O’Nan Moonshine

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(joint work with J. Duncan and K. Ono)

Universität zu Köln

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Monstrous Moonshine
- Preliminaries
- A connection between the Monster and modular functions

Other Moonshine

O’Nan Moonshine
- Rademacher sums
- Integrality
- Positivity

Traces of singular moduli

Arithmetic applications
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Classification of finite simple groups

Theorem

A finite simple group $G$ either belongs to one of 8 infinite families or is one of 26 sporadic simple groups,

Source: wikipedia
The Monster group $\mathbb{M}$

Some properties of the Monster

- The largest of the 26 sporadic finite simple groups
The Monster group $\mathbb{M}$

Some properties of the Monster

- The largest of the 26 sporadic finite simple groups
- $\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \cdot 10^{53}$
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Some properties of the Monster

- The largest of the 26 sporadic finite simple groups
- $\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \cdot 10^{53}$
- 194 conjugacy classes, hence 194 irreducible representations (over $\mathbb{C}$) with characters $\chi_1, \ldots, \chi_{194}$
Reminder: $\text{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$
Reminder: \( \text{SL}_2(\mathbb{R}) \) acts on the upper half-plane \( \mathcal{H} \) via

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\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}.
\]

Definition

Let \( \Gamma \leq \text{SL}_2(\mathbb{R}) \) be a discrete subgroup such that \( \text{vol}(\Gamma \setminus \mathcal{H}) < \infty \). A meromorphic function \( f : \mathcal{H} \to \hat{\mathbb{C}} \) is called a modular function for \( \Gamma \) if

\[
f(\gamma \tau) = f(\tau)
\]

for all \( \gamma \in \Gamma, \tau \in \mathcal{H} \) (growth condition at the boundary).
Reminder: $\text{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}$ via

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**Definition**

Let $\Gamma \leq \text{SL}_2(\mathbb{R})$ be a discrete subgroup such that $\text{vol}(\Gamma \backslash \mathcal{H}) < \infty$. A holomorphic function $f : \mathcal{H} \rightarrow \hat{\mathbb{C}}$ is called a (weakly holomorphic) modular form of weight $k$ for $\Gamma$ if

$$
f(\gamma \tau) = (c\tau + d)^k f(\tau)
$$

for all $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$, $\tau \in \mathcal{H}$ (+growth condition at the boundary). If $\text{Im}(\tau)^{\frac{k}{2}} f(\tau)$ is bounded on $\mathcal{H}$, we call $f$ a cusp form.
Facts

1. The quotient $\Gamma \backslash \mathcal{H}$ can be compactified to a Riemann surface $X(\Gamma)$ of genus $g$. Modular functions for $\Gamma$ define meromorphic functions on $X(\Gamma)$.
1 The quotient $\Gamma \backslash \mathcal{H}$ can be compactified to a Riemann surface $X(\Gamma)$ of genus $g$. Modular functions for $\Gamma$ define meromorphic functions on $X(\Gamma)$.

2 The field of meromorphic functions on $X(\Gamma)$ is isomorphic to an algebraic extension of $\mathbb{C}(x)$ of degree $g$. In particular if $g = 0$, it is isomorphic to $\mathbb{C}(x)$. 

**Facts**
Hauptmoduln

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1. The quotient $\Gamma \backslash \mathcal{H}$ can be compactified to a Riemann surface $X(\Gamma)$ of genus $g$. Modular functions for $\Gamma$ define meromorphic functions on $X(\Gamma)$.

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Definition

Let $\Gamma$ be as above such that $X(\Gamma)$ has genus 0 (+ mild extra conditions). A suitably normalized generator for the field of modular functions for $\Gamma$ is called the Hauptmodul for $\Gamma$. 
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   - Integrality
   - Positivity

4. **Traces of singular moduli**

5. **Arithmetic applications**
For \( N \in \mathbb{N} \) let

\[ \Gamma_0(N) := \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} \]

and

\[ \Gamma_0(p)^+ := N_{\text{SL}_2(\mathbb{R})}(\Gamma_0(p)) \quad (p \text{ prime}). \]
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**Theorem (A. Ogg)**

For $p$ prime, the Riemann surface $X(\Gamma_0(p)^+)$ has genus zero if and only if $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$. 
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For $p$ prime, the Riemann surface $X(\Gamma_0(p)^+)$ has genus zero if and only if $p$ divides $\#M$. 

**Question:** Why is this so?
The Jack Daniels problem

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Monstrous Moonshine I

Dimensions of irreducible representations:

\[ \chi_1(1) = 1, \quad \chi_2(1) = 196\,883, \quad \chi_3(1) = 21\,296\,876, \quad \chi_4(1) = 842\,609\,326. \]
Monstrous Moonshine I

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Hauptmodul for \( SL_2(\mathbb{Z}) \) \((q := e^{2\pi i \tau})\):

\[
J(\tau) = j(\tau) - 744 = \frac{E_4(\tau)^3}{\Delta(\tau)} - 744
\]

\[
= \sum_{n=-1}^{\infty} j_n q^n = q^{-1} + 196\,884 q + 21\,493\,760 q^2 + 864\,299\,970 q^3 + O(q^4).
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Observation (J. McKay & J. G. Thompson, 1979)

\[ j_1 = \chi_1(1) + \chi_2(1). \]
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j_2 = \chi_1(1) + \chi_2(1) + \chi_3(1)
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\[
j_3 = 2\chi_1(1) + 2\chi_2(1) + \chi_3(1) + \chi_4(1)
\]
Values of irreducible characters at other conjugacy classes.

