The u-Plane Integral And Indefinite Theta Functions

Gregory Moore
Rutgers University

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Much of this talk is review but I’ll mention some new results with my student Iurii Nidaiev at the end:

c. 1995: Mathematicians found mysterious relations of Donaldson invariants of 4-folds with $b_2^+ = 1$ to modular forms: Göttsche; Göttsche & Zagier; Fintushel & Stern…

Can physics explain this ???

1988: Witten reformulates Donaldson theory in terms of supersymmetric Yang-Mills theory.

1994: Seiberg & Witten initiate the solution of the IR LEET of N=2 supersymmetric field theories.
1994: Witten uses SW theory and his 1988 TFT interpretation to ``solve’’ Donaldson theory in terms of the much more tractable SW invariants.

1997: Moore & Witten: Give a more systematic account, in the process explaining the mysterious `95 math results. Moreover, mock modular forms appear in the evaluation of the Donaldson invariants for $\mathbb{CP}^2$.

2008: Malmendier & Ono, et. al. extend the connection of u-plane integrals for $\mathbb{CP}^2$ to mock modular forms.
2017: Very recently G. Korpas and J. Manschot revisited the u-plane integral, stressing the role of indefinite theta functions (and a modular completion.)

Today: I will review some of these developments.

But there has always been a nagging question – first mentioned to me by Witten in the course of our work:

Do other $N=2$ field theories, e.g. the superconformal ones, lead to new 4-manifold invariants?
This was the motivation for my work in 1998 with Marcos Marino (also with Marino and Peradze) on u-plane integrals for other theories.

Our conclusion was that Lagrangian theories will probably not lead to new four-manifold invariants, but superconformal points can still teach us new things about four-manifolds.

At the end I’ll present new results (with Iurii Nidaiev) on the partition function for the simplest AD theory on 4-manifolds with $b_2^+ > 0$. 
1. Introduction

2. Cambrian: Witten Reads Donaldson


4. Comments On The u-plane Integral

5. Other N=2 Theories

6. Holocene: AD3 Partition Function
Donaldson Invariants Of 4-folds

\( X \): Smooth, compact, \( \partial X = \emptyset \), oriented, \( (\pi_1(X) = 0) \)

\( P \to X \): Principal SU(2) or SO(3) bundle.

Give \( X \) a metric and consider moduli space of instantons:

\[ \mathcal{M} := \{ A: F + \ast F = 0 \} \mod \mathcal{G} \]

Donaldson defines cohomology classes in \( \mathcal{M} \) associated to points and surfaces in \( X \):

\[ \mu(pt) \; \& \; \mu(S) \]

\[ \varphi_D(pt^l S^r) := \int_{\mathcal{M}} \mu(pt)^l \mu(S)^r \]

Rational (often integer) valued.

Independent of metric! \( \Rightarrow \) smooth invariants of \( X \).
Witten’s Interpretation: 
Topologically Twisted SYM On X

Consider N=2 SYM theory on X for gauge group G

Choose an isomorphism of the principal $SO(3)_R$ bundle with that for $\Lambda^{2,+}(X)$

Set the R-symmetry connection = self-dual spin connection

This defines topological twisting: All fermion fields and susy operators become differential forms.
(In particular, the twisted theory is defined on non-spin manifolds.)

One supersymmetry operator $Q$ is scalar and $Q^2 = 0$

Formally: Correlation functions of $Q$-invariant operators localize to integrals over the finite-dimensional moduli spaces of G-ASD conns

Witten’s proposal: For $G = SU(2)$ correlation functions of $Q$-invariant operators are the Donaldson polynomials.
Local Observables

$U \in Inv(g) \Rightarrow U(\phi) \quad \phi \in \Omega^0(ad P \otimes \mathbb{C})$

$H_*(X, \mathbb{Z}) \xrightarrow{\text{Descent formalism}} H^*(\text{Fieldspace}; Q)$

