Recent progress in birational geometry

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Plan of talk

• Overview

• Pairs

• Examples of Fano’s

• Main results

• Work in progress

• Some ideas of proofs
Let $X$ be a projective variety with "good" singularities. We say $X$ is

- Fano if $K_X$ is anti-ample, e.g. $\mathbb{P}^n$
- Calabi-Yau if $K_X$ is trivial, e.g. abelian varieties
- canonically polarised if $K_X$ is ample

Such varieties are very interesting in

- birational/algebraic geometry (e.g. see below; derived categories),
- moduli theory (e.g., see below; varieties of general type; Hodge theory),
- differential geometry (e.g., Kähler-Einstein metrics, stability),
- arithmetic geometry (e.g., existence and density of rational points),
- mathematical physics (e.g., string theory, mirror symmetry).
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Conjecture (Minimal model and abundance)

Each variety $W$ is birational to a projective variety $Y$ with good singularities such that either

- $Y$ admits a Fano fibration,
- $Y$ admits a Calabi-Yau fibration,
- $Y$ is canonically polarised.

Known cases:
- Dimension 2: (Castelnuovo, Enriques)(Zariski, Kodaira, etc) 1900,
- Dimension 3: (Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov)(Fano, Hironaka, Iitaka, Iskovskikh-Manin, etc) 1970's-1990's,
- Any dimension for $W$ of general type (BCHM=B-Cascini-Hacon-McKernan, after Shokurov, etc) 2006.
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Birational geometry: overview – MMP

How to find such $Y$?

The required contractions [Kawamata, Shokurov] and flips [BCHM] exist. Important ingredient: the $C$-algebra $R = \bigoplus_{m \geq 0} H^0(mK_W)$ is finitely generated [BCHM].

Conjecture

• Termination: the program stops after finitely many steps
• Abundance: if $K_Y$ not ample, then $Y$ is fibred by Fano's and CY's.
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Run the MMP giving a sequence of birational transformations

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**Conjecture**

- **Termination:** *the program stops after finitely many steps*
- **Abundance:** *if $K_Y$ not ample, then $Y$ is fibred by Fano’s and CY’s.*
Birational classification of varieties (including MMP) involves many interesting problems/topics.

Show diagram.
Singularities of pairs

A pair \((X, B)\) consists of a normal variety \(X\) and a boundary divisor \(B\) with coefficients in \([0, 1]\). Singularities of \((X, B)\) are defined by taking a log resolution \(\phi: W \rightarrow X\) and writing \(K_W + B_W = \phi^*(K_X + B)\). The larger the coefficients of \(B_W\), the worse the singularities. Singularities are “good” if every coefficient of \(B_W\) is \(\leq 1\) (or \(< 1\)). \((X, B)\) is \(\epsilon\)-lc if every coefficient of \(B_W\) is \(\leq 1 - \epsilon\).

Example: \(X\) a smooth variety and \(B\) a simple normal crossing divisor.

Example: \(X\) a smooth surface and \(B\) a nodal curve.

Bad example: \(X\) a smooth surface and \(B\) a cuspidal curve.
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Example of Fano’s

For $n \geq 2$ consider $E \subset W_n \rightarrow X_n$ where $X_n$ is the cone over rational curve of deg $n$, and $W_n$ is blowup of vertex, $E$ is the exceptional curve. $W_n$ = projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. $E$ is the section given by the summand $\mathcal{O}_{\mathbb{P}^1}(-n)$. $X_n$ is obtained from $W_n$ by contracting $E$. $K_{W_n} + n - 2n E = f^* K_{X_n}$. $X_n$ is $2n$-lc Fano (as $n \rightarrow \infty$, singularities get deeper). 

\{X_n | n \in \mathbb{N}\} is not a bounded family.
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$W_n = \text{projective bundle of } O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n)$. 

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\downarrow & \\
\mathbb{P}^1 &
\end{align*}$$

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$W_n =$ projective bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$.

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\( X_n \) is \( \frac{2}{n} \)-lc Fano \hspace{1cm} \text{(as } n \to \infty, \text{ singularities get deeper)}.\]
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$X_n$ is $\frac{2}{n}$-lc Fano  (as $n \to \infty$, singularities get deeper).

$\{X_n \mid n \in \mathbb{N}\}$ is not a bounded family.
Main results: Fano varieties

Theorem (Boundedness of complements, \[B, 2016\])

For each \(d\) there is \(m\) such that if \(X\) is a klt Fano of dimension \(d\) then
\[
\text{h}^0(−mK_X) \neq 0.
\]
Moreover, \(|−mK_X|\) contains an element with good singularities.

This was conjectured by Shokurov (mid 1990's, originates in 1970's).
Proved in dimension 2 by Shokurov.
Partially proved in dimension 3 by Prokhorov-Shokurov.

Example:
\(X\) toric Fano, then can take \(m = 1\).