\[ \chi_1(2A) = 1, \quad \chi_2(2A) = 4371, \quad \chi_3(2A) = 91884, \quad \chi_4(2A) = 1139374. \]
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Hauptmodul for \( \Gamma_0(2)^+ \):

\[
J^+_2(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 2^{12} \frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}} + 24
\]

\[
= \sum_{n=-1}^{\infty} \alpha_n q^n = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + O(q^4).
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Monstrous Moonshine II

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There are 194 Hauptmoduln whose coefficients agree, as those of $J$ above, with character values of $\mathbb{M}$. 
Monstrous Moonshine II

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Definition
For a finite group $G$ let $V = \bigoplus_n V_n$ be a graded $G$-module, where all graded components $V_n$ are finite-dimensional. Then for each $g \in G$ we call the power series

$$\mathcal{T}_g(q) = \sum_n \text{tr}(g|V_n)q^n$$

the McKay-Thompson series of $g$ with respect to $V$. 
Monstrous Moonshine Conjecture

There is an infinite dimensional graded representation $V^h$ of $M$ whose McKay-Thompson series are the 194 Hauptmoduln found by Conway–Norton.
Monstrous Moonshine III

Monstrous Moonshine Conjecture

There is an infinite dimensional graded representation $V^\natural$ of $M$ whose McKay-Thompson series are the 194 Hauptmoduln found by Conway–Norton.

“Theorem” (Atkin–Fong–Smith, ∼ 1985)

The Moonshine module $V^\natural$ exists (abstract existence proof).
Monstrous Moonshine Conjecture

There is an infinite dimensional graded representation $V^\mathbb{H}$ of $\mathbb{M}$ whose McKay-Thompson series are the 194 Hauptmoduln found by Conway–Norton.

“Theorem” (Atkin–Fong–Smith, ∼ 1985)

The Moonshine module $V^\mathbb{H}$ exists (abstract existence proof).

Theorem (R. E. Borcherds, 1992)

The Moonshine module $V^\mathbb{H}$ is a vertex operator algebra constructed by Frenkel–Lepowsky–Meurman, whose automorphism group is isomorphic to $\mathbb{M}$. 
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There are 23 even unimodular lattices in dimension 24 with rootsystem of full rank, the Niemeier lattices. Examples: $A_1^{24}, A_2^{12}$. 
The Umbral groups

There are 23 even unimodular lattices in dimension 24 with rootsystem of full rank, the Niemeier lattices.
Examples: $A_1^{24}, A_2^{12}$.

For a Niemeier lattice $L$, its Umbral Group $G^L$ is defined as

$$G^L := \text{Aut}(L)/\text{Weyl}(L).$$

Examples: $G^{A_1^{24}} = M_{24}, \quad G^{A_2^{12}} = M_{12}$. 
Observation (Eguchi–Ooguri–Tachikawa, 2010)

Some dimensions of irreducible representations of $M_{24}$ are multiplicities of superconformal algebra characters of the K3 elliptic genus, which are known to be coefficients of a (vector-valued) mock theta function.
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Some dimensions of irreducible representations of $M_{24}$ are multiplicities of superconformal algebra characters of the K3 elliptic genus, which are known to be coefficients of a (vector-valued) mock theta function.

Theorem (T. Gannon, 2012)

There is an infinite-dimensional graded $M_{24}$-module whose McKay-Thompson series are specific (vector-valued) mock theta functions.
Umbral Moonshine Conjecture (Cheng–Duncan–Harvey, 2012)

For every Umbral Group $G^L$, there is an infinite-dimensional graded $G^L$-module whose McKay-Thompson series are specific (vector-valued) mock theta functions.
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For every Umbral Group $G^L$, there is an an infinite-dimensional graded $G^L$-module whose McKay-Thompson series are specific (vector-valued) mock theta functions.

Theorem (Duncan–Griffin–Ono, 2015)

The Umbral Moonshine conjecture is true.
Conjecture (Harvey–Rayhaun, 2015)

There is an infinite-dimensional graded $Th$-supermodule $W = \bigoplus_{m \equiv 0,1} W_m$, where $W_m = W^+_m \oplus W^-_m$ has vanishing odd part if $m$ is even and vice versa, whose McKay-Thompson series

$$T[g](\tau) = 2q^{-3} + \sum_{m=0}^{m \equiv 0,1} \text{str}(g|W_m)q^m$$

are specific weakly holomorphic weight $\frac{1}{2}$ modular forms.
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Theorem (Griffin–M., 2016)

The Thompson Moonshine Conjecture is true. Moreover, the occurring modular forms can be described systematically.
Finite simple groups