$\mu \quad \text{Localization identity}$

$g = \mathfrak{su}(2) \quad U = Tr_2 \left( \frac{\phi^2}{8\pi^2} \right) \quad U(S) \sim \int_S Tr(\phi F + \psi^2)$
Donaldson-Witten Partition Function

\[ Z_{DW}^\xi(p, s) = \langle e^{2p U + U(S)} \rangle_\Lambda \]

\[ = \Lambda^{-\frac{3}{4}(\chi + \sigma)} \sum \frac{\Lambda^{2\ell + r} p^\ell}{\ell! r!} \delta_D(pt^\ell S^r) \]

Mathai-Quillen & Atiyah-Jeffrey: Path integral formally localizes:

\[ \mathcal{M} \hookrightarrow \mathcal{A}/\mathcal{G} \]

Strategy: Evaluate in LEET: \( X \) compact \( \Rightarrow \)

Integrate over \( Q \)-invariant field configurations:
This includes an integral over vacua on \( \mathbb{R}^4 \)
Introduction

Cambrian: Witten Reads Donaldson

Carboniferous: SW Theory & u-plane Derivation Of Witten’s Conjecture.

Comments On The u-plane Integral

Other N=2 Theories

Holocene: AD3 Partition Function
Coulomb Branch Vacua On $\mathbb{R}^4$

$SU(2) \to U(1)$ by vev of adjoint Higgs field $\phi$: Order parameter: $u = \langle U(\phi) \rangle$

Coulomb branch: $\mathcal{B} = \mathfrak{t} \otimes \mathbb{C}/\mathcal{W} \cong \mathbb{C}$

$\text{adP} \to L^2 \oplus \mathcal{O} \oplus L^{-2}$

Photon: Connection $A$ on $L$  $U(1)$ VM: $(a^q, A, \chi, \psi, \eta)$

$a^q$: complex scalar field on $\mathbb{R}^4$:

Do path integral of quantum fluctuations around $a^q(x) = a + \delta a(x)$

What is the relation of $a = \langle a^q(x) \rangle$ to $u$?

What are the couplings in the LEET for the $U(1)$ VM?
**LEET: Constraints of N=2 SUSY**

General result on N=2 abelian gauge theory with\[ t \cong u(1) \oplus \cdots \oplus u(1): \text{ Action determined by a family of Abelian varieties and an ``N=2 central charge function''} :\]

\[ \mathcal{A} \to t \otimes \mathbb{C} - \mathcal{D} \]

\[ \Gamma := H_1(\mathcal{A}; \mathbb{Z}) \]

\[ Z: \Gamma \to \mathbb{C} \quad \langle dZ, dZ \rangle = 0 \]

**Duality Frame:** \[ \Gamma \cong \Gamma^{\text{electric}} \oplus \Gamma^{\text{magnetic}} \]

\[ a^I = Z(\alpha^I) \quad a_{D,I} := Z(\beta_I) = \left( \frac{\partial F}{\partial a^I} \right) \quad \tau_{IJ} := \frac{\partial a_{D,I}}{\partial a^J} \]

Action \[ \sim \int_X \bar{\tau}(F^+)^2 + \tau(F^-)^2 + d\alpha^q \ast (\text{Im } \tau)d\bar{a}^q + \cdots \]
Seiberg-Witten Theory: 1/2

For G=SU(2) SYM $\mathcal{A}$ is a family of elliptic curves:

$$E_u: \quad y^2 = x^2(x - u) + \frac{\Lambda^4}{4}x \quad u \in \mathbb{C}$$

$$Z(\gamma) = \oint_\gamma \lambda \quad \lambda = \frac{dx}{y} (x - u)$$

$u \to \infty$: Invariant cycle $A$:

$$a(u) = \oint_A \lambda$$

Choose B-cycle: $\Rightarrow \tau(a) \Rightarrow$ Action for LEET

LEET breaks down at $u = \pm \Lambda^2$ where $\text{Im}(\tau) \to 0$
LEET breaks down because there are new massless fields associated to BPS states

