Arithmetic topological quantum field theory?

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Plan

1. Schemes
2. Arithmetic Geometry
3. Fundamental Groups
4. Arithmetic Topology
5. Moduli Spaces
6. Arithmetic Chern-Simons Theory
7. Arithmetic Intersection Numbers
Recall that scheme theory associates to any ring $A$ a space $\text{Spec}(A)$ whose underlying set is the set of prime ideals in $A$, with the interpretation that $A$ is the ring of functions on $\text{Spec}(A)$:

An element $a \in A$ is the function $P \in \text{Spec}(A) \mapsto a \mod P \in A/P \subset k(P) := \text{Frac}(A/P)$.

Each $a \in A$ determines a zero set $Z(a)$ consisting of $P$ such that $a \in P$. This is a basic closed set in the Zariski topology on $\text{Spec}(A)$.

Its complement is a basic open set $U_a = \text{Spec}(A[1/a]) \subset \text{Spec}(A)$ consisting of prime ideals that do not contain $a$. 
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Examples:

Spec \((\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))\) is the variety given as the zero set of the \(f_i\), enriched by adding a point for each subvariety.

Spec \((\mathbb{Z})\) = \{\(0\)\} \cup \{2, 3, 5, 7, 11, \ldots, 37, \ldots, 691, \ldots, 1112707, \ldots\}

Spec \((\mathbb{Q}[x])\) = \{\(0\)\} \cup \{irreducible\ polynomials\} / \mathbb{Q} ×

If \(F\) is an algebraic number field, and \(O_F\) its ring of algebraic integers then determining the structure of Spec \((O_F)\) is a large part of a course on algebraic number theory.
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(4) If \( F \) is an algebraic number field, and \( \mathcal{O}_F \) its ring of algebraic integers then determining the structure of \( \text{Spec}(\mathcal{O}_F) \) is a large part of a course on algebraic number theory.
Schemes

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A general scheme is a pair

$$(X, \mathcal{O}_X)$$

where $X$ is a topological space, $\mathcal{O}_X$ is a sheaf of rings on $X$, and the pair is locally isomorphic to $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$. 
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A map

$$\text{Spec}(B) \longrightarrow \text{Spec}(A)$$

is induced by a ring homomorphism $A \longrightarrow B$, and all maps between schemes are locally of this form.
Let $x \in X$. Then

$$x \in \text{Spec}(A) \subset X,$$

so that $x$ corresponds to a prime ideal $P_x \subset A$. The ring $A/P_x$ is an integral domain with field of fractions denoted $k(x)$. 
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Thus, we get a map

$$\text{Spec}(k(x)) \longrightarrow \text{Spec}(A/P_x) \hookrightarrow \text{Spec}(A) \hookrightarrow X$$

whose image is the point $x$. We often identify the point $x$ with this map.
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The point \( x \) is closed if and only if \( P_x \) is a maximal ideal. Denote by

\[ X_0 \]

the set of closed points.
Arithmetic Geometry

Arithmetic geometry is the study of arithmetic geometries, i.e., schemes that are absolutely of finite type and maps between them. We will call them also arithmetic schemes. The objects of interest are finite unions of affine schemes of the form $\text{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m))$. The inclusion $\mathbb{Z} \subset \mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$ induces a map $\text{Spec}(\mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)) \to \text{Spec}(\mathbb{Z})$. 

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we need a ring homomorphism

\[ \mathbb{Z}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m) \longrightarrow \mathbb{Z}, \]

which corresponds to an integral solution to the equations

\[ f_1 = 0, f_2 = 0, \ldots, f_m = 0. \]
If $X$ is an arithmetic scheme and $x \in X_0$ a closed point, then $k(x)$ is finite.
If $X$ is an arithmetic scheme and $x \in X_0$ a closed point, then $k(x)$ is finite and one studies quantities like

$$\zeta(X, s) := \prod_{x \in X_0} \frac{1}{1 - |k(x)|^{-s}}.$$
Fundamental groups

A finite étale map $Y \to X$ is (essentially) one that is locally of the form $\text{Spec}(A[x]/(f(x))) \to \text{Spec}(A)$, where the discriminant $\Delta(f) \in A$ is a unit.

Example: For the map $\text{Spec}(C[x][y]/(y^n - x)) \to \text{Spec}(C[x])$, we have $\Delta = (-1)^n - x^n$. This is *not* a unit in $C[x]$, so the map is not étale.