Source: wikipedia
Some properties of ON

- One of the six pariah groups.
The O’Nan group ON

Some properties of ON

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- \( \# \text{ON} = 460 \, 815 \, 505 \, 920 = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31. \)
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- \# ON = 460 815 505 920 = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31.
- 30 conjugacy classes, hence 30 irreducible representations (over \mathbb{C}) with characters \chi_1, \ldots, \chi_{30}.
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Dimensions of irreducible representations:

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\begin{align*}
\chi_1(1) &= 1, & \chi_7(1) &= 26\,752, & \chi_{12}(1) &= 58\,311, & \chi_{18}(1) &= 85\,064.
\end{align*}
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#### Zagier’s basis of weight 3/2 forms:

\[
-g_4(\tau) = \sum_{n=-4}^{\infty} a_n q^n = -q^{-4} + 2 + 26\,752 q^3 + 143\,376 q^4 + 8\,288\,256 q^7 + O(q^8) \in M_{3/2}^{+}(\Gamma_0(4)).
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The O’Nan group ON

Some properties of ON

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Zagier’s basis of weight \( 3/2 \) forms:

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\]
Theorem 1 (Duncan-M.-Ono, 2017)

There is a (virtual) infinite-dimensional graded ON-module

\[ W := \bigoplus_{0 < m \equiv 0, 3 \pmod{4}} W_m \]

whose associated McKay-Thompson series are specific weight \( \frac{3}{2} \) (mock) modular forms.
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Remark

We have that

\[ \dim W_{163} = \frac{1}{2}(\alpha^2 + \alpha - 393768), \]

where

\[ \alpha = \left\lfloor e^{\pi \sqrt{163}} \right\rfloor = [262537412640768743.99999999999999642...]. \]
Proposition 1 (Duncan-M.-Ono, 2017)

The following are true.

1. For every conjugacy class \([g]\) of \(\text{ON}\) there is a unique mock modular form

\[
F_{[g]}(\tau) = -q^{-4} + 2 + \sum_{n=1} \ a_{[g]}(n)q^n
\]

of weight \(3/2\) for the group \(\Gamma_0(4o(g))\) satisfying the following conditions:

- \(F_{[g]}(\tau)\) lies in the Kohnen plus space, i.e., \(a_{[g]}(n) = 0\) if \(n \equiv 1, 2 \pmod{4}\).
- \(F_{[g]}(\tau)\) has a pole of order 4 at the cusp \(\infty\) and vanishes at essentially all other cusps.
- We have \(a_{[g]}(3) = \chi_7(g)\), and \(a_{[g]}(4) = \chi_1(g) + \chi_{12}(g) + \chi_{18}(g)\), and \(a_{[g]}(7)\) is more complicated.

2. The function \(F_{[g]}(\tau)\) above has integer Fourier coefficients. Furthermore, if \(o(g) \neq 16\), then it is modular.
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The relevant forms

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of weight 3/2 for the group \(\Gamma_0(4o(g))\) satisfying the following conditions:

1. \(F_{[g]}(\tau)\) lies in the Kohnen plus space, i.e., \(a_{[g]}(n) = 0\) if \(n \equiv 1, 2 \pmod{4}\).

2. \(F_{[g]}(\tau)\) has a pole of order 4 at the cusp \(\infty\) and vanishes at essentially all other cusps.

3. We have \(a_{[g]}(3) = \chi_7(g)\), and \(a_{[g]}(4) = \chi_1(g) + \chi_{12}(g) + \chi_{18}(g)\), and \(a_{[g]}(7) = \text{“more complicated”}\).
The relevant forms

Proposition 1 (Duncan-M.-Ono, 2017)

The following are true.

1. For every conjugacy class \([g]\) of ON there is a unique mock modular form

\[
F_{[g]}(\tau) = -q^{-4} + 2 + \sum_{n=1} a_{[g]}(n)q^n
\]

of weight 3/2 for the group \(\Gamma_0(4o(g))\) satisfying the following conditions:

1. \(F_{[g]}(\tau)\) lies in the Kohnen plus space, i.e., \(a_{[g]}(n) = 0\) if \(n \equiv 1, 2 \pmod{4}\).
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3. We have \(a_{[g]}(3) = \chi_7(g)\), and \(a_{[g]}(4) = \chi_1(g) + \chi_{12}(g) + \chi_{18}(g)\), and \(a_{[g]}(7)\) is "more complicated".

2. The function \(F_{[g]}(\tau)\) above has integer Fourier coefficients. Furthermore, if \(o(g) \neq 16\), then it is modular.
Strategy of the proof

- Take

\[ F_{[g]}(\tau) = -q^{-4} + 2 + \sum_{n=1}^{\infty} a_{[g]}(n)q^n \]
Strategy of the proof

- Take

\[ F_g(\tau) = -q^{-4} + 2 + \sum_{n=1}^{\infty} a_g(n)q^n \]

- Define \( \mathbb{C} \)-valued class function

\[ \omega_n : \text{ON} \to \mathbb{C}, \ g \mapsto a_g(n). \]
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- Take

\[ F_g(\tau) = -q^{-4} + 2 + \sum_{n=1}^{\infty} a_g(n)q^n \]

- Define \( \mathbb{C} \)-valued class function

\[ \omega_n : \text{ON} \to \mathbb{C}, \; g \mapsto a_g(n). \]

- Theorem 1 follows if we can show that

\[ \omega_n = \sum_{j=1}^{30} m_j(n)\chi_j, \]

with \( m_j(n) \in \mathbb{N}_0 \) for all \( j \) and (sufficiently large) \( n \).
Strategy of the proof

- Take
  \[ F[g](\tau) = -q^{-4} + 2 + \sum_{n=1}^{\infty} a[g](n)q^n \]

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- Idea of Thompson can reduce this to a finite computation.
Strategy of the proof

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- Idea of Thompson can reduce this to a finite computation.