LEET $u \in \mathcal{U}_{\Lambda^2}$:

$U(1)_D$ VM: $(a_D, A_D, \chi_D, \psi_D, \eta_D)$

Charge 1 HM: $(M = q \oplus \tilde{q}^*, \cdots)$

Similar story for LEET for $u \in \mathcal{U}_{-\Lambda^2}$
Apply SW LEET To $Z_{DW}^{\xi}$

$$Z_{DW}^{\xi}(p, s) = \langle e^{2p \cdot u + U(S)} \rangle_\Lambda$$

$$= \Lambda^{-\frac{3}{4}(\chi+\sigma)} \sum \frac{\Lambda^{2\ell+r} p^\ell}{\ell! \ r!} \omega_D(pt^\ell S^r)$$

$$Z_{DW}^{\xi}(p, s) = Z_u + Z_{\Lambda^2} + Z_{-\Lambda^2}$$
u-Plane Integral $Z_u$

Can be computed explicitly from QFT of LEET

Vanishes if $b_2^+ > 1$. When $b_2^+ = 1$:

$$Z_u = \int da \bar{d}a \left( \frac{du}{da} \right)^2 \frac{\chi}{2} \Delta^{\frac{\sigma}{8}} e^{2pu+S^2T(u)} \Psi$$

$$\Delta = (u - \Lambda^2)(u + \Lambda^2)$$

Contact term: $T(u) = \left( \frac{du}{da} \right)^2 E_2(\tau) - 8u$

Ψ: Sum over line bundles for the U(1) photon.
\[ \Psi = e^{y^{-1} \left( \frac{du}{da} \right)^2} s^2_+ \sum_{\lambda = \lambda_0 + H^2(X, \mathbb{Z})} y^{-\frac{1}{2}} e^{-i \pi \bar{\tau} \lambda^2_+ - i \pi \tau \lambda^2_-} \]

\[ (-1)^{w_2(X) \cdot (\lambda - \lambda_0)} e^{-i \left( \frac{du}{da} \right) s \cdot \lambda_-} \left( \frac{d\bar{\tau}}{d\bar{a}} \right) \left( \lambda_+ + \frac{1}{4\pi y} S_+ \left( \frac{du}{da} \right) \right) \]

\[ \tau = x + i \ y \]

\[ 2\lambda_0 \] is an integral lift of \[ \xi = w_2(P) \]

Metric dependent! \[ \lambda = \lambda_+ + \lambda_- \]
Metric Dependence

\[ b_2^+ = 1 \Rightarrow H^2(X) \text{ has hyperbolic signature} \]

All metric dependence enters through period point \( \omega \)

* \( \omega = \omega \quad \omega^2 = 1 \quad \text{Define: } \lambda_+ = \lambda \cdot \omega \)

\[ H^2(S^2 \times S^2) \cong \mathbb{II}^{1,1} \]

\[ e = PD(S^2 \times pt) \quad f = PD(pt \times S^2) \]

\[ \omega = \frac{1}{\sqrt{2}} (v e + \frac{1}{v} f) \]
Initial Comments On $Z_u$

$Z_u$ is a very subtle integral.

It is similar to (sometimes same as) a $\Theta$ lift.

The integrand is related to indefinite theta functions and for some manifolds it can be evaluated using mock modular forms.

We will return to all these issues.