However, $\text{Spec}(C[x,x^{-1}][y]/(y^n - x)) \to \text{Spec}(C[x,x^{-1}])$ is étale.
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Similarly,

$$\text{Spec}(\mathbb{F}_p[x, x^{-1}][y]/(y^n - x)) \rightarrow \text{Spec}(\mathbb{F}_p[x, x^{-1}])$$

is étale if $p \nmid n$. 
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One construction:

$$\pi_1(X) = \lim_{Y} \text{Aut}(Y/X),$$

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There is an equivalence of categories

Finite étale maps $Y \to X$

$\simeq$ Finite sets with continuous $\pi_1(X)$-action.

This depends only on general properties of the category of finite étale maps to $X$, in a manner similar to the Tannakian formalism for algebraic groups.
If

\[ X \longrightarrow Y, \]

then get a conjugacy class of homomorphisms

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For example, if \( x \in X \), then

\[ \text{Spec}(k(x)) \rightarrow X, \]

so get

\[ \pi_1(\text{Spec}(k(x))) \rightarrow \pi_1(X). \]
Fundamental Groups

Examples:

\[ \pi_1(X) \text{ is the pro-finite completion of the topological } \pi_1: \pi_1(X) = \lim_{\leftarrow} \pi_{\text{top}}(X)/N, \]
where \( N \) runs over normal finite index subgroups.

\[ X \text{ a variety over } \bar{\mathbb{Q}} \text{. Then } \pi_1(X) \cong \pi_1(X_{\mathbb{C}}). \]

\[ X \text{ a variety over } \bar{\mathbb{F}}_p. \text{ Then the fundamental group is similar to fundamental groups over } \mathbb{C}, \text{ except for the 'p-part.'} \]
Fundamental Groups

Examples:

$X$: a complex algebraic variety. Then $\pi_1(X)$ is the pro-finite completion of the topological $\pi_1$:

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\(X\): a variety over \(\overline{\mathbb{F}}_p\). Then the fundamental group is similar to fundamental groups over \(\mathbb{C}\), except for the ‘\(p\)-part.’
Fundamental Groups

\[ X = \text{Spec}(F), \text{ where } F \text{ is a field. Then} \]

\[ \pi_1(X) \cong \text{Gal}(\bar{F}/F), \]

where \( \bar{F} \) is a separable closure of \( F \).
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\[ X = \text{Spec}(\mathcal{O}_F) \text{ for an algebraic number field } F. \text{ Then} \]

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$X = \text{Spec}(\mathcal{O}_F)$ for an algebraic number field $F$. Then $\pi_1(X)^{ab} = \text{Cl}_F$, the ideal class group of $F$.

But

$$\pi_1(\text{Spec}(\mathcal{O}_F)) \simeq \varprojlim K \text{Gal}(K/F),$$

where $K$ runs over the extensions of $F$ that are unramified, i.e., such that $\text{Spec}(\mathcal{O}_K) \rightarrow \text{Spec}(\mathcal{O}_F)$ is étale. It can be infinite and quite complicated.
Fundamental Groups

\[ X = \text{Spec}(\mathbb{Z}). \text{ Then } \pi_1(X) = 1. \]
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[Golod-Shafarevich]
\[ X = \text{Spec}(\mathcal{O}_F) \text{ for } F = \mathbb{Q}(\sqrt{-2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}), \text{ then } \pi_1(X) \text{ is infinite.} \]
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[Kwang-Seob Kim]
$X = \text{Spec}(\mathcal{O}_F)$ for $F = \mathbb{Q}(\sqrt{653})$. Then (assuming GRH)

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[Kwang-Seob Kim]
\[ X = \text{Spec}(\mathcal{O}_F) \text{ for } F = \mathbb{Q}(\sqrt{1429}), \] Then (assuming GRH)
\[ \pi_1(X) \simeq PSL_2(\mathbb{F}_8) \times C_2. \]
$X_S = \text{Spec}(\mathcal{O}_F) \setminus S$, where $S$ is a finite set of prime ideals in $S$. Consider $K/F$ finite Galois extensions such that

$$
\text{Spec}(\mathcal{O}_K) \longrightarrow \text{Spec}(\mathcal{O}_F)
$$

is étale in the complement of $S$. Then

$$
\pi_1(X_S) \simeq \varprojlim_K \text{Gal}(K/F).
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Fundamental Groups

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This is from the map

$$\text{Spec}(k(x)) \to X,$$

functoriality

$$\pi_1(\text{Spec}(k(x))) \to \pi_1(X)$$

and the fact that

$$\pi_1(\text{Spec}(k(x))) = \text{Gal}(\overline{k(x)}/k(x)) = \langle Fr_x \rangle.$$
Arithmetic topology
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*F*: Algebraic number field.