BUT: There is a difference between 'finite' and 'feasible'.

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Let
\[ \Gamma_{K,K^2}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : |c| < K, |d| < K^2 \right\}. \]
Let
\[ \Gamma_{K,K^2}(N) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) : |c| < K, |d| < K^2 \right\} . \]

**Definition**

For \( \mu \in \mathbb{Z} \), \( k \in \frac{1}{2} \mathbb{Z} \), and \( \psi \) a multiplier system for \( \Gamma_0(N) \) of weight \( k \), we define the Rademacher sum
\[
R_{\psi,k}^{[\mu]}(\tau) := \lim_{K \to \infty} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_{K,K^2}(N)} \overline{\psi}(\gamma)(q^{\mu} |_{k}\gamma).
\]
Let
\[ \Gamma_{K,K^2}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : |c| < K, |d| < K^2 \right\}. \]

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\]

- Low-weight analogue of Poincaré series.
Let
\[ \Gamma_{K,K^2}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : |c| < K, |d| < K^2 \right\}. \]

**Definition**

For \( \mu \in \mathbb{Z}, k \in \frac{1}{2}\mathbb{Z}, \) and \( \psi \) a multiplier system for \( \Gamma_0(N) \) of weight \( k \), we define the **Rademacher sum**

\[ R^{[\mu]}_{\psi, k}(\tau) := \lim_{K \to \infty} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_{K,K^2}(N)} \overline{\psi}(\gamma)(q^\mu |k\gamma). \]

- Low-weight analogue of Poincaré series.
- Converges for \( k \geq 1 \), with regularization sometimes for \( k < 1 \).
Properties of Rademacher sums

Facts

Let $\mu < 0$.

- $R_{\psi, k}^{[\mu]}$ is a weight $k$ mock modular form for $\Gamma_0(N)$ with multiplier $\psi$ with shadow $R_{\psi, 2-k}^{[-\mu]} \in M_{2-k}(\Gamma_0(N), \overline{\psi})$. 
Properties of Rademacher sums

Facts

Let $\mu < 0$.

- $R_{\psi,k}^{[\mu]}$ is a weight $k$ mock modular form for $\Gamma_0(N)$ with multiplier $\psi$ with shadow $R_{\psi,2-k}^{[-\mu]} \in M_{2-k}(\Gamma_0(N), \overline{\psi})$.

- $R_{\psi,k}^{[\mu]}$ has a pole of order $|\mu|$ at $\infty$ and vanishes at all other cusps.
**Facts**

Let $\mu < 0$.

- $R^{[\mu]}_{\psi,k}$ is a weight $k$ mock modular form for $\Gamma_0(N)$ with multiplier $\psi$ with shadow $R^{[-\mu]}_{\psi,2-k} \in M_{2-k}(\Gamma_0(N), \overline{\psi})$.

- $R^{[\mu]}_{\psi,k}$ has a pole of order $|\mu|$ at $\infty$ and vanishes at all other cusps.

**“Definition”**

A hol. function $f : \mathcal{H} \to \mathbb{C}$ is called a mock modular form for $\Gamma_0(N)$ of weight $k$ and multiplier $\psi$ if there is a modular form $g \in M_{2-k}(\Gamma_0(N), \overline{\psi})$ s.t.

$$\hat{f}(\tau) := f(\tau) + \int_{-\infty}^{\infty} \frac{g(\overline{z})}{(z + \tau)^k} \, dz$$

transforms like a modular form. $g$ is called the shadow of $f$, $\hat{f}$ is the corresponding harmonic Maaß form.
Proof of Proposition 1.

- Let

\[ Z_{3,2}^{[\mu]} = R_{3,2}^{[\mu]}|_{\text{pr}}, \]

the projection to the Kohnen plus space.
Proof of Proposition 1.

- Let
  \[ Z_{\frac{3}{2},\psi}^{[\mu]} = R_{\frac{3}{2},\psi}^{[\mu]} | \text{pr}, \]
  the projection to the Kohnen plus space.
- \(-Z_{\frac{3}{2},1}^{-4} + 2Z_{\frac{3}{2},1}^{[0]}\) satisfies conditions 1 and 2 of Proposition 1.
Proof of Proposition 1.

- Let

\[ Z_{\frac{3}{2},\psi}^{[\mu]} = R_{\frac{3}{2},\psi}^{[\mu]} \mid \text{pr}, \]

the projection to the Kohnen plus space.

- \(-Z_{\frac{3}{2},1}^{[-4]} + 2Z_{\frac{3}{2},1}^{[0]}\) satisfies conditions 1 and 2 of Proposition 1

- Bruinier-Funke pairing yields that 1 and 2 determine a mock modular form uniquely up to addition of cusp forms, so choose cusp forms where possible to ensure 3.
Construction of $F[g]$

Proof of Proposition 1.