But first let’s finish writing down the full answer for the partition function.
Contributions From $\mathcal{U}_{\Lambda^2}$

Path integral for $U(1)_D$ VM + HM: General considerations imply:

$$\sum_{\lambda \in \frac{1}{2}w_2(X)+H^2(X,\mathbb{Z})} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_\lambda(p, S)$$

$$R_\lambda(p, S) = \text{Res} \left[ \left( \frac{da_D}{1 + \frac{d(\lambda)}{a_D^2}} \right) e^{2pu + S^2T(u) + i \left( \frac{du}{da_D} \right)s \cdot \lambda} C(u)^{\lambda^2} P(u)^{\sigma} E(u)^{\chi} \right]$$

$$d(\lambda) = \frac{(2\lambda)^2 - c^2}{4}$$

$$u = \Lambda^2 + \text{Series } a_D$$

$$c^2 = 2\chi + 3 \sigma$$

Deriving C,P,E From Wall-Crossing

\[
\frac{d}{d g_{\mu \nu}} Z_u = \int \text{Tot deriv} = \int_{\infty} du(\ldots) + \int_{\Lambda^2} du(\ldots) + \int_{-\Lambda^2} du(\ldots)
\]

\( Z_u \) piecewise constant: Discontinuous jumps across walls:

\[
\Delta_{\infty} Z_u: \quad W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \lambda_0 + H^2(X, \mathbb{Z})
\]

Precisely matches formula of Götttsche!

\[
\Delta_{\pm \Lambda^2} Z_u: \quad W(\lambda): \quad \lambda_+ = 0 \quad \lambda = \frac{1}{2} w_2(X) + H^2(X, \mathbb{Z})
\]

\[
\Delta_{\Lambda^2} Z_u + \Delta Z_{\Lambda^2} = 0 \Rightarrow \quad C(u), P(u), E(u)
\]
\[ Z_{DW} = \langle e^{p^O + I(S)} \rangle_{\text{micro}} = Z_{\text{Coulomb}} + Z_{\text{Higgs}} = Z_u + Z_{SW} \]

Donaldson polynomials do not jump at SW walls \( \Rightarrow \)
\[ 0 = \delta Z_{DW} = \delta Z_{\text{Coulomb}} + \delta Z_{\text{Higgs}} \]

\[ \mathcal{M}_{SW}(\lambda) = \{ (A^D_\mu, M_\alpha) : F_+(A^D) = \bar{M}M, \Phi M = 0 \} \]

\[ M \neq 0 \quad \quad M = 0 \quad \quad M \neq 0 \]
Witten Conjecture

⇒ a formula for all $X$ with $b_2^+ > 0$.

For $b_2^+(X) > 1$ (and Seiberg-Witten simple type) we derive the "Witten conjecture":

$$Z_D^\xi(p, s) = 2^{c^2-\chi_h} \left( \frac{1}{2} S^2 + 2p \right) \sum_{\lambda} SW(\lambda)e^{2\pi i \lambda \cdot \lambda_0}e^{2S \cdot \lambda} + e^{-\frac{1}{2} S^2 - 2p} \sum_{\lambda} SW(\lambda)e^{2\pi i \lambda \cdot \lambda_0}e^{-2iS \cdot \lambda}$$

$$\chi_h = \frac{\chi + \sigma}{4} \quad c^2 = 2\chi + 3 \sigma$$

Example: $X = K3, w_2(P) = 0$: $Z = \sinh\left(\frac{S^2}{2} + 2p\right)$

Major success in Physical Mathematics.
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Return To The u-Plane Integral

\[ Z_u = \int da \, d\tilde{a} \left( \frac{du}{da} \right)^{\frac{\chi}{2}} \Delta_{\frac{8}{\sigma}} e^{2pu+S^2T(u)} \Psi \]

\[ \Psi = e^{y^{-1} \left( \frac{du}{da} \right)^2 S_+^2} \sum_{\lambda=\lambda_0+H^2(X,Z)} y^{-\frac{1}{2}} e^{-i \pi \bar{\tau} \lambda_4^2-i \pi \tau \lambda_2^2} \]

\[ (-1)^{w_2(x) \cdot (\lambda-\lambda_0)} e^{-i \left( \frac{du}{da} \right) S \cdot \lambda_- \left( \frac{d\bar{\tau}}{d\tilde{a}} \right)} \left( \lambda_+ + \frac{i}{4\pi y} S_+ \left( \frac{du}{da} \right) \right) \]
$E_u: \quad y^2 = x^2(x - u) + \frac{\Lambda^4}{4}x \quad u \in \mathbb{C}$