*O*：<sup>F</sup>: Ring of algebraic integers in *F*.

*ν* ∈ Spec(*O*<sub>F</sub>) corresponding to non-zero prime ideal *P*<sub>ν</sub> ⊂ *O*<sub>ν</sub>. 

*O*<sub>ν</sub> = \( \lim \leftarrow_n O_F / P_ν^n \) completion of *O*<sub>F</sub> at *ν*. So *O*<sub>ν</sub> has maximal ideal *m*<sub>ν</sub> such that *m*<sub>ν</sub> = (π<sub>ν</sub>) and *O*<sub>ν</sub>/*m*<sub>ν</sub> = *k*(ν).

*F*<sub>ν</sub> = Frac(*O*<sub>ν</sub>) = *O*<sub>ν</sub>[1/π<sub>ν</sub>].
Arithmetic topology

$F$: Algebraic number field.

$\mathcal{O}_F$: Ring of algebraic integers in $F$.

$v \in \text{Spec}(\mathcal{O}_F)$ corresponding to non-zero prime ideal $P_v \subset \mathcal{O}_v$.

$\mathcal{O}_v = \lim_{\leftarrow n} \mathcal{O}_F/P_v^n$ completion of $\mathcal{O}_F$ at $v$. So $\mathcal{O}_v$ has maximal ideal $m_v$ such that $m_v = (\pi_v)$ and $\mathcal{O}_v/m_v = k(v)$.

$F_v = \text{Frac}(\mathcal{O}_v) = \mathcal{O}_v[1/\pi_v]$.

Consider $\mathbb{Q}, \mathbb{Z}, p, \mathbb{Z}_p, \mathbb{Q}_p$. 
Arithmetic topology

Review of some analogies:

\[
\text{Spec}(\mathcal{O}_F) \sim 3\text{-manifold } M \quad \text{Spec}(k(\mathcal{v})) \subset - \text{Spec}(\mathcal{O}_F) \sim \text{solid torus around knot (tubular neighbourhood)} \quad \text{Spec}(\mathcal{F}_v) \sim \text{knotted torus}
\]

\[
X = \text{Spec}(\mathcal{O}_F) \setminus S \sim 3\text{-manifold with boundary tori, one for each prime in } S
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Arithmetic topology

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\[ X_S = \text{Spec}(\mathcal{O}_F) \setminus S \sim [3\text{-manifold with boundary tori, one for each prime in } S] \]
\[ \pi_1(X_S) \sim \pi_1(3\text{-manifold with boundary}) \]

\[ [\text{Gal}(\bar{F}_v/F_v) = \pi_1(\text{Spec}(F_v)), \ v \in S] \sim \pi_1(\text{boundary tori}) \]

\[ X_S \sim \text{hyperbolic 3-manifold} \]

for \( S \) a sufficiently large finite set of primes.
Moduli Spaces

Given an arithmetic scheme $Y$ and a $p$-adic Lie group $A$, we will be interested in $M(Y, A) := \text{Hom}(\pi_1(Y), A) \mod A$, the moduli space of continuous homomorphisms $\rho: \pi_1(Y) \to A$ up to conjugation.

Key point of lecture: Even though an arithmetic scheme is quite different from a manifold, $M(Y, A)$ is structurally similar to moduli spaces of bundles in geometry and physics.
Given an arithmetic scheme $Y$ and a $p$-adic Lie group $A$, we will be interested in

$$M(Y, A) := \text{Hom}(\pi_1(Y), A)//A,$$

the moduli space of continuous homomorphisms

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up to conjugation.
Moduli Spaces

Given an arithmetic scheme $Y$ and a $p$-adic Lie group $A$, we will be interested in

$$M(Y, A) := \text{Hom}(\pi_1(Y), A)/\!\!/A,$$

the moduli space of continuous homomorphisms

$$\rho : \pi_1(Y) \longrightarrow A$$

up to conjugation.