Let

$$Z_{\frac{3}{2}, \psi}^{[\mu]} = R_{\frac{3}{2}, \psi}^{[\mu]} | \text{pr},$$

the projection to the Kohnen plus space.

$-Z_{\frac{3}{2}, 1}^{[-4]} + 2Z_{\frac{3}{2}, 1}^{[0]}$ satisfies conditions 1 and 2 of Proposition 1.

Bruinier-Funke pairing yields that 1 and 2 determine a mock modular form uniquely up to addition of cusp forms, so choose cusp forms where possible to ensure 3.

Check that coefficients are integers and that (almost) all are modular using again Bruinier-Funke.
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4. Traces of singular moduli

5. Arithmetic applications
Let

\[ F_{\chi_j}(\tau) := \frac{1}{\# \text{ON}} \sum_{g \in \text{ON}} \overline{\chi_j(g)} F_g(\tau) \]
Let

\[ F_{\chi_j}(\tau) := \frac{1}{\# ON} \sum_{g \in ON} \overline{\chi_j(g)F[g](\tau)^{\text{Schur}}} = -q^{-4} + 2 + \sum_{n=1} m_j(n)q^n. \]

Proposition 1 yields: \( F_{\chi_j} \) is a mock modular form of weight \( 3/2 \) of level \( N_{\chi_j} \) with rational coefficients and controllable shadow.
Let

$$F_{\chi_j}(\tau) := \frac{1}{\# \text{ON}} \sum_{g \in \text{ON}} \overline{\chi_j(g)F[g](\tau)}^{\text{Schur}} = -q^{-4} + 2 + \sum_{n=1} m_j(n)q^n.$$ 

- Proposition 1 yields: $F_{\chi_j}$ is a mock modular form of weight $3/2$ of level $N_{\chi_j}$ with rational coefficients and controllable shadow.
- Checking integrality naively by Sturm bound not feasible ($N_{\chi_1} = 10\,884\,720$).
Proposition 2

$F_{\chi_j}$ have all integer Fourier coefficients.
Proposition 2

The $F_{\chi_j}$ have all integer Fourier coefficients.

Proof.

- The $F_{[g]}$ satisfy numerous congruences modulo powers of $p | \# \text{ON}$ (proved by Sturm bound argument, $< 250$ coefficients to be checked).
Proposition 2

\( F_{\chi_j} \) have all integer Fourier coefficients.

Proof.

- The \( F_{[g]} \) satisfy numerous congruences modulo powers of \( p \mid \# \text{ON} \) (proved by Sturm bound argument, \(< 250\) coefficients to be checked).
- One can then verify directly that \( m_j(n) \) are \( p \)-integral for all \( p \mid \# \text{ON} \), hence by Proposition 1, the claim follows.
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Basic strategy

Fact

Given convergence, Rademacher sums have a Fourier expansion whose coefficients are given in terms of infinite sums of Kloosterman sums

\[ K_\psi(m, n, c) = \sum_{d \mid (c) \ast} \psi \left( \begin{pmatrix} \ast & \ast \\ c & d \end{pmatrix} \right) e^{2\pi i \frac{md+nd}{c}} \]

times \( I \)-Bessel functions.
Given convergence, Rademacher sums have a Fourier expansion whose coefficients are given in terms of infinite sums of Kloosterman sums:

\[ K_\psi(m, n, c) = \sum_{d \mid (c)^*} \psi \left( \begin{pmatrix} * & * \\ c & d \end{pmatrix} \right) e^{2\pi i \frac{md + nd}{c}} \]

times \( I \)-Bessel functions.

By the triangle inequality we have

\[ m_j(n) \geq \frac{|\text{str}(1|W_n)|}{\# \text{ON}} - \sum_{[g] \neq 1A} \frac{|\text{str}(g|W_n)|}{\# \text{ON}(g)} |\chi_j(g)|. \]
Basic strategy

Fact

Given convergence, Rademacher sums have a Fourier expansion whose coefficients are given in terms of infinite sums of Kloosterman sums

\[ K_\psi(m, n, c) = \sum_{d \mid (c)\ast} \psi \begin{pmatrix} \star & \star \\ c & d \end{pmatrix} e^{2\pi i \frac{md+nd}{c}} \]

times \text{I-Bessel functions.}

By the triangle inequality we have

\[ m_j(n) \geq \frac{|\text{str}(1|W_n)|}{\# \text{ON}} - \sum_{[g] \neq 1A} \frac{|\text{str}(g|W_n)|}{\# C_{\text{ON}}(g)} |\chi_j(g)|. \]

Show that from a certain point on, the first term dominates.
Proposition 3

The multiplicities $m_j(n)$ are all non-negative for $n \neq 7, 8, 12$. 
Proposition 3

The multiplicities \( m_j(n) \) are all non-negative for \( n \neq 7, 8, 12 \).

Ingredients of the proof.

- Careful, explicit estimates for Selberg-Kloosterman zeta functions.
Positivity of the multiplicities

Proposition 3
The multiplicities $m_j(n)$ are all non-negative for $n \neq 7, 8, 12$.