$P_0 = \frac{1}{2}(1, \sqrt{1 - u})$

$2P_0 = (0,0)$

$3P_0 = \frac{1}{2}(1, -\sqrt{1 - u})$

$4P_0 = \infty$
SW Curve And Modular Forms -2/2

\[ E_u: \quad y^2 = x^2(x - u) + \frac{\Lambda^4}{4}x \quad u \in \mathbb{C} \]

u-plane \cong \text{Modular curve for } \Gamma^0(4)

\[ u = \frac{1}{2} \left( \vartheta_2^4 + \vartheta_3^4 \right) \]

\[ \frac{da}{du} = \frac{1}{2} \vartheta_2 \vartheta_3 \]
Defining The Integral

Near each cusp expand in q-Puiseux series

Ingredients involve expansions in $q^{\frac{1}{8}}$ in arbitrarily high positive and negative powers.

$\sum_{\lambda}(\ldots)$ has the right period: Only powers of $q^{\frac{1}{4}}$ survive.

Expand in power series in $p^\ell S^r \Rightarrow$
bound on negative powers

Then near each cusp check all potentially divergent terms are killed by the integral over the phase.
\( Z_u \) Wants To Be A Total Derivative

\[
Z_u = \int d\tau d\bar{\tau} \mathcal{H}(\tau)\Psi_1
\]

\[
\Psi_1 = \sum_{\lambda} \frac{d}{d\bar{\tau}} (\mathcal{E}(\rho_\lambda^\omega)) e^{-i \pi \lambda^2 - i S \frac{du}{da} + 2 \pi i (\lambda - \lambda_0) \cdot w_2}
\]

\[
\mathcal{E}(r) = \int_{???}^r e^{-2\pi t^2} dt
\]

\[
\rho_\lambda^\omega = \sqrt{y} \lambda_+ - \frac{i}{4\pi \sqrt{y}} S_+ \frac{du}{da}
\]

??? Must be independent of \( \bar{\tau} \)

but can depend on \( \lambda, \omega, \tau \).

Choice is subtle: Want both convergence and modular invariance
Relation To Korpas & Manschot

KM modify the 2-observable: \[ U(S) \rightarrow U(S) + Q(*) \]

\[ Z_\omega^u - Z_{\omega^0}^u = \int d\tau d\bar{\tau} \mathcal{H}(\tau) \frac{d}{d\bar{\tau}} \hat{\Theta}^{\omega,\omega_0} \]

\[ \hat{\Theta}^{\omega,\omega_0} = \sum_{\lambda} \left( \varepsilon(r_\lambda^{\omega}) - \varepsilon(r_\lambda^{\omega_0}) \right) e^{\cdots} \]

\[ r_\lambda^{\omega} := \sqrt{y}\lambda_+ + \frac{1}{2\pi\sqrt{y}} S_+ \text{ Im} \left( \frac{du}{da} \right) \]

Modular completion of indefinite theta function of Vigneras, Zwegers, Zagier

Actually, there is no need to modify \( U(S) \): Just use \( \rho_\lambda^{\omega} \). Gives a different modular completion.
Wall-Crossing Formula

Gives a cleaner way to re-derive the old WCF for $Z_u$

$$Z_u^\omega - Z_u^{\omega_0} = \sum_{\text{cusps } u_s} \oint d\tau_s \mathcal{H}(\tau) \Theta^\omega,\omega_0$$

