Arithmetic topological quantum field theory?

Minhyong Kim

New York, September, 2017

Plan

1. Schemes
2. Arithmetic Geometry
3. Fundamental Groups
4. Arithmetic Topology
5. Moduli Spaces
6. Arithmetic Chern-Simons Theory
7. Arithmetic Intersection Numbers

Schemes

Schemes

Recall that scheme theory associates to any ring A a space

$$\mathrm{Spec}(A)$$

whose underlying set is the set of prime ideals in A, with the interpretation that A is the ring of functions on $\mathrm{Spec}(A)$:

An element $a \in A$ is the function

$$P \in \mathrm{Spec}(A) \mapsto [a \mod P] \in A/P \subset k(P) := Frac(A/P).$$

Schemes

Recall that scheme theory associates to any ring A a space

$$\mathrm{Spec}(A)$$

whose underlying set is the set of prime ideals in A, with the interpretation that A is the ring of functions on $\mathrm{Spec}(A)$:

An element $a \in A$ is the function

$$P \in \mathrm{Spec}(A) \mapsto [a \mod P] \in A/P \subset k(P) := Frac(A/P).$$

Each $a \in A$ determines a zero set $Z(a)$ consisting of P such that $a \in P$. This is a basic closed set in the Zariski topology on $\mathrm{Spec}(A)$.

Schemes

Recall that scheme theory associates to any ring A a space

$$\mathrm{Spec}(A)$$

whose underlying set is the set of prime ideals in A, with the interpretation that A is the ring of functions on $\mathrm{Spec}(A)$:

An element $a \in A$ is the function

$$P \in \mathrm{Spec}(A) \mapsto [a \mod P] \in A/P \subset k(P) := Frac(A/P).$$

Each $a \in A$ determines a zero set $Z(a)$ consisting of P such that $a \in P$. This is a basic closed set in the Zariski topology on $\mathrm{Spec}(A)$. Its complement is a basic open set

$$U_a = \mathrm{Spec}(A[1/a]) \subset \mathrm{Spec}(A)$$

consisting of prime ideals that do not contain a.

Schemes

Examples:

Schemes

Examples:

(1)

$$\mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$$

is the variety given as the zero set of the f_i, enriched by adding a point for each subvariety.

Schemes

Examples:

(1)

$$\mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$$

is the variety given as the zero set of the f_i, enriched by adding a point for each subvariety.

(2)

$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{2, 3, 5, 7, 11, \ldots, 37, \ldots, 691, \ldots, 1112707, \ldots\}$

Schemes

Examples:

(1)

$$\mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$$

is the variety given as the zero set of the f_i, enriched by adding a point for each subvariety.

(2)

$$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{2, 3, 5, 7, 11, \ldots, 37, \ldots, 691, \ldots, 1112707, \ldots\}$$

(3)

$$\mathrm{Spec}(\mathbb{Q}[x]) = \{(0)\} \cup \{\text{irreducible polynomials}\}/\mathbb{Q}^\times$$

Schemes

Examples:

(1)

$$\mathrm{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$$

is the variety given as the zero set of the f_i, enriched by adding a point for each subvariety.

(2)

$$\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{2, 3, 5, 7, 11, \ldots, 37, \ldots, 691, \ldots, 1112707, \ldots\}$$

(3)

$$\mathrm{Spec}(\mathbb{Q}[x]) = \{(0)\} \cup \{\text{irreducible polynomials}\}/\mathbb{Q}^\times$$

(4) If F is an algebraic number field, and $\mathcal{O}_F$ its ring of algebraic integers then determining the structure of $\mathrm{Spec}(\mathcal{O}_F)$ is a large part of a course on algebraic number theory.

Schemes

Since $A[1/a]$ is the ring of functions on $\mathrm{Spec}(A[1/a])$, this association naturally determines a sheaf of rings

$$\mathcal{O}_{\mathrm{Spec}(A)}$$

on $\mathrm{Spec}(A)$.

Schemes

Since $A[1/a]$ is the ring of functions on $\mathrm{Spec}(A[1/a])$, this association naturally determines a sheaf of rings

$$\mathcal{O}_{\mathrm{Spec}(A)}$$

on $\mathrm{Spec}(A)$.

A general scheme is a pair

$$(X, \mathcal{O}_X)$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of rings on X, and the pair is locally isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

Schemes

Since $A[1/a]$ is the ring of functions on $\mathrm{Spec}(A[1/a])$, this association naturally determines a sheaf of rings

$$\mathcal{O}_{\mathrm{Spec}(A)}$$

on $\mathrm{Spec}(A)$.

A general scheme is a pair

$$(X, \mathcal{O}_X)$$

where X is a topological space, $\mathcal{O}_X$ is a sheaf of rings on X, and the pair is locally isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

A map

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is induced by a ring homomorphism $A \longrightarrow B$, and all maps between schemes are locally of this form.

Schemes

Let $x \in X$. Then

$$x \in \mathrm{Spec}(A) \subset X,$$

so that x corresponds to a prime ideal $P_x \subset A$. The ring A/P_x is an integral domain with field of fractions denoted $k(x)$.

Schemes

Let $x \in X$. Then

$$x \in \mathrm{Spec}(A) \subset X,$$

so that x corresponds to a prime ideal $P_x \subset A$. The ring A/P_x is an integral domain with field of fractions denoted $k(x)$.