Key point of lecture:

*Even though an arithmetic scheme is quite different from a manifold, $M(Y, A)$ is structurally similar to moduli spaces of bundles in geometry and physics.*
Local functions:

A point $x \in Y_0$ together with a representation $V$ of $A$ on a $k$-vector space defines a function

$$Tr_{x,V} : M(Y, A) \rightarrow k;$$

$$\rho \mapsto Tr(\rho(Fr_x)|V).$$
Moduli Spaces

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Two global functions:

(1) $L$-functions;
Local functions:

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Tr_{x,V} : M(Y,A) \longrightarrow k;
\]

\[
\rho \mapsto Tr(\rho(Fr_x)|V).
\]

Two global functions:

(1) \( L \)-functions;

(2) Chern-Simons functions.
Moduli Spaces

For $X = \text{Spec}(\mathcal{O}_F)$, can define an arithmetic Chern-Simons functional

$$\mathcal{CS}_c : M(X, A) \to \mathbb{Q}_p.$$
For $X = \text{Spec}(\mathcal{O}_F)$, can define an \textit{arithmetic Chern-Simons functional}

$$\mathcal{CS}_c : M(X, A) \to \mathbb{Q}_p.$$  

Would like to compute

$$\int_{\rho \in M(X, A)} \exp Tr_{x_1, V_1}(\rho) \cdots \exp Tr_{x_m, V_m}(\rho) \exp (2\pi i \mathcal{CS}(\rho)) d\rho.$$
For $X = \text{Spec}(O_F)$, can define an arithmetic Chern-Simons functional

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At the moment, can define $\mathcal{CS}_c$, but can compute integrals only in the finite abelian case.
Assume now that $n \geq 2$ and $F$ is an algebraic number field such that $\exp \left( \frac{2\pi i}{n^2} \right) \in F$.

Let $X = \text{Spec}(\mathcal{O}_F)$. 
Assume now that \( n \geq 2 \) and \( F \) is an algebraic number field such that \( \exp \left( \frac{2\pi i}{n^2} \right) \in F \).

Let \( X = \text{Spec}(\mathcal{O}_F) \).

Basic cohomological fact:

\[
H^3(X, \mathbb{Z}/n) \cong \frac{1}{n} \mathbb{Z}/\mathbb{Z}.
\]
Assume now that $n \geq 2$ and $F$ is an algebraic number field such that $\exp\left(\frac{2\pi i}{n^2}\right) \in F$.

Let $X = \text{Spec}(\mathcal{O}_F)$.

Basic cohomological fact:

$$H^3(X, \mathbb{Z}/n) \cong \frac{1}{n} \mathbb{Z}/\mathbb{Z}.$$ 

We get the map

$$\text{inv} : H^3(\pi_1(X), \mathbb{Z}/n) \longrightarrow H^3(X, \mu_n) \cong \frac{1}{n} \mathbb{Z}/\mathbb{Z}.$$
Arithmetic Chern-Simons Functionals (Finite Case)

The CS-functional will depend on the choice of a cohomology class

\[ c \in H^3(A, \mathbb{Z}/n). \]
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\[ c \in H^3(A, \mathbb{Z}/n). \]

Then
\[ \text{CS} : M(X, A) \to \frac{1}{n} \mathbb{Z}/\mathbb{Z} \]
is defined by
\[ \rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \text{inv}(\rho^*(c)). \]
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\[ c \in H^3(A, \mathbb{Z}/n). \]

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is defined by

\[ \rho \mapsto \rho^*(c) \in H^3(\pi_1(X), \mathbb{Z}/n) \mapsto \text{inv}(\rho^*(c)). \]

\[ \exp(2\pi i \text{CS}) : M(X, A) \longrightarrow S^1. \]
Arithmetic Chern-Simons Functionals (Finite Case)

Example:
Let \( A = \mathbb{Z}/n \). Then

\[
M(X, \mathbb{Z}/n) = \text{Hom}(Cl_X, \mathbb{Z}/n).
\]

Take \( c \in H^3(A, \mathbb{Z}/n) \) to be given as

\[
a \cup \delta a,
\]

where \( a \in H^1(A, \mathbb{Z}/n) \) is the class coming from the identity map, while

\[
\delta : H^1(A, \mathbb{Z}/n) \longrightarrow H^2(A, \mathbb{Z}/n)
\]

is the Bockstein map coming from the extension

\[
0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n \longrightarrow 0.
\]

Then

\[
\text{CS}_{a \cup \delta a}(\rho) = \text{inv}(\rho^*(a) \cup \rho^*(\delta a)).
\]
Finite Arithmetic Chern-Simons Functionals with Boundaries

\[ X_S = \text{Spec}(\mathcal{O}_F[1/S]) \] for a finite set \( S \) of places;
\[ \partial X_S = \bigsqcup_{v \in S} \text{Spec}(F_v). \]

\[ \pi_S := \pi_1(X_S), \quad \pi_v := \text{Gal}(\bar{F}_v/F_v), \]

and fix a tuple of homomorphisms

\[ i_S = (i_v : \pi_v \rightarrow \pi_S)_{v \in S} \]

corresponding to embeddings \( \bar{F} \rightarrow \bar{F}_v \).