$u_s \in \{\infty, \Lambda^2, -\Lambda^2\}$ for $SU(2), N_f = 0$

Contribution of the cusp $u_s$:

$$\sum_\lambda e^{i\pi(\lambda-\lambda_0)w_2} (\text{sign}(\lambda \cdot \omega) - \text{sign}(\lambda \cdot \omega_0)) \left[ \frac{d\sigma}{d\tau_s} \left( \frac{da_s}{du} \right)^{b_2^2} \Delta^g e^{2pu + S^2\tau} e^{-iS \cdot \lambda \frac{du}{da_s} q_s} \right]_{q^0}$$
Three Techniques For Explicit Evaluations Of $Z_u$

Put $\omega_0$ in a vanishing chamber

Use unfolding/method of orbits

Sometimes we can find a convergent modular invariant completion of the anti-derivative and write the integral as a total derivative.
Vanishing Chamber

\[ \mathcal{D}_D(p^L S^r) = 0 \]
Donaldson Invariants For $S^2 \times S^2$

$H^2(X, \mathbb{Z}) \cong II^{1,1}$ \hspace{1cm} $Z_{SW} = 0$

\[ Z_u = \text{cnst.} \left[ g \ h \ \cot \left( \frac{S \cdot e}{2h} \right) \right]_{q^0} \]

\[ Z_u = \text{cnst.} \left[ g \ h \ \cot \left( \frac{S \cdot f}{2h} \right) \right]_{q^0} \]

\[ g = (u^2 - 1)e^{2pu+S^2 T} \]

\[ h = \frac{da}{du} \]
Donaldson Invariants For $\mathbb{CP}^2$

No chambers. $S = S_+ \ & \ Z_{SW} = 0$.

Can choose $w_2(P) = 0 \ or \ 1$

Case $w_2(P) = 0$ (Moore & Witten)

$$\frac{(S, \omega)}{32\sqrt{2\pi}} \int_{\Gamma_0(4)\backslash \mathcal{H}} \frac{dx\,dy}{y^{3/2}} \frac{\vartheta_4^9}{(\frac{1}{2}\vartheta_2\vartheta_3)^4} \exp\left\{2pu - \frac{1}{24} S^2 \left[\frac{\hat{E}_2}{h(\tau)^2} - 8u\right]\right\} \vartheta_4$$

Zagier’s weight $(3/2,0)$ form for $\Gamma_0(4)$

$$G(\tau, y) = \sum_{n \geq 0} \mathcal{H}(n)q^n + \sum_{f = -\infty}^{\infty} q^{-f^2} \frac{1}{16\pi y^{1/2}} \int_1^\infty e^{-4\pi f^2 uy} \frac{du}{u^{3/2}}$$

$$\frac{\partial}{\partial \tau} G = \frac{1}{32\pi i} \frac{1}{y^{3/2}} \vartheta_3(2\tau)$$
Donaldson Invariants For $\mathbb{CP}^2$

\[
\frac{(S, \omega)}{32\sqrt{2}\pi} \int_{\Gamma^0(4) \backslash \mathcal{H}} \frac{dx \, dy}{y^{3/2}} \frac{\vartheta_4^9}{(\frac{1}{2}\vartheta_2 \vartheta_3)^4} \exp \left\{ 2pu - \frac{1}{24}S^2 \left[ \frac{\hat{E}_2}{h(\tau)^2} - 8u \right] \right\} \overline{\vartheta_4}
\]

Act with differential operators to modify the shadow by powers of $\hat{E}_2$

\[
Z_D = -4\sqrt{2}\pi (S, \omega) \sum_{j=0}^{\infty} \frac{\Gamma(3/2)}{j! \Gamma(3/2+j)} \left( \frac{S^2}{2} \right)^j \sum_I \frac{f_I}{h_I^{2j}} (q \frac{d}{dq})^j \mathcal{H}_I(\tau) \bigg|_{q^0}
\]

Case $w_2(P) = 1$ (Malmendier & Ono, 2008 et seq.)