Thus, we get a map

$$\mathrm{Spec}(k(x)) \longrightarrow \mathrm{Spec}(A/P_x) \hookrightarrow \mathrm{Spec}(A) \hookrightarrow X$$

whose image is the point x. We often identify the point x with this map.

Schemes

Let $x \in X$. Then

$$x \in \mathrm{Spec}(A) \subset X,$$

so that x corresponds to a prime ideal $P_x \subset A$. The ring A/P_x is an integral domain with field of fractions denoted $k(x)$.

Thus, we get a map

$$\mathrm{Spec}(k(x)) \longrightarrow \mathrm{Spec}(A/P_x) \hookrightarrow \mathrm{Spec}(A) \hookrightarrow X$$

whose image is the point x. We often identify the point x with this map.

The point x is closed if and only if P_x is a maximal ideal. Denote by

$$X_0$$

the set of closed points.

Arithmetic Geometry

Arithmetic Geometry

Arithmetic geometry is the study of arithmetic geometries,

Arithmetic Geometry

Arithmetic geometry is the study of arithmetic geometries, i.e., schemes that are absolutely of finite type and maps between them. We will call them also *arithmetic schemes*.

The objects of interest are finite unions of affine schemes of the form

$$\mathrm{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)).$$

Arithmetic Geometry

Arithmetic geometry is the study of arithmetic geometries, i.e., schemes that are absolutely of finite type and maps between them. We will call them also *arithmetic schemes*.

The objects of interest are finite unions of affine schemes of the form

$$\mathrm{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)).$$

The inclusion

$$\mathbb{Z} \longrightarrow \mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$$

induces a map

$$\mathrm{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)) \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

Arithmetic Geometry

To get a map

$$\mathrm{Spec}(\mathbb{Z}) \longrightarrow \mathrm{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$$

we need a ring homomorphism

$$\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)) \longrightarrow \mathbb{Z},$$

which corresponds to an integral solution to the equations

$$f_1 = 0, f_2 = 0, \ldots, f_m = 0.$$

Arithmetic Geometry

If X is an arithmetic scheme and $x \in X_0$ a closed point, then $k(x)$ is finite

Arithmetic Geometry

If X is an arithmetic scheme and $x \in X_0$ a closed point, then $k(x)$ is finiteand one studies quantities like

$$\zeta(X, s) := \prod_{x \in X_0} \frac{1}{1 - |k(x)|^{-s}}.$$

Fundamental groups

Fundamental groups

A finite étale map $Y \longrightarrow X$ is (essentially) one that is locally of the form

$$\mathrm{Spec}(A[x]/(f(x))) \longrightarrow \mathrm{Spec}(A),$$

where the discriminant $\Delta(f) \in A$ is a unit.

Fundamental groups

A finite étale map $Y \longrightarrow X$ is (essentially) one that is locally of the form

$$\mathrm{Spec}(A[x]/(f(x))) \longrightarrow \mathrm{Spec}(A),$$

where the discriminant $\Delta(f) \in A$ is a unit.

Example:

For the map

$$\mathrm{Spec}(\mathbb{C}[x][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{C}[x]),$$

we have

$$\Delta = (-1)^{n-1} n^n x^{n-1}.$$

Fundamental groups

A finite étale map $Y \longrightarrow X$ is (essentially) one that is locally of the form

$$\mathrm{Spec}(A[x]/(f(x))) \longrightarrow \mathrm{Spec}(A),$$

where the discriminant $\Delta(f) \in A$ is a unit.

Example:

For the map

$$\mathrm{Spec}(\mathbb{C}[x][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{C}[x]),$$

we have

$$\Delta = (-1)^{n-1} n^n x^{n-1}.$$

This is *not* a unit in $\mathbb{C}[x]$, so the map is not étale.

Fundamental groups

A finite étale map $Y \longrightarrow X$ is (essentially) one that is locally of the form

$$\mathrm{Spec}(A[x]/(f(x))) \longrightarrow \mathrm{Spec}(A),$$

where the discriminant $\Delta(f) \in A$ is a unit.

Example:

For the map

$$\mathrm{Spec}(\mathbb{C}[x][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{C}[x]),$$

we have

$$\Delta = (-1)^{n-1} n^n x^{n-1}.$$

This is *not* a unit in $\mathbb{C}[x]$, so the map is not étale.

However,

$$\mathrm{Spec}(\mathbb{C}[x, x^{-1}][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{C}[x, x^{-1}])$$

is étale.

Fundamental groups

Similarly,

$$\mathrm{Spec}(\mathbb{F}_p[x, x^{-1}][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{F}_p[x, x^{-1}])$$

is étale if $p \nmid n$.

Fundamental groups

Similarly,

$$\mathrm{Spec}(\mathbb{F}_p[x, x^{-1}][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{F}_p[x, x^{-1}])$$

is étale if $p \nmid n$.

The map

$$\mathrm{Spec}(\mathbb{Z}[x]/(x^2 + 23)) \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

is not étale,

Fundamental groups

Similarly,

$$\mathrm{Spec}(\mathbb{F}_p[x, x^{-1}][y]/(y^n - x)) \longrightarrow \mathrm{Spec}(\mathbb{F}_p[x, x^{-1}])$$

is étale if $p \nmid n$.

The map

$$\mathrm{Spec}(\mathbb{Z}[x]/(x^2 + 23)) \longrightarrow \mathrm{Spec}(\mathbb{Z})$$

is not étale,

But

$$\mathrm{Spec}(\mathbb{Z}[1/46][x]/(x^2 + 23)) \longrightarrow \mathrm{Spec}(\mathbb{Z}[1/46])$$

is étale.