Ingredients of the proof.
- Careful, explicit estimates for Selberg-Kloosterman zeta functions.
- Write Kloosterman sums as sums over a sparse set, i.e. equivalence classes of binary quadratic forms.
Proposition 3

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Ingredients of the proof.

- Careful, explicit estimates for Selberg-Kloosterman zeta functions.
- Write Kloosterman sums as sums over a sparse set, i.e. equivalence classes of binary quadratic forms.
- Explicit estimates for coefficients of weight $3/2$ cusp forms

\[ \Rightarrow \quad m_j(n) \geq 0 \text{ for } n \geq 109. \]
Proposition 3

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Ingredients of the proof.

- Careful, explicit estimates for Selberg-Kloosterman zeta functions.
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- Explicit estimates for coefficients of weight 3/2 cusp forms

$$\Rightarrow m_j(n) \geq 0 \text{ for } n \geq 109.$$

- Check rest by inspection.
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Fact

Let $Q = [a, b, c]$ be a quadratic form of discriminant $D < 0$ and $\tau_Q \in \mathcal{H}$, such that $a\tau_Q^2 + b\tau_Q + c = 0$. Then $J(\tau_Q)$ is a real-algebraic integer of degree $h(D)$.

E.g.:

\[
J \left( \frac{1 + \sqrt{-163}}{2} \right) = -262\,537\,412\,640\,768\,744
\]

\[
J \left( \frac{1 + \sqrt{-15}}{2} \right) = -\frac{192\,513 + 85\,995\sqrt{5}}{2}.
\]
Fact

Let \( Q = [a, b, c] \) be a quadratic form of discriminant \( D < 0 \) and \( \tau_Q \in \mathcal{H} \), such that \( a\tau^2_Q + b\tau_Q + c = 0 \). Then \( J(\tau_Q) \) is a real-algebraic integer of degree \( h(D) \).

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J \left( \frac{1 + \sqrt{-163}}{2} \right) = -262537412640768744
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- Play an important role in explicit class field theory ("Kronecker’s Jugendtraum", Hilbert’s 12\(^{th}\) problem)
Fact

Let $Q = [a, b, c]$ be a quadratic form of discriminant $D < 0$ and $\tau_Q \in \mathbb{H}$, such that $a\tau_Q^2 + b\tau_Q + c = 0$. Then $J(\tau_Q)$ is a real-algebraic integer of degree $h(D)$.

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- Play an important role in explicit class field theory (“Kronecker’s Jugendtraum”, Hilbert’s 12th problem)
- Similar results are true for Hauptmoduln of genus 0 congruence subgroups and other modular functions
For a function $f : \mathcal{H} \to \mathbb{C}$, a discriminant $-D < 0$ and $N \in \mathbb{N}$ define

$$\text{Tr}^{(N)}_D(f) := \sum_{Q \in Q^{(N)}_{-D}/\Gamma_0(N)} \frac{f(\tau_Q)}{\omega^{(N)}(Q)},$$

where

- $Q^{(N)}_{-D} = \{[a, b, c] : b^2 - 4ac = -D \text{ and } N \mid a\}$,
- $\omega^{(N)}(Q) = \frac{1}{2} \cdot \# \text{Stab}_{\Gamma_0(N)}(Q)$. 
Theorem (D. Zagier)

\[- q^{-1} + 2 + \sum_{D \equiv 0,3 \pmod{4}} \text{Tr}^{(1)}_{D}(J)q^{D} \]

\[= - q^{-1} + 2 - 248q^{3} + 492q^{4} - 4119q^{7} + O(q^{8}) \in M_{3/2}^{1,+}(\Gamma_{0}(4)). \]
Theorem (D. Zagier)

\[- q^{-1} + 2 + \sum_{D \equiv 0, 3} \text{Tr}_{D}^{(1)}(J)q^D\]

\[= - q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + O(q^8) \in M_{3/2}^{1,+}(\Gamma_0(4)).\]

Can be extended to more general modular functions with vanishing constant terms (Bruinier-Funke, Miller-Pixton,...)
Generating functions

**Theorem (D. Zagier)**

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\[= - q^{-1} + 2 - 248q^{3} + 492q^{4} - 4119q^{7} + O(q^{8}) \in \mathcal{M}_{3/2}^{1,+}(\Gamma_0(4)). \]

Can be extended to more general modular functions with vanishing constant terms (Bruinier-Funke, Miller-Pixton,...)