Also considered massless matter; Noted that the mock modular form for $N_f = 0$ same as in Mathieu Moonshine.
What About Other N=2 Theories?

Central Question: Given the successful application of N=2 SYM for SU(2) to the theory of 4-manifold invariants, are there interesting applications of OTHER N=2 field theories?

Topological twisting just depends on $SU(2)_R$ symmetry and makes sense for any N=2 field theory.

$$Z^T := \langle e^{U+U(S)} \rangle^T$$

The u-plane integral makes sense for ANY family of Seiberg-Witten curves.
N=2 Theories

Lagrangian theories: Compact Lie group $G$, quaternionic representation $\mathcal{R}$ with $G$-invariant metric,

$$\tau_0 \in \prod_{\text{simple factors}} \mathcal{H} \quad m \in \text{Lie}(G_f) \quad G_f = Z(G) \subset O(\mathcal{R})$$

Class S: Theories associated to Hitchin systems on Riemann surfaces.

Superconformal theories

Couple to N=2 supergravity
Lagrangian Theories

Superconformal Theories

Class S Theories
SU(2) With Fundamental Hypers

Moore & Witten

\[ \mathcal{R} = N_{fl}(2 \oplus 2^*) \quad Spin(2N_{fl}) \subset G_f \]

Mass parameters \( m_f \in \mathbb{C}, \ f = 1, \ldots, N_{fl} \)

Must take \( \xi = w_2(X) \)

Seiberg-Witten: \( E_u : \ y^2 = x^3 + a_2x^2 + a_4x + a_6 \)

\( a_k \): Polynomials in \( \Lambda, m_f, u \)

\( Z_u \) has exactly the same expression as before but now, e.g. \( da/du \) depends on \( m_f \)

New ingredient: \( \mathcal{D} \) has \( 2 + N_{fl} \) cusps \( u_s \):

\[ \Delta = \prod_s (u - u_s) \]
For example, for $SU(2)$, $N_f = 1$ with large mass:
Analog Of Witten Conjecture

\[ b_2^+ > 1 \quad Z(p; s; m_f) = \sum_{j=1}^{2+N_{fl}} Z(p, s; m_f; u_j) \]

\[ Z(p, s; m_f; u_j) = \tilde{\alpha}^\chi \tilde{\beta}^\sigma \sum_{\lambda} SW(\lambda) e^{2\pi i \lambda \cdot \lambda_0} R_j(p, s) \]

\( X \) is SWST \( \Rightarrow \)

\[ R_j(p, s) = \kappa_j^\chi_h \left( \frac{du}{da} \right)^{\chi_h + \sigma} \exp \left( 2p \, u_j + S^2 T(u_j) - i \left( \frac{du}{da} \right)_j S \cdot \lambda \right) \]

\[ u = u_j + \kappa_j q_j + \mathcal{O}(q_j^2) \]

Everything computable explicitly as functions of the masses from first order degeneration of the SW curve.
\[ R_j(p, s) = \kappa_j^{\chi_h} \left( \frac{du}{da} \right)^{\chi_h + \sigma} \exp \left( 2p \, u_j + S^2 T(u_j) - i \left( \frac{du}{da} \right)_j \cdot S \cdot \lambda \right) \]

\[ u = u_j + \kappa_j q_j + O(q_j^2) \]

\[ y^2 = 4x^3 - g_2(u, m)x - g_3(u, m) \]

Assume a simple zero for \( \Delta \) as \( u \to u_j \)

Choose local duality frame with \( a_j \to 0 \)

Nonvanishing period: \( \frac{da_j}{du} \) and \( \frac{da_{j,D}}{du} \to i \infty \):

\[ \left( \frac{da_j}{du} \right)^2 \bigg|_{u_j} \sim \left( \frac{g_2}{g_3} \right) \bigg|_{u_j} \]

\[ \kappa_j \sim \left( \frac{g_2^3}{\Delta'} \right) \bigg|_{u_j} \]
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Superconformal Points