Fundamental Groups

A connected scheme X has a fundamental group $\pi_1(X)$ that classifies finite étale maps.

Fundamental Groups

A connected scheme X has a fundamental group $\pi_1(X)$ that classifies finite étale maps.

One construction:

$$\pi_1(X) = \varprojlim_{Y} \mathrm{Aut}(Y/X),$$

as Y runs over the connected Galois finite étale covers of X.

Fundamental Groups

A connected scheme X has a fundamental group $\pi_1(X)$ that classifies finite étale maps.

One construction:

$$\pi_1(X) = \varprojlim_{Y} \mathrm{Aut}(Y/X),$$

as Y runs over the connected Galois finite étale covers of X.

There is an equivalence of categories

$$\text{Finite étale maps } Y \longrightarrow X$$

$$\simeq \text{Finite sets with continuous } \pi_1(X)\text{-action.}$$

This depends only on general properties of the category of finite étale maps to X, in a manner similar to the Tannakian formalism for algebraic groups.

Fundamental Groups

If

$$X \longrightarrow Y,$$

then get a conjugacy class of homomorphisms

$$\pi_1(X) \longrightarrow \pi_1(Y).$$

Fundamental Groups

If

$$X \longrightarrow Y,$$

then get a conjugacy class of homomorphisms

$$\pi_1(X) \longrightarrow \pi_1(Y).$$

For example, if $x \in X$, then

$$\mathrm{Spec}(k(x)) \longrightarrow X,$$

so get

$$\pi_1(\mathrm{Spec}(k(x))) \longrightarrow \pi_1(X).$$

Fundamental Groups

Examples:

Fundamental Groups

Examples:

X: a complex algebraic variety. Then $\pi_1(X)$ is the pro-finite completion of the topological π_1:

$$\pi_1(X) = \varprojlim \pi_1^{top}(X)/N,$$

where N runs over normal finite index subgroups.

Fundamental Groups

Examples:

X: a complex algebraic variety. Then $\pi_1(X)$ is the pro-finite completion of the topological π_1:

$$\pi_1(X) = \varprojlim \pi_1^{top}(X)/N,$$

where N runs over normal finite index subgroups.

X: a variety over $\bar{\mathbb{Q}}$. Then

$$\pi_1(X) \simeq \pi_1(X_{\mathbb{C}}).$$

Fundamental Groups

Examples:

X: a complex algebraic variety. Then $\pi_1(X)$ is the pro-finite completion of the topological π_1:

$$\pi_1(X) = \varprojlim \pi_1^{top}(X)/N,$$

where N runs over normal finite index subgroups.

X: a variety over $\bar{\mathbb{Q}}$. Then

$$\pi_1(X) \simeq \pi_1(X_{\mathbb{C}}).$$

X: a variety over $\bar{\mathbb{F}}_p$. Then the fundamental group is similar to fundamental groups over $\mathbb{C}$, except for the 'p-part.'

Fundamental Groups

$X = \mathrm{Spec}(F)$, where F is a field. Then

$$\pi_1(X) \simeq \mathrm{Gal}(\bar{F}/F),$$

where $\bar{F}$ is a separable closure of F.

Fundamental Groups

$X = \mathrm{Spec}(F)$, where F is a field. Then

$$\pi_1(X) \simeq \mathrm{Gal}(\bar{F}/F),$$

where $\bar{F}$ is a separable closure of F.

$X = \mathrm{Spec}(\mathcal{O}_F)$ for an algebraic number field F. Then $\pi_1(X)^{ab} = Cl_F$, the ideal class group of F.

Fundamental Groups

$X = \mathrm{Spec}(F)$, where F is a field. Then

$$\pi_1(X) \simeq \mathrm{Gal}(\bar{F}/F),$$

where $\bar{F}$ is a separable closure of F.

$X = \mathrm{Spec}(\mathcal{O}_F)$ for an algebraic number field F. Then $\pi_1(X)^{ab} = Cl_F$, the ideal class group of F.

But

$$\pi_1(\mathrm{Spec}(\mathcal{O}_F)) \simeq \varprojlim_K \mathrm{Gal}(K/F),$$

where K runs over the extensions of F that are unramified, i.e., such that $\mathrm{Spec}(\mathcal{O}_K) \longrightarrow \mathrm{Spec}(\mathcal{O}_F)$ is étale. It can be infinite and quite complicated.

Fundamental Groups

$X = \mathrm{Spec}(\mathbb{Z})$. Then $\pi_1(X) = 1$.

Fundamental Groups

$X = \mathrm{Spec}(\mathbb{Z})$. Then $\pi_1(X) = 1$.

[Golod-Shafarevich]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13})$, then $\pi_1(X)$ is infinite.

Fundamental Groups

$X = \mathrm{Spec}(\mathbb{Z})$. Then $\pi_1(X) = 1$.

[Golod-Shafarevich]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13})$, then $\pi_1(X)$ is infinite.

[Kwang-Seob Kim]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{653})$. Then (assuming GRH)

$$\pi_1(X) \simeq A_5.$$

Fundamental Groups

$X = \mathrm{Spec}(\mathbb{Z})$. Then $\pi_1(X) = 1$.

[Golod-Shafarevich]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13})$, then $\pi_1(X)$ is infinite.

