbb{E} \left[ f(\sqrt{t}Z_1 + \sqrt{1-t}Z_2) f(\sqrt{t}Z_1 + \sqrt{1-t}Z_3) \right]$$

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where $Z_1, Z_2, Z_3$ are standard Gaussian random vectors.
For a set $A \subset \mathbb{R}^n$, the quantity $Q_t(A) = Q_t(1_A)$ is sometimes referred to as the noise stability of $A$.

**Theorem: Borell’s noise stability inequality**

If $A \subset \mathbb{R}^n$ and $H$ is a half-space such that $\gamma(A) = \gamma(H)$ then

$$Q_t(H) \geq Q_t(A)$$

- The limit case $t \to 1$ is the Gaussian Isoperimetric inequality of Borell, Sudakov-Tsirelson.
- The limit case $t \to 0$ boils down to understanding $|\int_A x d\gamma|^2$.
- **Claim:** Out of all possible sets whose Gaussian measure is exactly $\gamma(A)$, the quantity $|\int_A x d\gamma|$ is maximized on half-spaces.
- We would like to somehow use this principle for larger values of $t$. 
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Ronen Eldan  Stochastic calculus and geometric inequalities
For a set \( A \subseteq \mathbb{R}^n \), the quantity \( Q_t(A) = Q_t(\mathbf{1}_A) \) is sometimes referred to as the noise stability of \( A \).

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Stochastic calculus and Geometric inequalities
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A stochastic approach for the proof

We begin by writing

\[ Q_t(A) = \mathbb{P}(\sqrt{t}Z_1 + \sqrt{1-t}Z_2 \in A \text{ and } \sqrt{t}Z_1 + \sqrt{1-t}Z_3 \in A) \]

\[ = \mathbb{E}_{Z_1}[\mathbb{P}(\sqrt{t}Z_1 + \sqrt{1-t}Z_2 \in A \text{ and } \sqrt{t}Z_1 + \sqrt{1-t}Z_3 \in A|Z_1)] \]

\[ = \mathbb{E}_{Z_1} \left[ \mathbb{P}(\sqrt{t}Z_1 + \sqrt{1-t}Z_2 \in A|Z_1)^2 \right]. \]

Now let \( W_t \) be a standard Brownian motion. Then

\((W_t, W_1) \sim (\sqrt{t}Z_1, \sqrt{t}Z_1 + \sqrt{1-t}Z_2).\)

So,

\[ Q_t(A) = \mathbb{E} \left[ \mathbb{P}(W_1 \in A|W_t)^2 \right] \]
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Consider the Doob martingale

\[ M_t = \mathbb{P}(W_1 \in A \mid W_t). \]

Then,

\[ Q_t(A) = \mathbb{E}(M_t^2). \]

Since \( M_t \) is a martingale, we have that

\[ \mathbb{E}(M_t^2) - M_0^2 = \text{Var}(M_t) = \mathbb{E}[M]_t \]

where \([M]_t\) is the quadratic variation of the process \( M_t \). The last equality is a continuous version of the fact that

\[ \mathbb{E} \left( \sum \Delta_i \right)^2 = \mathbb{E} \sum (\Delta_i)^2 \]

where \( \Delta_i \) are martingale increments.

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Calculating the stability of a set under Gaussian perturbations boils down to calculating the quadratic variation of \( M_t \).
Recall that for small $t$, one has $Q_t(A) \sim t \left| \int_A xd\gamma \right|^2$. So we also have that

$$\frac{d}{dt} [M]_t \bigg|_{t=0} = \left( \int_A xd\gamma \right)^2.$$

But up to some renormalization of time and space, different values of $t$ should behave essentially the same!

And indeed, a straightforward calculation, using Itô’s formula, gives

$$\frac{d}{dt} [M]_t = (1 - t)^{-1} \left| \int_{A - \frac{W_t}{\sqrt{1-t}}} xd\gamma(x) \right|^2.$$

Note also that

$$\gamma \left( \frac{A - W_t}{\sqrt{1-t}} \right) = M_t.$$
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$$\gamma \left( \frac{A - W_t}{\sqrt{1 - t}} \right) = M_t.$$
Let $N_t$ be the martingale defined in the same way as $M_t$ only that $A$ is replaced by the half-space $H$. The above claim implies that

$$\frac{d}{dt} [M]_t \leq \frac{d}{dt} [N]_t \bigg|_{N_t = M_t}.$$ 

The Dambis / Dubins-Schwarz theorem

Every continuous martingale is a time change of a Brownian motion.