Assume \( S \) contains all places dividing \( n \).

Now \( c \in Z^3(A, \mathbb{Z}/n) \) will denote a 3-cocycle.
In addition to the global moduli space 

\[ M(X_S, A) := \text{Hom}(\pi_S, A) // A \]

we have the local moduli space 

\[ M(\partial X_S, A) := \{ \phi_S = (\phi_v)_{v \in S} \mid \phi_v : \pi_v \to A \} // A. \]

Thus, we get a restriction map 

\[ r = i^*_S : M(X_S, A) \longrightarrow M(\partial X_S, A) \]
More cohomological facts:
More cohomological facts:

\[ H^i(\pi_v, \mathbb{Z}/n) = 0, \quad \forall i > 2; \]

\[ H^2(\pi_v, \mathbb{Z}/n) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}. \]

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More cohomological facts:

\[ H^i(\pi_v, \mathbb{Z}/n) = 0, \quad \forall i > 2; \]

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\[ H^i(\pi_S, \mathbb{Z}/n) = 0, \quad \forall i > 2; \]

Reciprocity sequence:

\[ 0 \rightarrow H^2(\pi_S, \mathbb{Z}/n) \rightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \rightarrow \sum \frac{1}{n} \mathbb{Z}/\mathbb{Z} \rightarrow 0. \]
Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v \in S}$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial.
Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v \in S}$, each $\phi_v^*(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$d^{-1}(\phi_v^*(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n),$$

the solutions to

$$d\alpha_v = \phi_v^*(c),$$

form a torsor for

$$Z^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n) = H^2(\pi_v, \mathbb{Z}/n) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$
Finite Arithmetic Chern-Simons Functionals with Boundaries

For any $\phi_S = (\phi_v)_{v \in S}$, each $\phi^*_v(c) \in Z^3(\pi_v, \mathbb{Z}/n)$ is trivial. Thus,

$$d^{-1}(\phi^*_v(c)) \in C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n),$$

the solutions to

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Hence,

$$\prod_{v \in S} d^{-1}(\phi^*_v(c))$$

is a torsor for

$$\prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \cong \prod_{v \in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z}.$$
Finite Arithmetic Chern-Simons Functionals with Boundaries

We push this out using the sum map

$$\Sigma : \prod_v \frac{1}{n}\mathbb{Z}/\mathbb{Z} \rightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$$

to get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T}(\phi_S) := \Sigma_\ast(\prod_v d^{-1}(\phi_v))$$.

As $\phi_S$ varies, we get a $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$-torsor

$$\mathcal{T} \rightarrow M(\partial X_S, A)$$

over the local moduli space.
We push this out using the sum map

\[ \Sigma : \prod_v \frac{1}{n} \mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{n} \mathbb{Z}/\mathbb{Z} \]

to get a \( \frac{1}{n} \mathbb{Z}/\mathbb{Z} \)-torsor

\[ \mathcal{T}(\phi_S) := \Sigma_*(\prod_v d^{-1}(\phi_v)). \]

As \( \phi_S \) varies, we get a \( \frac{1}{n} \mathbb{Z}/\mathbb{Z} \)-torsor

\[ \mathcal{T} \longrightarrow M(\partial X_S, A) \]

over the local moduli space.

Can also exponentiate to get an \( S^1 \)-bundle \( \exp(\mathcal{T}) \).
Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in M(X_S, A)$, we can solve

$$d\beta = \rho^*(c) \in Z^3(\pi_S, \mathbb{Z}/n),$$

and put

$$\text{CS}(\rho) = \Sigma_*(r(\beta)) \in T(r(\rho)).$$
Finite Arithmetic Chern-Simons Functionals with Boundaries

If $\rho \in M(X_S, A)$, we can solve

$$d\beta = \rho^*(c) \in \mathbb{Z}^3(\pi_S, \mathbb{Z}/n),$$

and put

$$\text{CS}(\rho) = \sum_{\ast}(r(\beta)) \in T(r(\rho)).$$

Lemma

$\text{CS}(\rho)$ is independent of the choice of $\beta$.

This follows immediately from the reciprocity sequence

$$0 \rightarrow H^2(\pi_S, \mathbb{Z}/n) \rightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \rightarrow \sum_{\prod_{v \in S}} \frac{1}{n} \mathbb{Z}/\mathbb{Z} \rightarrow 0.$$
Finite Arithmetic Chern-Simons Functionals with Boundaries

If \( \rho \in M(X_S, A) \), we can solve

\[
d\beta = \rho^*(c) \in \mathbb{Z}^3(\pi_S, \mathbb{Z}/n),
\]

and put

\[
\mathcal{CS}(\rho) = \Sigma_*(r(\beta)) \in \mathcal{T}(r(\rho)).
\]

Lemma
\( \mathcal{CS}(\rho) \) is independent of the choice of \( \beta \).