Consider $N_{fl} = 1$. At a critical point $m = m_*$ two singularities $u_\pm$ collide at $u = u_*$ and the SW curve becomes a cusp: $y^2 = x^3$ [Argyres,Plesser,Seiberg,Witten]

Two mutually nonlocal BPS states have vanishing mass:

$$\int_{\gamma_1} \lambda \to 0 \quad \int_{\gamma_2} \lambda \to 0 \quad \gamma_1 \cdot \gamma_2 \neq 0$$

Physically: No local Lagrangian for the LEET: Signals a nontrivial superconformal field theory Appears in the IR in the limit $m \to m_*$
AD3 From $SU(2) \, N_f = 1$

SW curve in the scaling region:

$$y^2 = x^3 - 3 \Lambda_{AD}^2 x + u_{AD}$$

$$\Lambda_{AD}^2 \sim (m - m_*)$$
The $SU(2) \ N_f = 1$ u-plane integral has a nontrivial contribution from the scaling region $u_\pm \to u_*$.

$$\lim_{m \to m_*} Z_u - \int du \ d\bar{u} \ \lim_{m \to m_*} \ Measure(u, \bar{u}; m) \neq 0$$

Limit and integration commute except in an infinitesimal region around $u_*$.

Attribute the discrepancy to the contribution of the AD3 theory.
AD3 Partition Function - 2

1. Limit \( m \to m_* \) exists.

2. The partition function is a sum over all \( Q \)-invariant field configurations.

3. Scaling region near \( u_* \) governed by AD3 theory.

\[
\lim_{m \to m_*} Z^{SU(2),N_f=1} \text{``contains'' the AD partition function}
\]

Extract it from the scaling region. Our result:

\[
Z^{AD3} = \lim_{\Lambda_{AD} \to 0} \left( Z_u^{AD3-family} + Z_{SW}^{AD3-family} \right)
\]

Claim: This is the AD3 TFT on \( X \) for \( b_2^+ > 0 \).
1. Existence of limit is highly nontrivial. It follows from "superconformal simple type sum rules":

Theorem [MMP, 1998] If the superconformal simple type sum rules hold:

a.) $\chi_h - c^2 - 3 \leq 0$

b.) $\sum_{\lambda} SW(\lambda)e^{2\pi i \lambda \cdot \lambda_0} \lambda^k = 0 \quad 0 \leq k \leq \chi_h - c^2 - 4$

Then the limit $m \to m_*$ exists

It is now a rigorous theorem that SWST $\Rightarrow$ SCST
2. For $b_2^+ > 0$ recover the expected selection rule:

$$\langle U^\ell U(S)^r \rangle \neq 0 \text{ only for } \frac{6\ell + r}{5} = \frac{\chi_h - c^2}{5}$$

3. For $b_2^+ = 1$ correlation functions have continuous metric dependence – a hallmark of a superconformal theory (M&W $SU(2), N_f = 4$)

Explicitly, if $X$ of $SWST$ and $b_2^+ > 1$:

$$Z^{AD3} = \frac{1}{r!} K_1 K_2^\chi K_3^\sigma \sum_\lambda e^{2\pi i \lambda \cdot \lambda_0} SW(\lambda)(\lambda \cdot S)^{r-2}(24(\lambda \cdot S)^2 + S^2)$$

$$r = \chi_h - c^2 - 2$$
Vafa-Witten Partition Functions
Discussions with G. Korpas & J. Manschot

VW twist of N=4 SYM formally computes the "Euler character" of instanton moduli space.

(Not really a topological invariant.)

Physics suggests the partition function is modular (S-duality) and holomorphic.

Surprise! Computations of Klyachko and Yoshioka show that the holomorphic generating function is only mock modular.

This has never been properly derived from a physical argument. We are trying to fill that gap.