**Definition/Theorem**

For $N \in \mathbb{N}$, we call $H^{(N)}(D) := \text{Tr}^{(N)}_{D}(1)$ the generalized Hurwitz class number and set

\[\mathcal{H}^{(N)}(\tau) := - \frac{[\Gamma(1) : \Gamma_0(N)]}{12} + \sum_{D} H^{(N)}(D)q^{D}. \]

$\mathcal{H}^{(N)}$ is a weight $3/2$ mock modular form of level $4N$. 
Proposition 5

Let $N \in \mathbb{N}$ such that $X_0(N)$ has genus 0 and

$$\text{Tr}_4^{(N)}(D) := \frac{1}{2} \left( \text{Tr}_D^{(N)}(J_2^{(N)}) - \text{Tr}_D^{(N/d)}(J^{(N/d)}) \right),$$

where $J^{(N)}$ denotes the Hauptmodul for $\Gamma_0(N)$ and $J_2^{(N)} = q^{-2} + O(q)$ is the unique modular function for $\Gamma_0(N)$ with this Fourier expansion at infinity and no poles anywhere else and $d := \gcd(N, 2)$. Then we have

$$\mathcal{F}^{(N)}(\tau) := -q^{-4} + \text{const} + \sum_{D > 0} \text{Tr}_4^{(N)}(D)q^D$$

$$= R_{3/2, 4o(g)}^{[-4], +}(\tau) - \frac{c_2}{2} \mathcal{H}^{(N)}(\tau) + \frac{c_1}{2} \mathcal{H}^{(N/d)}(\tau)$$

for some rational numbers $c_1$ and $c_2$. In particular, the function $\mathcal{F}^{(N)}$ has integer Fourier coefficients.
There is an analogue of Proposition 5 for $N$ where $\text{genus}(X_0(N)) > 0$, but $\text{genus}(X_0^+(N)) = 0$. 

$W_e$ can express character values of $\text{ON}$ in terms of traces of singular moduli, generalized class numbers and coefficients of cusp forms.

Example

$F_1 A = T(1)$,

$F_7 AB = T(7) + 4H(1) - 4H(7)$,

$F_{11} A = T(11) + 125H(1) - 65H(11) - 45G(11)$,

where $G(11)(\tau) = q^{3} - q^{4} - q^{11} + O(q^{12}) \in S^+_{3/2}(44)$. 

M. H. Mertens (U. Köln)
There is an analogue of Proposition 5 for $N$ where $\text{genus}(X_0(N)) > 0$, but $\text{genus}(X_0^+(N)) = 0$. 

$\Rightarrow$ We can express character values of ON in terms of traces of singular moduli, generalized class numbers and coefficients of cusp forms.
There is an analogue of Proposition 5 for $N$ where $\text{genus}(X_0(N)) > 0$, but $\text{genus}(X_0^+(N)) = 0$.

We can express character values of ON in terms of traces of singular moduli, generalized class numbers and coefficients of cusp forms.

Example

\[
\begin{align*}
F_{1A} &= \mathcal{T}^{(1)}, \\
F_{7AB} &= \mathcal{T}^{(7)} + 4\mathcal{H}^{(1)} - 4\mathcal{H}^{(7)}, \\
F_{11A} &= \mathcal{T}^{(11,+)} + \frac{12}{5}\mathcal{H}^{(1)} - \frac{6}{5}\mathcal{H}^{(11)} - \frac{4}{5}\mathcal{G}^{(11)},
\end{align*}
\]

where $\mathcal{G}^{(11)}(\tau) = q^3 - q^4 - q^{11} + O(q^{12}) \in S_{3/2}^+(44)$. 
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5 Arithmetic applications
Theorem 2 (Duncan–M.–Ono, 2017)

Suppose that $-D < 0$ is a fundamental discriminant. Then the following are true:

1. If $-D < -8$ is even and $g_2 \in \text{ON}$ has order 2, then $\dim W_D \equiv \text{tr}(g_2 | W_D) \equiv -24 \, H(D) \pmod{2^4}$.

2. If $p \in \{3, 5, 7\}$, $(−D_p) = -1$ and $g_p \in \text{ON}$ has order $p$, then $\dim W_D \equiv \text{tr}(g_p | W_D) \equiv \begin{cases} -24 \, H(D) \pmod{3^2} & \text{if } p = 3, \\ -24 \, H(D) \pmod{p} & \text{if } p = 5, 7. \end{cases}$
Theorem 2 (Duncan–M.–Ono, 2017)

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1. If $-D < -8$ is even and $g_2 \in \text{ON}$ has order 2, then
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   \]
Theorem 2 (Duncan–M.–Ono, 2017)

Suppose that $-D < 0$ is a fundamental discriminant. Then the following are true:

1. If $-D < -8$ is even and $g_2 \in \text{ON}$ has order 2, then
   \[ \dim W_D \equiv \text{tr}(g_2|W_D) \equiv -24H(D) \pmod{2^4}. \]

2. If $p \in \{3, 5, 7\}$, \( \left( \frac{-D}{p} \right) = -1 \) and $g_p \in \text{ON}$ has order $p$, then
   \[ \dim W_D \equiv \text{tr}(g_p|W_D) \equiv \begin{cases} -24H(D) \pmod{3^2} & \text{if } p = 3, \\ -24H(D) \pmod{p} & \text{if } p = 5, 7. \end{cases} \]
Conjecture (Birch and Swinnerton-Dyer)

Let $E/\mathbb{Q}$ be an elliptic curve. Then we have that

$$
\frac{L^{(r)}(E, 1)}{r! \Omega_E} = \frac{\# \Sha(E) \cdot \text{Reg}(E) \prod_{\ell} c_{\ell}(E)}{\left(\#E(\mathbb{Q})_{\text{tors}}\right)^2},
$$

where $r$ denotes the order of vanishing of $L(E, s)$ at $s = 1$, which equals the Mordell–Weil rank of $E$. 
The BSD-conjecture and Waldspurger’s theorem

**Conjecture (Birch and Swinnerton-Dyer)**

Let $E/\mathbb{Q}$ be an elliptic curve. Then we have that

$$\frac{L^{(r)}(E, 1)}{r!\Omega_E} = \frac{\#\Sha(E) \cdot \Reg(E) \prod_{\ell} c_{\ell}(E)}{(\#E(\mathbb{Q})_{\text{tors}})^2},$$

where $r$ denotes the order of vanishing of $L(E, s)$ at $s = 1$, which equals the Mordell–Weil rank of $E$.

