Perverse Sheaves on Semi-abelian Varieties: Structure and Applications

Laurentiu Maxim
(joint work with Yongqiang Liu and Botong Wang)

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Let $X$ be a smooth connected complex quasi-projective variety with $b_1(X) > 0$. 

The (identity component of the) moduli space of rank-one $C$-local systems on $X$ is defined as:

$$\text{Char}(X) := \text{Hom}(H_1(X, \mathbb{Z})/\text{Torsion}, C^*) \cong (C^*)^{b_1(X)}$$

**Definition**

The $i$-th cohomology jumping locus of $X$ is defined as:

$$V_i(X) = \{ \rho \in \text{Char}(X) | H^i(X, L_{\rho}) \neq 0 \}$$

$V_i(X)$ are closed subvarieties of $\text{Char}(X)$ and homotopy invariants of $X$. 

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where $L_\rho$ is the rank-one $\mathbb{C}$-local system on $X$ associated to the representation $\rho \in \text{Char}(X)$.
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$$1 \rightarrow T \rightarrow G \rightarrow A \rightarrow 1,$$

where $A$ is an abelian variety of dimension $g$ and $T \cong (\mathbb{C}^\ast)^m$ is an algebraic affine torus of dimension $m$. 
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$$1 \to T \to G \to A \to 1,$$

where $A$ is an abelian variety of dimension $g$ and $T \cong (\mathbb{C}^*)^m$ is an algebraic affine torus of dimension $m$. In particular,

$$\pi_1(G) \cong \mathbb{Z}^{m+2g}, \text{ with } \dim G = m + g.$$
Albanese map. Albanese variety

Definition

Let $X$ be a smooth complex quasi-projective variety. The \textit{Albanese map} of $X$ is a morphism $\text{alb} : X \to \text{Alb}(X)$ from $X$ to a semi-abelian variety $\text{Alb}(X)$. 

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$\text{Alb}(X)$ is called the *Albanese variety* associated to $X$. 

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In particular,

$$\text{Char}(X) \cong \text{Char}(\text{Alb}(X)).$$
By the projection formula, for any $\rho \in \text{Char}(X) \cong \text{Char}(\text{Alb}(X))$: 

$$H^i(X, L_\rho) \cong H^i(\text{Alb}(X), (R\text{alb}_* \mathbb{C}_X) \otimes L_\rho).$$
Constructible complexes enter the scene

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If alb is proper (e.g., $X$ is projective), the BBDG decomposition theorem yields that $R\text{alb}_* \mathbb{C}_X$ is a direct sum of (shifted) perverse sheaves.
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This motivates the study of cohomology jumping loci of constructible complexes (resp., perverse sheaves) on semi-abelian varieties.
Definition

Let $\mathcal{F} \in D^b_c(G, \mathbb{C})$ be a bounded constructible complex of \mathbb{C}-sheaves on a semi-abelian variety $G$. The degree $i$ cohomology jumping locus of $\mathcal{F}$ is defined as:

$$V^i(G, \mathcal{F}) := \{ \rho \in \text{Char}(G) \mid H^i(G, \mathcal{F} \otimes \mathbb{C} L_\rho) \neq 0 \}.$$
Cohomology jump loci of constructible complexes

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**Theorem (Budur-Wang)**

*Each $\mathcal{V}^i(G, \mathcal{F})$ is a finite union of translated subtori of $\text{Char}(G)$.*
Mellin transformation

\[ \text{Char}(G) = \text{Spec } \Gamma_G, \text{ with } \Gamma_G := \mathbb{C}[\pi_1(G)] \cong \mathbb{C}[t_1^{\pm1}, \cdots, t_{m+2g}^{\pm1}] \]
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Let \( \mathcal{L}_G \) be the (universal) rank 1 local system of \( \Gamma_G \)-modules on \( G \), defined by mapping the generators of \( \pi_1(G) \cong \mathbb{Z}^{m+2g} \) to the multiplication by the corresponding variables of \( \Gamma_G \).
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The *Mellin transformation* \( \mathcal{M}_* : D^b_c(G, \mathbb{C}) \to D^b_{coh}(\Gamma_G) \) is given by

\[
\mathcal{M}_*(\mathcal{F}) := \text{Ra}_*(\mathcal{L}_G \otimes_{\mathbb{C}} \mathcal{F}),
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where \( a : G \to \text{pt} \) is the constant map, and \( D^b_{coh}(\Gamma_G) \) denotes the bounded coherent complexes of \( \Gamma_G \)-modules.