Theorem (Effective birationality, \[B, 2016\])

For each \(d, \epsilon > 0\) there is \(m\) such that if \(X\) is \(\epsilon\)-lc Fano of dimension \(d\), then
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|−mK_X|\ defines a birational map.
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Main results: Fano varieties

Theorem (Boundedness of complements, [B, 2016])

For each $d$ there is $m$ such that if $X$ is a klt Fano of dimension $d$ then $h^0(-mK_X) \neq 0$.

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Theorem (Boundedness of singular Fano’s, [B, 2016])

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Known as Borisov-Alexeev-Borisov conjecture (from early 1990’s).
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For a pair $(X, B)$ and $R$-divisor $A$ define $\text{lct}(X, B, |A|_R) = \sup\{s | (X, B+sN) \text{ is lc for every } 0 \leq N \sim R_A\}$.

**Theorem (Boundedness of lc thresholds [B, 2016])**

For each $d$, $r$, $\epsilon > 0$ there is $t > 0$ such that if
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then $\text{lct}(X, B, |A|_R) \geq t$.

In particular, if $A \sim R M + L$ were $M$, $L \geq 0$, then $\text{lct}(X, B, |M|_R) \geq t$.

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Theorem (Mori-Prokhorov)

Let $X$ be a 3-fold with terminal sing, $f : X \to Z$ a Mori fibre space;

- if $Z$ is a surface, then $Z$ has canonical sing;
- if $Z$ is a curve, then multiplicities of fibres of $f$ are bounded.

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For each $d, \epsilon > 0$, there is $\delta > 0$ such that if $X$ is $\epsilon$-lc of dim $d$ and $f : X \to Z$ is a Mori fibre space, then $Z$ is $\delta$-lc.
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then we can write

$$K_X + B \sim R f^*(K_Z + B_Z + M_Z)$$

such that $(Z, B_Z + M_Z)$ is $\delta$-lc.

Theorem (B, 2012)

Shokurov conjecture holds if $(F, \text{Supp } B|_F)$ belongs to a bounded family where $F$ is general fibre.

Note: BAB implies $F$ belongs to a bounded family.
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Complements

Let $X$ be a Fano variety. An $m$-complement is of the form $K_X + \Delta$ where $(X, \Delta)$ has lc singularities, $m(K_X + \Delta) \sim 0$.

Note that $m\Delta \in |-mK_X|$.

Example: $X = \mathbb{P}^1$, $\Delta = x_1 + x_2$ with $x_i$ distinct points, then $K_X + \Delta$ is a 1-complement.

Example: $X \subset \mathbb{P}^3$ a cubic surface, $\Delta$ a general hyperplane section, then $K_X + \Delta$ is a 1-complement.
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Theorem (Boundedness of complements [B, 2016])

For each $d$ there is $m$ such that any klt Fano variety $X$ of dimension $d$ has an $m$-complement.

Some ideas of the proof:

We can change $X$ birationally and find $B$ such that $(X, B)$ has lc singularities and either

1. $B$ has a component $S$ with coefficient 1 and $-K_{X} + B$ ample,
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3. $X$ is $\epsilon$-lc for fixed $\epsilon > 0$.

These cases require very different inductive treatment.

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$(Z, B_Z + M_Z)$ is a generalised pair; developed in [B-Zhang, 2014].
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Show $F$ is birational to a bounded $F'$.

Show we can make $(F, \Theta_F + P_F)$ bad singularities.

This gives divisors on $F'$ with bounded "degree" but unbounded lc thresholds.

This contradicts (a special case of) the theorem on boundedness of lc thresholds.
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Difficulty with induction: $F$ may not be Fano, singularities of $(F, \Theta_F + P_F)$ hard to control.

Show $F$ is birational to a bounded $F'$.

Show we can make $(F, \Theta_F + P_F)$ bad singularities.

This gives divisors on $F'$ with bounded "degree" but unbounded lc thresholds.

This contradicts (a special case of) the theorem on boundedness of lc thresholds.
Theorem (Boundedness of lc thresholds [B, 2016])

For each $d, r, \epsilon > 0$ there is $t > 0$ such that if

- $(X, B)$ is projective $\epsilon$-lc of dimension $d$,
- $A$ is very ample with $A^d \leq r$, and
- $A - B$ is ample,

then

$$\operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geq t.$$
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Some ideas of proof of this theorem:
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Pick $0 \leq N \sim_\mathbb{R} A$. 
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Some ideas of proof of this theorem:

Pick $0 \leq N \sim_{\mathbb{R}} A$.

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# Boundedness of singularities

## Theorem (Boundedness of lc thresholds [B, 2016])

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Pick $0 \leq N \sim_{\mathbb{R}} A$.

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We need to bounded multiplicity of $T$ in $\phi^* N$ on resolutions $\phi: V \to X$. 

Boundedness of singularities

Use a local-global type of complement to produce \( \Lambda \) such that \((X, \Lambda)\) is lc and \(a(T, X, \Lambda) = 0\).
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Reduce to the case when $(X, \Lambda)$ is log smooth and $T$ is reduced.
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By means of finite maps onto $\mathbb{P}^d$ reduce to a similar problem on $\mathbb{P}^d$. 

Finally the problem is reduced to boundedness of $\epsilon$-lc toric Fano varieties of dim $d$ which is well-known.
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