[Kwang-Seob Kim]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{653})$. Then (assuming GRH)

$$\pi_1(X) \simeq A_5.$$

[Kwang-Seob Kim]
$X = \mathrm{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{1429})$, Then (assuming GRH)

$$\pi_1(X) \simeq PSL_2(\mathbb{F}_8) \times C_2.$$

Fundamental Groups

$X_S = \mathrm{Spec}(\mathcal{O}_F) \setminus S$, where S is a finite set of prime ideals in S. Consider K/F finite Galois extensions such that

$$\mathrm{Spec}(\mathcal{O}_K) \longrightarrow \mathrm{Spec}(\mathcal{O}_F)$$

is étale in the complement of S. Then

$$\pi_1(X_S) \simeq \varprojlim_K \mathrm{Gal}(K/F).$$

Fundamental Groups

When X is a scheme of finite type, then for every closed point $x \in X_0$, get a conjugacy class of elements

$$[Fr_x] \subset \pi_1(X)$$

Fundamental Groups

When X is a scheme of finite type, then for every closed point $x \in X_0$, get a conjugacy class of elements

$$[Fr_x] \subset \pi_1(X)$$

This is from the map

$$\mathrm{Spec}(k(x)) \longrightarrow X,$$

functoriality

$$\pi_1(\mathrm{Spec}(k(x))) \longrightarrow \pi_1(X)$$

and the fact that

$$\pi_1(\mathrm{Spec}(k(x))) = \mathrm{Gal}(\overline{k(x)}/k(x)) = < Fr_x > .$$

Arithmetic topology

Arithmetic topology

F: Algebraic number field.

$\mathcal{O}_F$: Ring of algebraic integers in F.

$v \in \mathrm{Spec}(\mathcal{O}_F)$ corresponding to non-zero prime ideal $P_v \subset \mathcal{O}_v$.

$\mathcal{O}_v = \varprojlim_n \mathcal{O}_F/P_v^n$ completion of O_F at v. So $\mathcal{O}_v$ has maximal ideal m_v such that $m_v = (\pi_v)$ and $\mathcal{O}_v/m_v = k(v)$.

$F_v = Frac(\mathcal{O}_v) = \mathcal{O}_v[1/\pi_v]$.

Arithmetic topology

F: Algebraic number field.

$\mathcal{O}_F$: Ring of algebraic integers in F.

$v \in \mathrm{Spec}(\mathcal{O}_F)$ corresponding to non-zero prime ideal $P_v \subset \mathcal{O}_v$.

$\mathcal{O}_v = \varprojlim_n \mathcal{O}_F/P_v^n$ completion of O_F at v. So $\mathcal{O}_v$ has maximal ideal m_v such that $m_v = (\pi_v)$ and $\mathcal{O}_v/m_v = k(v)$.

$F_v = Frac(\mathcal{O}_v) = \mathcal{O}_v[1/\pi_v]$.

Consider $\mathbb{Q}$, $\mathbb{Z}$, p, $\mathbb{Z}_p$, $\mathbb{Q}_p$.

Arithmetic topology

Review of some analogies:

Arithmetic topology

Review of some analogies:

$$X = \mathrm{Spec}(\mathcal{O}_F) \sim \text{3-manifold} \quad M$$

Arithmetic topology

Review of some analogies:

$$X = \mathrm{Spec}(\mathcal{O}_F) \sim \text{3-manifold} \quad M$$

$$[\mathrm{Spec}(k(v)) \hookrightarrow \mathrm{Spec}(\mathcal{O}_F)] \sim [\text{knot } S^1 \hookrightarrow M]$$

Arithmetic topology

Review of some analogies:

$$X = \mathrm{Spec}(\mathcal{O}_F) \sim \text{3-manifold} \quad M$$

$$[\mathrm{Spec}(k(v)) \hookrightarrow \mathrm{Spec}(\mathcal{O}_F)] \sim [\text{knot } S^1 \hookrightarrow M]$$

$$\mathrm{Spec}(\mathcal{O}_v) \sim [\text{solid torus around knot (tubular neighbourhood)}]$$

Arithmetic topology

Review of some analogies:

$$X = \mathrm{Spec}(\mathcal{O}_F) \sim \text{3-manifold} \quad M$$

$$[\mathrm{Spec}(k(v)) \hookrightarrow \mathrm{Spec}(\mathcal{O}_F)] \sim [\text{knot } S^1 \hookrightarrow M]$$

$$\mathrm{Spec}(\mathcal{O}_v) \sim [\text{solid torus around knot (tubular neighbourhood)}]$$

$$\mathrm{Spec}(F_v) \sim [\text{knotted torus}]$$

Arithmetic topology

Review of some analogies:

$$X = \mathrm{Spec}(\mathcal{O}_F) \sim \text{3-manifold} \quad M$$

$$[\mathrm{Spec}(k(v)) \hookrightarrow \mathrm{Spec}(\mathcal{O}_F)] \sim [\text{knot } S^1 \hookrightarrow M]$$

$$\mathrm{Spec}(\mathcal{O}_v) \sim [\text{solid torus around knot (tubular neighbourhood)}]$$

$$\mathrm{Spec}(F_v) \sim [\text{knotted torus}]$$

$$X_S = \mathrm{Spec}(\mathcal{O}_F) \setminus S$$

$$\sim [\text{3-manifold with boundary tori, one for each prime in } S]$$

Arithmetic Topology

$$\pi_1(X_S) \sim \pi_1(\text{3-manifold with boundary})$$

$$[\mathrm{Gal}(\bar{F}_v/F_v) = \pi_1(\mathrm{Spec}(F_v)), \quad v \in S] \sim \pi_1(\text{boundary tori})$$

$$X_S \sim \text{hyperbolic 3-manifold}$$

for S a sufficiently large finite set of primes.