**Theorem (Waldspurger, Kohnen)**

Let $N \in \mathbb{N}$ be odd and square-free, $f \in S_{k+\frac{1}{2}}^+(\Gamma_0(4N))$ be a newform and $F \in S_{2k}(\Gamma_0(N))$ the image of $f$ under the Shimura correspondence. For a suitable fundamental discriminant $D$ we have

$$\langle f, f \rangle = \frac{\langle F, F \rangle \pi^k}{2^{\omega(N)}(k-1)!|D|^{k-\frac{1}{2}} L(F, D; k)} \cdot |b_f(|D|)|^2.$$
Quadratic twists

Connection through Modularity Theorem:

**Lemma**

Let $-D < 0$ be a suitable fundamental discriminant and $E/\mathbb{Q}$ an elliptic curve of odd, square-free conductor $N$. Denote the weight 2 newform associated to $E$ by $F_E \in S_2(N)$ and its Shintani lift by

$$f_E(\tau) = \sum_{n=3}^{\infty} b_E(n) q^n \in S_{3/2}^+(4N).$$

If $E(-D)$ denotes the quadratic twist of $E$ by $-D$, we have

$$\frac{L(E(-D), 1)}{\Omega_{E(-D)}} = C_E \cdot |b_E(D)|^2,$$

where $C_E$ is a constant depending on $E$, but not (really) on $D$. 
Theorem 3 (Duncan–M.–Ono, 2017)

Assume BSD. If $p = 11$ or $19$ and $-D < 0$ is a fundamental discriminant for which $\left(\frac{-D}{p}\right) = -1$, and $g_p \in \text{ON}$ has order $p$, then the following are true.
Theorem 3 (Duncan–M.–Ono, 2017)

Assume BSD. If $p = 11$ or $19$ and $-D < 0$ is a fundamental discriminant for which $\left( \frac{-D}{p} \right) = -1$, and $g_p \in \text{ON}$ has order $p$, then the following are true.

1. $\text{Sel}(E_p(-D))[p] \neq \{0\}$ if and only if

$$\dim W_D \equiv \text{tr}(g_p | W_D) \equiv -24H(D) \pmod{p}.$$
Theorem 3 (Duncan–M.–Ono, 2017)

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   \]

2. Suppose further that $L(E_p(-D), 1) \neq 0$. Then $p | \#\text{III}(E_p(-D))$ if and only if
   \[
   \dim W_D \equiv \text{tr}(g_p|W_D) \equiv -24H(D) \pmod{p}.
   \]
Theorem 4 (Duncan–M.–Ono, 2017)

Let $N \in \{14, 15\}$ and write $N = p'p$, $p' < p$, and let $\delta_p := \frac{p-1}{2}$. If $-D < 0$ is a fundamental discriminant for which $\left(\frac{-D}{p}\right) = -1$, then the following are true.
Theorem 4 (Duncan–M.–Ono, 2017)

Let \( N \in \{14, 15\} \) and write \( N = p'p, \ p' < p, \) and let \( \delta_p := \frac{p-1}{2}. \) If \( -D < 0 \) is a fundamental discriminant for which \( \left( \frac{-D}{p} \right) = -1, \) then the following are true.

1. \( \text{Sel}(E_N(-D))[p] \neq \{0\} \) if and only if

\[
\text{tr}(g_{p'}|W_D) \equiv \text{tr}(g_N|W_D) \equiv \delta_p \cdot (H(D) - \delta_p H(p')(D)) \pmod{p}.
\]
Theorem 4 (Duncan–M.–Ono, 2017)

Let $N \in \{14, 15\}$ and write $N = p'p$, $p' < p$, and let $\delta_p := \frac{p-1}{2}$. If $-D < 0$ is a fundamental discriminant for which $\left( \frac{-D}{p} \right) = -1$, then the following are true.

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   \]

2. Suppose further that $L(E_N(-D), 1) \neq 0$. Then $p \mid \#\text{III}(E_N(-D))$ if and only if
   \[
   \text{tr}(g_{p'}|W_D) \equiv \text{tr}(g_N|W_D) \equiv \delta_p \cdot (H(D) - \delta_p H^{(p')} (D)) \pmod{p}.
   \]
Table: Examples for the curve $E_{14}$

$$H_{14}(D) := \delta_7(H(D) - \delta_7 H^{(2)}(D)),$$
$$\text{tr}_2(D) := \text{tr}(g_2|W_D),$$
$$\text{Diff}_{14}(D) := H_{14}(D) - \text{tr}_2(D)$$
Thank you for your attention.