We may couple the processes $M_t$ and $N_t$ via the Dambis / Dubins-Schwartz theorem so that

$$M_t = M_0 + B([M]_t), \quad N_t = N_0 + B([N]_t)$$

where $B(\cdot)$ is a standard (1-dimensional) Brownian motion.

Under this coupling, we have that $[N]_t \geq [M]_t$ for all $t$, almost surely. This finishes the proof.
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Robustness for Borell’s inequality

Consider the 

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\text{deficit } \delta(A) := Q_t(H) - Q_t(A).
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**Theorem (Moseel-Neeman, ’12)**

If \( A \subset \mathbb{R}^n \) and \( H \) is the half-space minimizing \( \gamma(A \Delta H) \), such that \( \gamma(A) = \gamma(H) \) then for some \( C = C(\gamma(A), t) > 0, \)

\[
C^{-1} \gamma(A \Delta H)^4 \leq \delta(A).
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- For \( A \subset \mathbb{R}^n \) set \( \varepsilon(A) = \left| \int_H x d\gamma(x) \right|^2 - \left| \int_A x d\gamma(x) \right|^2 \) where \( H \) is a half-space having \( \gamma(H) = \gamma(A) \).

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If \( A \subset \mathbb{R}^n \) and \( H \) is a half-space such that \( \gamma(A) = \gamma(H) \) then for some \( C = C(\gamma(A), t) > 0, \)

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Let go back to the (measure-valued) process $W_t | \mathcal{F}_t$. Denote its density by $F_t$. An easy calculation gives

$$dF_t(x) = (1 - t)^{-1/2} F_t(x) \langle x - W_t, dW_t \rangle$$

If we denote $V_t^{(1)} = \int_{A - W_t / \sqrt{1-t}} x d\gamma(x)$, then

$$dV_t^{(1)} = \left( \int_{A - W_t / \sqrt{1-t}} x \otimes x d\gamma(x) \right) dW_t$$

and so on.
Some ideas for the proof: considering the second derivatives

- We define

\[ \epsilon_t = \left| \int_{\frac{H-W_t}{\sqrt{1-t}}} y d\gamma(y) \right|^2 - \left| \int_{\frac{A-W_t}{\sqrt{1-t}}} y d\gamma(y) \right|^2. \]

- According to the above, the process \( \frac{d}{dt} [N_t - [M]_t] \) is large given that \( \epsilon_t \) is bounded away from 0.

- We want to show: \( \epsilon_0 \) is large \( \Rightarrow \epsilon_t > c\epsilon_0 \) in an interval \( t \in [0, \delta] \) for \( \delta \) large enough.

- It turns out to be useful to consider **second derivatives**, hence to calculate \( d\epsilon_t \).

- In analogy to the log-Laplace transform, the second derivative has to do with the covariance matrices of \( \gamma \) restricted to these sets.
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**Theorem - hypercontractivity of the operator $P_t$ (Gross, Nelson,...)**

For any $p > 1$ and $t > 0$, there exists a constant $q > p$ satisfying

$$
\| P_t f \|_{L_q(\gamma)} \leq \| f \|_{L_p(\gamma)}.
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for all $f \in L_p(\gamma)$

- This fact has applications to several fields such as analysis of PDEs and quantum information theory.
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- It is well-known that the operator $P_t$ has the following smoothing property,

**Theorem - hypercontractivity of the operator $P_t$ (Gross, Nelson,...)**

For any $p > 1$ and $t > 0$, there exists a constant $q > p$ satisfying

$$\|P_t f\|_{L_q(\gamma)} \leq \|f\|_{L_p(\gamma)}.$$  

for all $f \in L_p(\gamma)$

- This fact has applications to several fields such as analysis of PDEs and quantum information theory.
- Does $P_t$ admit any regularization properties over $L_1$?
What about $L_1$ functions?

Question (Talagrand)

Is it true that for every non-negative function $f$ such that $\mathbb{E}f(\Gamma) = 1$ one has

$$\mathbb{E} \left[ P_t[f](\Gamma) \mathbf{1}_{\{ P_t[f](\Gamma) \in [\alpha, 2\alpha] \}} \right] \leq \frac{C(t)}{\alpha} g(\alpha)$$

for some function $g(\alpha)$ satisfying $g(\alpha) \to 0$ as $\alpha \to \infty$?