Theorem (Gabber-Loeser ’96, Liu-M.-Wang ’17)

If \( G = T \) is a complex affine torus, then:

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\mathcal{F} \in \text{Perv}(T, \mathbb{C}) \iff H^i(\mathcal{M}_*(\mathcal{F})) = 0 \text{ for all } i \neq 0.
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(By the projection formula) cohomology jump loci of $\mathcal{F}$ are determined by those of $\mathcal{M}_*(\mathcal{F})$:

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where if $R$ is a Noetherian domain and $E^\bullet$ is a bounded complex of $R$-modules with finitely generated cohomology, we set

$$\mathcal{V}^i(E^\bullet) := \{ \chi \in \text{Spec } R \mid H^i(F^\bullet \otimes_R R/\chi) \neq 0 \},$$

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with $F^\bullet$ a bounded above finitely generated free resolution of $E^\bullet$. So, understanding $\mathcal{V}^i(G, \mathcal{F})$ is now a commutative algebra problem!
Theorem (Liu-M.-Wang ’18)

For any \( \mathbb{C} \)-perverse sheaf \( \mathcal{P} \) on a semi-abelian variety \( G \), the cohomology jump loci of \( \mathcal{P} \) satisfy the following properties:

(i) \textit{Propagation property}:

\[
\mathcal{V}^{-m-g}(G, \mathcal{P}) \subseteq \cdots \subseteq \mathcal{V}^0(G, \mathcal{P}) \supseteq \mathcal{V}^1(G, \mathcal{P}) \supseteq \cdots \supseteq \mathcal{V}^g(G, \mathcal{P}).
\]

Moreover, \( \mathcal{V}^i(G, \mathcal{P}) = \emptyset \) if \( i \notin [-m-g, g] \).

(ii) \textit{Codimension lower bound}: for any \( i \geq 0 \),

\[
\text{codim}\mathcal{V}^i(G, \mathcal{P}) \geq 2i \quad \text{and} \quad \text{codim}\mathcal{V}^{-i}(G, \mathcal{P}) \geq i.
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Remark (Equivalent formulation of the propagation property)

Let $\mathcal{P}$ be a $\mathbb{C}$-perverse sheaf so that not all $H^i(G, \mathcal{P})$ are zero.
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The propagation property is equivalent to: $k_+ \geq 0$, $k_- \leq 0$ and

$$H^j(G, \mathcal{P}) \neq 0 \iff k_- \leq j \leq k_+.$$
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(If $G = A$ is an abelian variety, a similar result was proved by Weissauer.)
Generic vanishing

Corollary (Kramer ’14, Liu-M.-Wang ’17, Franecki-Kapranov ’00)

For any $\mathbb{C}$-perverse sheaf $\mathcal{P}$ on a semi-abelian variety $G$,

$$H^i(G, \mathcal{P} \otimes_{\mathbb{C}} L_{\rho}) = 0$$

for any generic rank-one $\mathbb{C}$-local system $L_{\rho}$ and all $i \neq 0$. 

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In particular,

\[
\chi(G, \mathcal{P}) \geq 0.
\]

Moreover, the equality holds if and only if \( V^0(G, \mathcal{P}) \neq \text{Char}(G) \).
Corollary (Liu-M.-Wang ’18)

Let $X$ be a smooth quasi-projective variety of complex dimension $n$. Assume that $R \text{alb}_* \mathbb{C}_X[n]$ is a perverse sheaf on $\text{Alb}(X)$ (e.g., alb is proper and semi-small). Then:

1. Propagation property: $V^n(X) \supseteq V^{n-1}(X) \supseteq \cdots \supseteq V^0(X) = \{1\}$ and $V^n(X) \supseteq V^{n+1}(X) \supseteq \cdots \supseteq V^{2n}(X)$.