Moduli Spaces

Moduli Spaces

Given an arithmetic scheme Y and a p-adic Lie group A, we will be interested in

$$M(Y, A) := \mathrm{Hom}(\pi_1(Y), A)//A,$$

the moduli space of continuous homomorphisms

$$\rho : \pi_1(Y) \longrightarrow A$$

up to conjugation.

Moduli Spaces

Given an arithmetic scheme Y and a p-adic Lie group A, we will be interested in

$$M(Y, A) := \mathrm{Hom}(\pi_1(Y), A)//A,$$

the moduli space of continuous homomorphisms

$$\rho : \pi_1(Y) \longrightarrow A$$

up to conjugation.

Key point of lecture:

Even though an arithmetic scheme is quite different from a manifold, $M(Y, A)$ is structurally similar to moduli spaces of bundles in geometry and physics.

Moduli Spaces

Local functions:

A point $x \in Y_0$ together with a representation V of A on a k-vector space defines a function

$$Tr_{x,V} : M(Y, A) \longrightarrow k;$$

$$\rho \mapsto Tr(\rho(Fr_x)|V).$$

Moduli Spaces

Local functions:

A point $x \in Y_0$ together with a representation V of A on a k-vector space defines a function

$$Tr_{x,V} : M(Y, A) \longrightarrow k;$$

$$\rho \mapsto Tr(\rho(Fr_x)|V).$$

Two global functions:

(1) L-functions;

Moduli Spaces

Local functions:

A point $x \in Y_0$ together with a representation V of A on a k-vector space defines a function

$$Tr_{x,V} : M(Y, A) \longrightarrow k;$$

$$\rho \mapsto Tr(\rho(Fr_x)|V).$$

Two global functions:

(1) L-functions;

(2) Chern-Simons functions.

Moduli Spaces

For $X = \mathrm{Spec}(\mathcal{O}_F)$, can define an *arithmetic Chern-Simons functional*

$$\mathbb{CS}_c : M(X, A) \longrightarrow \mathbb{Q}_p.$$

Moduli Spaces

For $X = \mathrm{Spec}(\mathcal{O}_F)$, can define an *arithmetic Chern-Simons functional*

$$\mathbb{CS}_c : M(X, A) \longrightarrow \mathbb{Q}_p.$$

Would like to compute

$$\int_{\rho \in M(X,A)} \exp Tr_{x_1, V_1}(\rho) \cdots \exp Tr_{x_m, V_m}(\rho) \exp\left(2\pi i \mathbb{CS}(\rho)\right) d\rho.$$

Moduli Spaces

For $X = \mathrm{Spec}(\mathcal{O}_F)$, can define an *arithmetic Chern-Simons functional*

$$\mathbb{CS}_c : M(X, A) \longrightarrow \mathbb{Q}_p.$$

Would like to compute

$$\int_{\rho \in M(X,A)} \exp Tr_{x_1, V_1}(\rho) \cdots \exp Tr_{x_m, V_m}(\rho) \exp\left(2\pi i \mathbb{CS}(\rho)\right) d\rho.$$

At the moment, can define $\mathbb{CS}_c$, but can compute integrals only in the finite abelian case.

Arithmetic Chern-Simons Functionals (Finite Case)

Assume now that $n \geq 2$ and F is an algebraic number field such that $\exp(2\pi i/n^2) \in F$.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$.

Arithmetic Chern-Simons Functionals (Finite Case)

Assume now that $n \geq 2$ and F is an algebraic number field such that $\exp(2\pi i/n^2) \in F$.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$.

Basic cohomological fact:

$$H^3(X, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Arithmetic Chern-Simons Functionals (Finite Case)

Assume now that $n \geq 2$ and F is an algebraic number field such that $\exp(2\pi i/n^2) \in F$.

Let $X = \mathrm{Spec}(\mathcal{O}_F)$.

Basic cohomological fact:

$$H^3(X, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

We get the map

$$\mathrm{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Arithmetic Chern-Simons Functionals (Finite Case)

The CS-functional will depend on the choice of a cohomology class

$$c \in H^3(A, \mathbb{Z}/n).$$

Arithmetic Chern-Simons Functionals (Finite Case)

The CS-functional will depend on the choice of a cohomology class

$$c \in H^3(A, \mathbb{Z}/n).$$

Then

$$\mathbb{CS} : M(X, A) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

Arithmetic Chern-Simons Functionals (Finite Case)

The CS-functional will depend on the choice of a cohomology class

$$c \in H^3(A, \mathbb{Z}/n).$$

Then

$$\mathbb{CS} : M(X, A) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is defined by

$$\rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \mathrm{inv}(\rho^*(c)).$$

$$\exp(2\pi i \mathbb{CS}) : M(X, A) \longrightarrow S^1.$$

Arithmetic Chern-Simons Functionals (Finite Case)

Example:

Let $A = \mathbb{Z}/n$. Then

$$M(X, \mathbb{Z}/n) = \mathrm{Hom}(Cl_X, \mathbb{Z}/n).$$

Take $c \in H^3(A, \mathbb{Z}/n)$ to be given as

$$a \cup \delta a,$$

where $a \in H^1(A, \mathbb{Z}/_n)$ is the class coming from the identity map, while

$$\delta : H^1(A, \mathbb{Z}/n) \longrightarrow H^2(A, \mathbb{Z}/n)$$

is the Bockstein map coming from the extension

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

Then

$$\mathbb{CS}_{a\cup\delta a}(\rho) = \mathrm{inv}(\rho^*(a) \cup \rho^*(\delta a)).$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

$X_S = \mathrm{Spec}(\mathcal{O}_F[1/S])$ for a finite set S of places;

$\partial X_S = \coprod_{v \in S} \mathrm{Spec}(F_v)$.

$$\pi_S := \pi_1(X_S), \quad \pi_v := \mathrm{Gal}(\bar{F}_v/F_v),$$

and fix a tuple of homomorphisms

$$i_S = (i_v : \pi_v \longrightarrow \pi_S)_{v \in S}$$

corresponding to embeddings $\bar{F} \hookrightarrow \bar{F}_v$.

Assume S contains all places dividing n.

Now $c \in Z^3(A, \mathbb{Z}/n)$ will denote a 3-cocycle.

Finite Arithmetic Chern-Simons Functionals with Boundaries

In addition to the global moduli space

$$M(X_S, A) := \mathrm{Hom}(\pi_S, A)//A$$

we have the local moduli space

$$M(\partial X_S, A) := \{\phi_S = (\phi_v)_{v \in S} \mid \phi_v : \pi_v \longrightarrow A\}//A.$$

Thus, we get a restriction map

$$r = i_S^* : M(X_S, A) \longrightarrow M(\partial X_S, A)$$

More cohomological facts:

Finite Arithmetic Chern-Simons Functionals with Boundaries

More cohomological facts:

$$H^i(\pi_v, \mathbb{Z}/n) = 0, \quad \forall i > 2;$$

$$H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

$$H^i(\pi_S, \mathbb{Z}/n) = 0, \quad \forall i > 2;$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

More cohomological facts:

$$H^i(\pi_v, \mathbb{Z}/n) = 0, \quad \forall i > 2;$$

$$H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

$$H^i(\pi_S, \mathbb{Z}/n) = 0, \quad \forall i > 2;$$

Reciprocity sequence:

$$0 \longrightarrow H^2(\pi_S, \mathbb{Z}/n) \longrightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\sum} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v \in S}$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial.

Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v \in S}$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$d^{-1}(\phi_v^*(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n),$$

the solutions to

$$d\alpha_v = \phi_v^*(c),$$

form a torsor for

$$Z^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n) = H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v\in S}$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$d^{-1}(\phi_v^*(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n),$$

the solutions to

$$d\alpha_v = \phi_v^*(c),$$

form a torsor for

$$Z^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n) = H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Hence,

$$\prod_{v\in S} d^{-1}(\phi_v^*(c))$$

is a torsor for

$$\prod_{v\in S} H^2(\pi_v, \mathbb{Z}/n) \simeq \prod_{v\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

We push this out using the sum map

$$\Sigma : \prod_v \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T}(\phi_S) := \Sigma_*(\prod_v d^{-1}(\phi_v)).$$

As ϕ_S varies, we get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T} \longrightarrow M(\partial X_S, A)$$

over the local moduli space.

Finite Arithmetic Chern-Simons Functionals with Boundaries

We push this out using the sum map

$$\Sigma : \prod_v \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T}(\phi_S) := \Sigma_*(\prod_v d^{-1}(\phi_v)).$$

As ϕ_S varies, we get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T} \longrightarrow M(\partial X_S, A)$$

over the local moduli space.

Can also exponentiate to get an S^1-bundle $\exp(\mathcal{T})$.

Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in M(X_S, A)$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_S, \mathbb{Z}/n),$$

and put

$$\mathbb{CS}(\rho) = \Sigma_*(r(\beta)) \in \mathcal{T}(r(\rho)).$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in M(X_S, A)$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_S, \mathbb{Z}/n),$$

and put

$$\mathbb{CS}(\rho) = \Sigma_*(r(\beta)) \in \mathcal{T}(r(\rho)).$$

Lemma

$\mathbb{CS}(\rho)$ is independent of the choice of β.

This follows immediately from the reciprocity sequence

$$0 \longrightarrow H^2(\pi_S, \mathbb{Z}/n) \longrightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\Sigma} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in M(X_S, A)$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_S, \mathbb{Z}/n),$$

and put

$$\mathbb{CS}(\rho) = \Sigma_*(r(\beta)) \in \mathcal{T}(r(\rho)).$$

Lemma

$\mathbb{CS}(\rho)$ *is independent of the choice of* β.

This follows immediately from the reciprocity sequence

$$0 \longrightarrow H^2(\pi_S, \mathbb{Z}/n) \longrightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \xrightarrow{\Sigma} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

Thus, as ρ varies, we get a canonical section

$$\mathbb{CS} \in \Gamma(M(X_S, A), r^*(\mathcal{T})).$$

Finite Arithmetic Chern-Simons Functionals with Boundaries

In topological Chern-Simons theory, one takes an integral

$$\int_{\{\rho \mid (\rho|\partial X_S)=\rho_S\}} \exp(2\pi i \mathbb{CS}(\rho))d\rho$$

and then let ρ_S vary to get a section of

$$\exp(\mathcal{T}) \longrightarrow M(\partial X_S, A).$$

More precisely, from the view of topological quantum field theory, this is the state

$$\Psi(X_S) \in V(\partial X_S) := \Gamma(M(\partial X_S, A), \exp(\mathcal{T}))$$

on ∂X_S that the theory assigns to X_S.