This follows immediately from the reciprocity sequence

\[
0 \rightarrow H^2(\pi_S, \mathbb{Z}/n) \rightarrow \prod_{v \in S} H^2(\pi_v, \mathbb{Z}/n) \rightarrow \sum \frac{1}{n} \mathbb{Z}/\mathbb{Z} \rightarrow 0.
\]

Thus, as \( \rho \) varies, we get a canonical section

\[
\mathcal{CS} \in \Gamma(M(X_S, A), r^*(\mathcal{T})).
\]
Finite Arithmetic Chern-Simons Functionals with Boundaries

In topological Chern-Simons theory, one takes an integral

\[ \int_{\{ \rho \mid (\rho|\partial X_S) = \rho_S \}} \exp(2\pi i \mathcal{CS}(\rho)) d\rho \]

and then let \( \rho_S \) vary to get a section of

\[ \exp(\mathcal{T}) \longrightarrow M(\partial X_S, A). \]

More precisely, from the view of topological quantum field theory, this is the state

\[ \Psi(X_S) \in V(\partial X_S) := \Gamma(M(\partial X_S, A), \exp(\mathcal{T})) \]

on \( \partial X_S \) that the theory assigns to \( X_S \).

At the moment, can define a finite-coefficient analogue of this construction.
Computing Chern-Simons: Decomposition Formula

We have the natural map \( \pi \circ S \circ \pi \). Thus, we get the map

\[
\rho \rightarrow \rho \circ q \circ S
\]

\( \mathrm{CS}(\rho \circ q \circ S) \in T(r(\rho)) \).

On the other hand, for each \( v \in S \), we get a composed representation \( \rho_{un}^v \):

\[
\pi_{un}^v - \pi \rho - A,
\]

where \( \pi_{un}^v \cong \mathrm{Gal}(\bar{k}^v / k_v) \) is the unramified quotient of \( \pi_v \).
Computing Chern-Simons: Decomposition Formula

We have the natural map

\[ \pi_S \xrightarrow{q_S} \pi. \]
Computing Chern-Simons: Decomposition Formula

We have the natural map

\[ \pi_S \xrightarrow{q_S} \pi. \]

Thus, we get the map

\[ M(X, A) \xrightarrow{\rho \mapsto \rho \circ q_S} M(X_S, A) \]

\[ \text{CS}(\rho \circ q_S) \in T(r(\rho)). \]
Computing Chern-Simons: Decomposition Formula

We have the natural map

$$\pi_S \xrightarrow{q_S} \pi.$$ 

Thus, we get the map

$$M(X, A) \longrightarrow M(X_S, A)$$

$$\rho \mapsto \rho \circ q_S.$$ 

$$\text{CS}(\rho \circ q_S) \in T(r(\rho)).$$

On the other hand, for each $v \in S$, we get a composed representation

$$\rho_v^{un} : \pi_v^{un} \longrightarrow \pi \xrightarrow{\rho} A,$$

where $\pi_v^{un} \simeq \text{Gal}(\bar{k}_v/k_v)$ is the unramified quotient of $\pi_v$. 
Computing Chern-Simons: Decomposition Formula

By solving
\[ d\beta_v = (\rho_v^{un})^*(c) \]
with
\[ \beta_v \in C^2(\pi_v^{un}, \mathbb{Z}/n)/B^2(\pi_v^{un}, \mathbb{Z}/n) \longrightarrow C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n) \]
for each \( v \), we get another element
\[ \sum_v (\beta_v) \in \mathcal{T}(r(\rho)). \]

This is independent of the choice of \( \beta_v \) because
\[ H^2(\pi_v^{un}, \mathbb{Z}/n) = 0. \]
Computing Chern-Simons: Decomposition Formula

By solving

\[ d\beta_v = (\rho^u_n)^*(c) \]

with

\[ \beta_v \in C^2(\pi^u_n, \mathbb{Z}/n)/B^2(\pi^u_n, \mathbb{Z}/n) \rightarrow C^2(\pi_v, \mathbb{Z}/n)/B^2(\pi_v, \mathbb{Z}/n) \]

for each \( v \), we get another element

\[ \sum_v (\beta_v) \in T(\tau(\rho)). \]