2. Codimension lower bound: for any $i \geq 0$, $\text{codim} V^{n-i}(X) \geq i$ and $\text{codim} V^{n+i}(X) \geq 2i$.

3. Generic vanishing: $H^i(X, L^\rho) = 0$ for generic $\rho \in \text{Char}(X)$ and all $i \neq n$.

4. Signed Euler characteristic property: $(-1)^n \cdot \chi(X) \geq 0$.

5. Betti property: $b_i(X) > 0$ for any $i \in [0, n]$, and $b_1(X) \geq n$. 

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Corollary (Liu-M.-Wang ’18)

Let $X$ be a smooth quasi-projective variety with proper Albanese map (e.g., $X$ is projective), and assume that $X$ is homotopy equivalent to a torus. Then $X$ is isomorphic to a semi-abelian variety.
Homological duality

Definition (Denham-Suciu-Yuzvinsky ’15)

A connected finite CW complex $X$, with $H := H_1(X, \mathbb{Z})$, is an **abelian duality space of dimension $n$** if:

(a) $H^i(X, \mathbb{Z}[H]) = 0$ for $i \neq n$,
(b) $H^n(X, \mathbb{Z}[H])$ is a (non-zero) torsion-free $\mathbb{Z}$-module.
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In what follows we work with the full character variety

$$\text{Char}(X) = \text{Hom}(H, \mathbb{C}^*).$$
Using properties of the Mellin transformation, we get:

**Theorem (Liu-M.-Wang ’18)**

Let $X$ be an $n$-dimensional smooth complex quasi-projective variety, which is homotopy equivalent to an $n$-dimensional CW complex (e.g., $X$ is affine). Suppose the Albanese map $\text{alb}$ is proper and semi-small (e.g., a closed embedding), or $\text{alb}$ is quasi-finite. Then $X$ is an abelian duality space of dimension $n$. 

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Example (Very affine manifolds)

Let $X$ be an $n$-dimensional very affine manifold, i.e., a smooth closed subvariety of a complex affine torus $T = (\mathbb{C}^*)^m$ (e.g., the complement of an essential hyperplane / toric arrangement). Then $X$ is an abelian duality space of dimension $n$. (Here, alb is proper and semi-small.)
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Example (Elliptic arrangement complements)

Let $E$ be an elliptic curve, and let $\mathcal{A}$ be an essential elliptic arrangement in $E^{\times n}$ with complement $X := E^{\times n} \setminus \mathcal{A}$. 
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Let $E$ be an elliptic curve, and let $A$ be an essential elliptic arrangement in $E^{\times n}$ with complement $X := E^{\times n} \setminus A$. Then $X$ is a complex $n$-dimensional affine variety, and by the universal property of the Albanese map, the natural embedding $X \hookrightarrow E^{\times n}$ factors through $\text{alb} : X \rightarrow \text{Alb}(X)$. Hence the Albanese map $\text{alb} : X \rightarrow \text{Alb}(X)$ is also an embedding (hence quasi-finite).
Example (Very affine manifolds)

Let $X$ be an $n$-dimensional very affine manifold, i.e., a smooth closed subvariety of a complex affine torus $T = (\mathbb{C}^*)^m$ (e.g., the complement of an essential hyperplane / toric arrangement). Then $X$ is an abelian duality space of dimension $n$. (Here, alb is proper and semi-small.)