At the moment, can define a finite-coefficient analogue of this construction.

Computing Chern-Simons: Decomposition Formula

Computing Chern-Simons: Decomposition Formula

We have the natural map

$$\pi_S \xrightarrow{q_S} \pi.$$

Computing Chern-Simons: Decomposition Formula

We have the natural map

$$\pi_S \xrightarrow{q_S} \pi.$$

Thus, we get the map

$$M(X, A) \longrightarrow M(X_S, A)$$

$$\rho \mapsto \rho \circ q_S.$$

$$\mathbb{CS}(\rho \circ q_S) \in \mathcal{T}(r(\rho)).$$

Computing Chern-Simons: Decomposition Formula

We have the natural map

$$\pi_S \xrightarrow{q_S} \pi.$$

Thus, we get the map

$$M(X, A) \longrightarrow M(X_S, A)$$

$$\rho \mapsto \rho \circ q_S.$$

$$\mathbb{CS}(\rho \circ q_S) \in \mathcal{T}(r(\rho)).$$

On the other hand, for each $v \in S$, we get a composed representation

$$\rho_v^{un} : \pi_v^{un} \longrightarrow \pi \xrightarrow{\rho} A,$$

where $\pi_v^{un} \simeq \mathrm{Gal}(\bar{k_v}/k_v)$ is the unramified quotient of π_v.

Computing Chern-Simons: Decomposition Formula

By solving

$$d\beta_v = (\rho_v^{un})^*(c)$$

with

$$\beta_v \in C^2(\pi_v^{un}, \mathbb{Z}/n)/B^2(\pi_v^{un}, \mathbb{Z}/n) \longrightarrow C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n)$$

for each v, we get another element

$$\sum_v (\beta_v) \in \mathcal{T}(r(\rho)).$$

This is independent of the choice of β_v because

$$H^2(\pi_v^{un}, \mathbb{Z}/n) = 0.$$

Computing Chern-Simons: Decomposition Formula

By solving

$$d\beta_v = (\rho_v^{un})^*(c)$$

with

$$\beta_v \in C^2(\pi_v^{un}, \mathbb{Z}/n)/B^2(\pi_v^{un}, \mathbb{Z}/n) \longrightarrow C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n)$$

for each v, we get another element

$$\sum_v (\beta_v) \in \mathcal{T}(r(\rho)).$$

This is independent of the choice of β_v because

$$H^2(\pi_v^{un}, \mathbb{Z}/n) = 0.$$

Thus, we can take the difference

$$\mathbb{CS}(\rho \circ q_S) - \sum_v (\beta_v) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

.

Computing Chern-Simons: Decomposition Formula

Theorem (w/ H. Chung, D. Kim, J. Park, and H. Yoo)

$$\mathbb{CS}(\rho) = \mathbb{CS}(\rho \circ q_S) - \sum_v (\beta_v).$$

This is an analogue of the *decomposition formula* in Chern-Simons theory, and gives us a way to compute the values.

Computing Chern-Simons: Decomposition Formula

Theorem (w/ H. Chung, D. Kim, J. Park, and H. Yoo)

$$\mathbb{CS}(\rho) = \mathbb{CS}(\rho \circ q_S) - \sum_v (\beta_v).$$

This is an analogue of the *decomposition formula* in Chern-Simons theory, and gives us a way to compute the values.

Key Point:

> $\mathbb{CS}(\rho)$ *is the difference between a global ramified trivialisation and a local unramified trivialisation.*

Chern-Simons Invariant: Examples

[Joint work with H. Chung, D. Kim, J. Park, and H. Yoo]

$A = \mathbb{Z}/2$.

Let $p \equiv 1 \mod 4$ be a prime and $F_t = \mathbb{Q}(\sqrt{-pt})$, where t is a positive square-free integer prime to p. Then $F_t(\sqrt{p})/F_t$ is unramified, giving us a character

$$\rho_t : \pi_1(\mathcal{O}_{F_t}) \longrightarrow \mathbb{Z}/2.$$

Chern-Simons Invariant: Examples

[Joint work with H. Chung, D. Kim, J. Park, and H. Yoo]
$A = \mathbb{Z}/2$.

Let $p \equiv 1 \mod 4$ be a prime and $F_t = \mathbb{Q}(\sqrt{-pt})$, where t is a positive square-free integer prime to p. Then $F_t(\sqrt{p})/F_t$ is unramified, giving us a character

$$\rho_t : \pi_1(\mathcal{O}_{F_t}) \longrightarrow \mathbb{Z}/2.$$

Then

Proposition

$$\mathbb{CS}_{a\cup\delta a}(\rho_t) = 1/2 \Leftrightarrow \left(\frac{t}{p}\right) = -1.$$

Corollary

If $\left(\frac{t}{p}\right) = -1$, *then* $F_t(\sqrt{p})/F$ *does not embed in an* $\mathbb{Z}/4$ *unramified extension of* F_t.

Arithmetic linking numbers

Arithmetic linking numbers

$X = \operatorname{Spec}(\mathcal{O}_F)$, where F is a totally complex number field with a fixed trivialisation $\mathbb{Z}/n \simeq \mu_n \subset F$.