This is independent of the choice of \( \beta_v \) because

\[ H^2(\pi^u_n, \mathbb{Z}/n) = 0. \]

Thus, we can take the difference

\[ \text{CS}(\rho \circ qs) - \sum_v (\beta_v) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z} \]
Computing Chern-Simons: Decomposition Formula

Theorem (w/ H. Chung, D. Kim, J. Park, and H. Yoo)

\[ \text{CS}(\rho) = \text{CS}(\rho \circ q_s) - \sum_v (\beta_v). \]

This is an analogue of the decomposition formula in Chern-Simons theory, and gives us a way to compute the values.
Computing Chern-Simons: Decomposition Formula

Theorem (w/ H. Chung, D. Kim, J. Park, and H. Yoo)

$$\text{CS}(\rho) = \text{CS}(\rho \circ q_S) - \sum_v (\beta_v).$$

This is an analogue of the decomposition formula in Chern-Simons theory, and gives us a way to compute the values.

Key Point:

$$\text{CS}(\rho) \text{ is the difference between a global ramified trivialisation and a local unramified trivialisation.}$$
Chern-Simons Invariant: Examples

[Joint work with H. Chung, D. Kim, J. Park, and H. Yoo]

$A = \mathbb{Z}/2$.

Let $p \equiv 1 \mod 4$ be a prime and $F_t = \mathbb{Q}(\sqrt{-pt})$, where $t$ is a positive square-free integer prime to $p$. Then $F_t(\sqrt{p})/F_t$ is unramified, giving us a character

$$\rho_t : \pi_1(O_{F_t}) \longrightarrow \mathbb{Z}/2.$$
Chern-Simons Invariant: Examples

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$$\rho_t : \pi_1(O_{F_t}) \longrightarrow \mathbb{Z}/2.$$ 

Then

Proposition

$$\mathbb{CS}_{a \cup \delta a}(\rho_t) = 1/2 \iff \left(\frac{t}{p}\right) = -1.$$ 

Corollary

If $\left(\frac{t}{p}\right) = -1$, then $F_t(\sqrt{p})/F$ does not embed in an $\mathbb{Z}/4$ unramified extension of $F_t$. 
Arithmetic linking numbers

$X = \operatorname{Spec}(O_F)$, where $F$ is a totally complex number field with a fixed trivialisation $\mathbb{Z}/n \cong \mu_n \subset F$.

Would like to define arithmetic linking numbers using a complex $[d : \Omega^1_X[n] - \Omega^2_X[n]] = [d : H^1(X, \mathbb{Z}/n) - \operatorname{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)]$.

The differential here is obtained as a composition of the Bockstein map $\delta : H^1(X, \mathbb{Z}/n) - H^2(X, \mathbb{Z}/n)$ and a map $H^2(X, \mathbb{Z}/n) - \operatorname{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)$ induced by cup product and duality.
Arithmetic linking numbers

\[ X = \text{Spec}(\mathcal{O}_F), \text{ where } F \text{ is a totally complex number field with a fixed trivialisation } \mathbb{Z}/n \cong \mu_n \subset F. \]

Would like to define arithmetic linking numbers using a complex

\[
[d : \Omega^1_X[n] \to \Omega^2_X[n]]
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$X = \text{Spec}(\mathcal{O}_F)$, where $F$ is a totally complex number field with a fixed trivialisation $\mathbb{Z}/n \simeq \mu_n \subset F$.

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$$= [d : H^1(X, \mathbb{Z}/n) \longrightarrow \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)].$$
Arithmetic linking numbers

\( X = \text{Spec}(\mathcal{O}_F) \), where \( F \) is a totally complex number field with a fixed trivialisation \( \mathbb{Z}/n \cong \mu_n \subset F \).

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The differential here is obtained as a composition of the Bockstein map

\[
\delta : H^1(X, \mathbb{Z}/n) \longrightarrow H^2(X, \mathbb{Z}/n)
\]

and a map

\[
H^2(X, \mathbb{Z}/n) \longrightarrow \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m)
\]

induced by cup product and duality.
Arithmetic linking numbers

That is, recall the perfect (Artin-Verdier) duality pairing

\[ \langle \cdot, \cdot \rangle : H^1(X, \mathbb{Z}/n) \times \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m) \rightarrow \frac{1}{n} \mathbb{Z}/\mathbb{Z} \]

and the cup product pairing

\[ \cup : H^1(X, \mathbb{Z}/n) \times H^2(X, \mathbb{Z}/n) \rightarrow H^3(X, \mathbb{Z}/n) \simeq \frac{1}{n} \mathbb{Z}/\mathbb{Z} \]