Example (Elliptic arrangement complements)

Let $E$ be an elliptic curve, and let $\mathcal{A}$ be an essential elliptic arrangement in $E\times^n$ with complement $X := E\times^n \setminus \mathcal{A}$. Then $X$ is a complex $n$-dimensional affine variety, and by the universal property of the Albanese map, the natural embedding $X \hookrightarrow E\times^n$ factors through $\text{alb} : X \to \text{Alb}(X)$. Hence the Albanese map $\text{alb} : X \to \text{Alb}(X)$ is also an embedding (hence quasi-finite). So $X$ is an abelian duality space of dimension $n$. 
Theorem (Denham-Suciu-Yuzvinsky ’15, Liu-M.-Wang ’17)

Let $X$ be an abelian duality space $X$ of dimension $n$. Then:

(i) **Propagation property:** $\mathcal{V}^n(X) \supseteq \mathcal{V}^{n-1}(X) \supseteq \cdots \supseteq \mathcal{V}^0(X)$.

(ii) **Codimension lower bound:** for any $i \geq 0$,

$$\text{codim} \mathcal{V}^{n-i}(X) = b_1(X) - \dim \mathcal{V}^{n-i}(X) \geq i.$$ 

(iii) **Generic vanishing:** $H^i(X, L_\rho) = 0$ for $\rho$ generic and all $i \neq n$.

(iv) **Signed Euler characteristic property:**

$$(-1)^n \chi(X) \geq 0.$$ 

(v) **Betti property:** $b_i(X) > 0$, for $0 \leq i \leq n$, and $b_1(X) \geq n$. 

Laurentiu Maxim
The following provides a new topological characterization of compact complex tori and, resp., abelian varieties in terms of homological duality properties:
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Let $X$ be a compact Kähler manifold. Then $X$ is an abelian duality space if and only if $X$ is a compact complex torus.

In particular, abelian varieties are the only complex projective manifolds that are abelian duality spaces.
The following provides a new topological characterization of compact complex tori and, resp., abelian varieties in terms of homological duality properties:

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Characterization of perverse sheaves

Theorem (Schnell '15)

If $A$ is an abelian variety and $\mathcal{F} \in D^b_c(A, \mathbb{C})$, then:

$$\mathcal{F} \in \text{Perv}(A, \mathbb{C}) \iff \forall i \in \mathbb{Z} : \text{codim} \mathcal{V}^i(A, \mathcal{P}) \geq |2i|.$$
If $1 \to T \to G \to A \to 1$ defines a semi-abelian variety $G$, and
$
\Gamma_G := \mathbb{C}[\pi_1(G)], \quad \Gamma_T = \mathbb{C}[\pi_1(T)] \quad \text{and} \quad \Gamma_A = \mathbb{C}[\pi_1(A)],$
then $\text{Spec } \Gamma_G, \text{ Spec } \Gamma_T$ and $\text{Spec } \Gamma_A$ are affine tori fitting into a short exact sequence of linear algebraic groups.

Definition

Let $V$ be an irreducible subvariety of $\text{Spec } \Gamma_G$. Define:

- torus dimension: $\dim_t V = \dim p(V)$,
- abelian dimension: $\dim_a V = \frac{1}{2}(\dim V - \dim_t V)$,
- semi-abelian dimension: $\dim_{sa} V = \dim_t V + \dim_a V$,

- $\text{codim}_t V = m - \dim_t V$,
- $\text{codim}_a V = g - \dim_a V$,
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Remark

1. If $G = T$ is a complex affine torus: $\dim_{sa}(V) = \dim(V)$, $\text{codim}_{sa}(V) = \text{codim}(V)$, $\dim_a(V) = \text{codim}_a(V) = 0$. 

Theorem (Liu-M.-Wang '18)

A constructible complex $F \in D^b_{\text{ct}}(G, \mathbb{C})$ is perverse on $G$ ⇐⇒ (i) $\text{codim}_a V_i(G, F) \geq i$ for any $i \geq 0$, and (ii) $\text{codim}_{sa} V_i(G, F) \geq -i$ for any $i \leq 0$. 

Corollary

$F \in D^b_{\text{ct}}(T, \mathbb{C})$ is perverse on a complex affine torus $T$ ⇐⇒ (i) For any $i > 0$: $V_i(T, F) = \emptyset$, and (ii) For any $i \leq 0$: $\text{codim}_a V_i(T, F) \geq -i$. 

Laurentiu Maxim
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Happy Birthday, Sylvain!