Would like to define arithmetic linking numbers using a complex

$$[d : \Omega^1_X[n] \longrightarrow \Omega^2_X[n]]$$

Arithmetic linking numbers

$X = \mathrm{Spec}(\mathcal{O}_F)$, where F is a totally complex number field with a fixed trivialisation $\mathbb{Z}/n \simeq \mu_n \subset F$.

Would like to define arithmetic linking numbers using a complex

$$[d : \Omega^1_X[n] \longrightarrow \Omega^2_X[n]]$$

$$= [d : H^1(X, \mathbb{Z}/n) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)].$$

Arithmetic linking numbers

$X = \mathrm{Spec}(\mathcal{O}_F)$, where F is a totally complex number field with a fixed trivialisation $\mathbb{Z}/n \simeq \mu_n \subset F$.

Would like to define arithmetic linking numbers using a complex

$$[d : \Omega^1_X[n] \longrightarrow \Omega^2_X[n]]$$

$$= [d : H^1(X, \mathbb{Z}/n) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)].$$

The differential here is obtained as a composition of the Bockstein map

$$\delta : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)$$

and a map

$$H^2(X, \mathbb{Z}/n) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)$$

induced by cup product and duality.

Arithmetic linking numbers

That is, recall the perfect (Artin-Verdier) duality pairing

$$\langle \cdot, \cdot \rangle : H^1(X, \mathbb{Z}/n) \times \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

and the cup product pairing

$$\cup : H^1(X, \mathbb{Z}/n) \times H^2(X, \mathbb{Z}/n) \longrightarrow H^3(X, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

inducing a map

$$r : H^2(X, \mathbb{Z}/n) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m),$$

which we use to define $d := r \circ \delta$.

Arithmetic linking numbers

Remarks:

–Bockstein map as a differential is partly inspired by one of the constructions of the De Rham-Witt complex.

–Examining the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/n \arrow[r] \arrow[d, "\simeq"'] & \mu_{n^2} \arrow[r] \arrow[d] & \mathbb{Z}/n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu_n \arrow[r] & \mathbb{G}_m \arrow[r, "(\cdot)^n"] & \mathbb{G}_m \arrow[r] & 0
\end{tikzcd}$$

we see that the Bockstein map is also induced by the Chern class map for $\mathbb{G}_m$-torsors.

Arithmetic linking numbers

Given an ideal I, we can define its class

$$[I]_n \in \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m) \simeq \mathrm{Hom}(H^1(M, \mathbb{Z}/n), \frac{1}{n}\mathbb{Z}/\mathbb{Z})$$

$$\simeq Cl(\mathcal{O}_F)/n.$$

Say I is *n-homologically trivial*, if $[I]_n$ is in the image of

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow \mathrm{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m).$$

Arithmetic linking numbers

If I and J are n-homologically trivial, then we define their mod n height pairing or linking number by

$$h_n(I, J) := \langle d^{-1}[I]_n, [J]_n \rangle.$$

Lemma

The pairing is well-defined, and

$$h_n(I, J) = h_n(J, I).$$

Arithmetic linking numbers

Proposition

Let I, J be ideals in $\mathcal{O}_F$ supported on X_S that are n-torsion in the Picard group of X_S. Let $f \in F^*$ such that $div(f|X_S) = I^n$. Let T be the support of J, π_v be a uniformiser at v, and $e_v = ord_v(J)$.

Arithmetic linking numbers

Proposition

Let I, J be ideals in $\mathcal{O}_F$ supported on X_S that are n-torsion in the Picard group of X_S. Let $f \in F^*$ such that $div(f|X_S) = I^n$.
Let T be the support of J, π_v be a uniformiser at v, and $e_v = ord_v(J)$.
Then

$$ht_n(I, J) = \sum_{v \in T} (f_v, \pi_v^{e_v})_n,$$

where the bracket denotes the $n-$th power residue symbol.

Arithmetic linking numbers

Can define also the mod n abelian Chern-Simons invariant of X:

$$Z(X,n) := \sum_{\rho \in H^1(X,\mathbb{Z}/n)} \exp(2\pi i \mathbb{CS}(\rho)).$$

Also with a linear term:

$$\sum_{\rho \in H^1(X,\ \mathbb{Z}/n)} \exp[2\pi i(\mathbb{CS}(\rho) + \sum_j \langle \rho, [\xi_j]_n \rangle)]$$

for a finite set $\{\xi_j\}$ of homologically trivial ideals.

Arithmetic linking numbers

Let $n = p$, a prime. Let $a = \dim H^1(X, \mathbb{Z}/p)$ and $b = \dim Ker(d)$. Denote by $\bar{d}$ the induced isomorphism

$$\bar{d} : H^1(X, \mathbb{Z}/p)/K \simeq Im(d).$$

Arithmetic linking numbers

Let $n = p$, a prime. Let $a = \dim H^1(X, \mathbb{Z}/p)$ and $b = \dim Ker(d)$. Denote by $\bar{d}$ the induced isomorphism

$$\bar{d} : H^1(X, \mathbb{Z}/p)/K \simeq Im(d).$$

Then

$$\sum_{\rho \in H^1(X,\ \mathbb{Z}/p)} \exp[2\pi i(\mathbb{CS}(\rho) + \sum_j \langle \rho, [\xi_j]_p \rangle)]$$

$$= p^{(a+b)/2} \left(\frac{\det(\bar{d})}{p} \right) i^{[\frac{(a-b)(p-1)^2}{4}]} \exp[-2\pi i(\frac{1}{4} \sum_{i,\ j} ht_p(\xi_i, \xi_j))]$$