Theorem (Ball, Barthe, Bednorz, Oleszkiewicz and Wolff 2010)

For every non-negative function $f$ such that $\mathbb{E}f(\Gamma) = 1$ one has

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A logarithmic anti-concentration result

Theorem (E., Lee 2014)

If \( g \) is a positive function satisfying \( \mathbb{E}[g(\Gamma)] = 1 \) and

\[
\nabla^2 \log g(x) \succeq -\beta \text{Id}, \quad \forall x \in \mathbb{R}^n
\]

then

\[
\mathbb{E} \left[ g(\Gamma) \mathbf{1}_{g(\Gamma) \in [s, 2s]} \right] \leq \frac{C(\beta)(\log \log s)^4}{(\log s)^{1/2}}, \quad \forall s \geq 2
\]

- It is easy to check that \( P_t[f] \) satisfies (1) with \( \beta = \frac{1}{t} \).
- Talagrand’s conjecture follows with \( \varphi(\alpha) \sim \frac{\log^4 \log \alpha}{\sqrt{\log \alpha}} \).
- The dependence on \( \alpha \) is optimal up to the log log factor, which was very recently removed by J. Lehec.
We have a positive function $g(x)$ satisfying $\mathbb{E}[g(\Gamma)] = 1$ and $\nabla^2 \log g(x) \succeq -\beta \text{Id}$.

Define a measure $\mu$ by $\frac{d\mu}{d\gamma}(x) = g(x)$. Suppose by contradiction that $\mu(E \setminus F) = \Omega(1)$ where $E = \{x; \ g(x) > \alpha\}$ and $F = \{x; g(x) > 2\alpha\}$.

Define

$$T(x) = x + \frac{\nabla \log g(x)}{|\nabla \log g(x)|^2} \log 2$$

and $E' = T(E)$.

We would like to show: (i) $E'' \subset F$, (ii) $\mu(E') \approx \mu(E)$.

The Hessian estimate gives $\log g(T(x)) \geq \log g(x) + \log 2 - \frac{\beta}{|\nabla \log g(x)|^2}$.
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Ronen Eldan

Stochastic calculus and Geometric inequalities
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Now consider the change of measure $\frac{dP}{dQ} = M_1 = g(W_1)$ so that under the measure $P$, $W_1$ has the distribution $gd\gamma$.

According to Girsanov’s theorem, under this change of measure the process $W_t$ satisfies

$$dW_t = dB_t + \nu_t dt, \quad \nu_t = \nabla \log P_t[g](B_t)$$

where $B_t$ is a $Q$-Brownian motion.

Föllmer’s drift was used by J. Lehec (2010) to give stochastic proofs of numerous inequalities in Gaussian space such as Shannon’s inequality, Log-Sobolev, Brascamp Lieb inequalities etc (uses some ideas of Borell, 2000).
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Let $X_t$ be a Brownian motion with respect to a probability space $(\Omega, P)$ and suppose that

$$Y_t = X_t + \int_0^t v_s ds$$

for some adapted drift $v_t$. **Girsanov’s theorem** states that the measures associated to the processes $B_t, W_t$ are absolutely continuous with respect to each other and that if one defines the change of measure

$$\frac{dQ}{dP} = \exp \left( - \int_0^1 \langle v_t, dX_t \rangle - \frac{1}{2} \int_0^1 |v_t|^2 dt \right)$$

then the process $Y_t$ becomes a Brownian motion with respect to the measure $Q$. 

Perturbing Fölmer’s drift

**Idea:** instead of perturbing the point $x$, we define an adapted perturbation in the space of paths.

Recall that $dW_t = dB_t + v_t dt$. We define

$$dW_t^\delta = dB_t + (1 + \delta)v_t dt$$

Girsanov’s theorem gives us a precise formula for the measure of the "sets" $E_\delta$.

The main difficulty becomes proving that this perturbation is correlated with the gradient of the function.

In other words that $\left\langle \int_0^1 v_t, v_1 \right\rangle$ is rather large, with high probability.

**Fact**

The process $v_t$ is a martingale.
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Some other recent applications of stochastic calculus to geometric results

- Some new inequalities regarding diffusions with respect to convex potential functions (Borell ’00).
- New proofs of classical functional inequalities such as Shannon’s inequality, Talagrand’s transportation-entropy inequality and the Brascamp-Lieb inequalities (Lehec, 2011).
- Derivation of isoperimetric and concentration inequalities over convex bodies. In particular allows a reduction of the KLS conjecture to the variance conjecture, up to logarithmic factors (E. 2012).
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- Unlike in the case of the semi-group approach where all the expressions represent expectations with respect to the diffusion, this approach allows a path-wise analysis.
- Itô’s formula allows us to do precise calculations (e.g., of expressions for derivatives with respect to time).
- As we saw above, other theorems like Dubins-Schwarz and Girsanov’s formula can also be useful.
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Thank you!