inducing a map

\[ r : H^2(X, \mathbb{Z}/n) \rightarrow \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m), \]

which we use to define \( d := r \circ \delta. \)
Arithmetic linking numbers

Remarks:
– Bockstein map as a differential is partly inspired by one of the constructions of the De Rham-Witt complex.
– Examining the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}/n & \rightarrow & \mu_n^2 & \rightarrow & \mathbb{Z}/n & \rightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mu_n & \rightarrow & \mathbb{G}_m & \rightarrow & \mathbb{G}_m & \rightarrow & 0
\end{array}
\]

we see that the Bockstein map is also induced by the Chern class map for $\mathbb{G}_m$-torsors.
Arithmetic linking numbers

Given an ideal $I$, we can define its class

$$[I]_n \in \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m) \cong \text{Hom}(H^1(M, \mathbb{Z}/n), \frac{1}{n}\mathbb{Z}/\mathbb{Z})$$

$$\cong \text{Cl}(\mathcal{O}_F)/n.$$ 

Say $I$ is $n$-homologically trivial, if $[I]_n$ is in the image of

$$d : H^1(X, \mathbb{Z}/n) \longrightarrow \text{Ext}^2_X(\mathbb{Z}/n, \mathbb{G}_m).$$
Arithmetic linking numbers

If $I$ and $J$ are $n$-homologically trivial, then we define their mod $n$ height pairing or linking number by

$$h_n(I, J) := \langle d^{-1}[I]_n, [J]_n \rangle.$$ 

**Lemma**

The pairing is well-defined, and

$$h_n(I, J) = h_n(J, I).$$
Arithmetic linking numbers

Proposition

Let $I, J$ be ideals in $O_F$ supported on $X_S$ that are $n$-torsion in the Picard group of $X_S$. Let $f \in F^*$ such that $\text{div}(f|_{X_S}) = I^n$. Let $T$ be the support of $J$, $\pi_v$ be a uniformiser at $v$, and $e_v = \text{ord}_v(J)$. Then

$$ht_n(I, J) = \sum_{v \in T} (f_v, \pi_v)^n,$$

where the bracket denotes the $n$-th power residue symbol.
Arithmetic linking numbers

Proposition

Let $I, J$ be ideals in $\mathcal{O}_F$ supported on $X_S$ that are $n$-torsion in the Picard group of $X_S$. Let $f \in F^*$ such that $\text{div}(f|X_S) = I^n$. Let $T$ be the support of $J$, $\pi_v$ be a uniformiser at $v$, and $e_v = \text{ord}_v(J)$. Then

$$ht_n(I, J) = \sum_{v \in T} (f_v, \pi_v^{e_v})_n,$$

where the bracket denotes the $n$–th power residue symbol.
Arithmetic linking numbers

Can define also the mod $n$ abelian Chern-Simons invariant of $X$:

$$Z(X, n) := \sum_{\rho \in H^1(X, \mathbb{Z}/n)} \exp(2\pi i \mathbb{CS} (\rho)).$$

Also with a linear term:

$$\sum_{\rho \in H^1(X, \mathbb{Z}/n)} \exp[2\pi i (\mathbb{CS} (\rho) + \sum_j \langle \rho, [\xi_j]_n \rangle)]$$

for a finite set $\{\xi_j\}$ of homologically trivial ideals.
Arithmetic linking numbers

Let $n = p$, a prime. Let $a = \dim H^1(X, \mathbb{Z}/p)$ and $b = \dim \ker(d)$. Denote by $\bar{d}$ the induced isomorphism

$$\bar{d} : H^1(X, \mathbb{Z}/p)/K \simeq \text{Im}(d).$$
Let $n = p$, a prime. Let $a = \dim H^1(X, \mathbb{Z}/p)$ and $b = \dim \ker(d)$. Denote by $\bar{d}$ the induced isomorphism

$$\bar{d} : H^1(X, \mathbb{Z}/p)/K \simeq \text{Im}(d).$$

Then

$$\sum_{\rho \in H^1(X, \mathbb{Z}/p)} \exp\left[2\pi i (\mathcal{CS}(\rho) + \sum_j \langle \rho, [\xi_j]_p \rangle)\right]$$

$$= p^{(a+b)/2} \left( \frac{\det(\bar{d})}{p} \right) i^{(a-b)(p-1)^2/4} \exp\left[-2\pi i \frac{1}{4} \sum_{i, j} h_{tp}(\xi_i, \xi_j)\right]$$