Applications of Balmer–Witt Groups to Stratified Spaces

Jon Woolf

August, 2018
Part I

Balmer–Witt Groups
Forms in triangulated categories with duality

Suppose $C$ is a triangulated category in which $2$ is invertible, and

$$D : C^{\text{op}} \to C$$

is an exact duality. Note $D(-)[n]$ is also a duality for any $n \in \mathbb{Z}$.

- A non-degenerate symmetric form is an isomorphism
  $$\sigma : c \to Dc \quad \text{with} \quad \sigma = D\sigma$$

- It is metabolic if there is a triangle
  $$b \xrightarrow{\lambda} c \xrightarrow{D\lambda \cdot \sigma} Db \to c[1]$$
  in which case $\lambda : b \to c$ is a lagrangian for $\sigma$.

- Forms $\sigma$ and $\sigma'$ are isometric if $\sigma' = \alpha \cdot \sigma \cdot \alpha^{-1}$ for some $\alpha$. 

Balmer–Witt Groups

Consider shifted duality $D(−)[−n]$ for $n \in \mathbb{Z}$. The Balmer–Witt group is the monoid quotient

$$W_n(C) = \frac{\{\text{isometry classes of non-deg symmetric forms}\}}{\{\text{isometry classes of metabolic forms}\}}$$

It is an abelian group since $\sigma \oplus (−\sigma)$ is metabolic.
Balmer–Witt Groups

Consider shifted duality $D(-)[-n]$ for $n \in \mathbb{Z}$. The Balmer–Witt group is the monoid quotient

$$W_n(C) = \frac{\{\text{isometry classes of non-deg symmetric forms}\}}{\{\text{isometry classes of metabolic forms}\}}$$

It is an abelian group since $\sigma \oplus (-\sigma)$ is metabolic.

Remarks

- Balmer uses cohomological indexing: $W^n(C) = W_{-n}(C)$
- $W_*(-)$ functorial for exact functors commuting with duality
- Unlike in the abelian / exact case, $[\sigma] = 0 \iff \sigma$ metabolic
- There are isomorphisms $W_n(C) \cong W_{n+4}(C): \sigma \mapsto \sigma[2]$
Self-dual t–structures

A bounded t-structure $C^{\leq 0} \subset C$ is

- **self-dual** if $C^{\leq 0} = D(C^{\geq 0})$
- equivalently $C^{\heartsuit} = C^{\leq 0} \cap C^{\geq 0}$ abelian with exact duality

**Theorem (Schürmann–W, cf Balmer)**

Suppose $C$ has a self-dual t–structure with heart $C^{\heartsuit}$. Then

\[ W_n(C) \cong \begin{cases} W_+(C^{\heartsuit}) & n \equiv 0 \mod 4 \\ W_-(C^{\heartsuit}) & n \equiv 2 \mod 4 \\ 0 & \text{otherwise} \end{cases} \]

**Example (Witt groups of field $\mathbb{Q}$)**

\[ W_n(Db(\mathbb{Q})) \cong \begin{cases} W(\mathbb{Q}) & n \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases} \]
Self-dual $t$–structures

A bounded $t$-structure $C^{\leq 0} \subset C$ is

- self-dual if $C^{\leq 0} = D(C^{\geq 0})$
- equivalently $C^{\heartsuit} = C^{\leq 0} \cap C^{\geq 0}$ abelian with exact duality

**Theorem (Schürmann–W, cf Balmer)**

*Suppose $C$ has a self-dual $t$–structure with heart $C^{\heartsuit}$. Then*

$$W_n(C) \cong \begin{cases} 
W_+(C^{\heartsuit}) & n = 0 \mod 4 \\
W_-(C^{\heartsuit}) & n = 2 \mod 4 \\
0 & \text{otherwise}
\end{cases}$$

**Example (Witt groups of field $\mathbb{Q}$)**

$$W_n(D^b(\mathbb{Q})) \sim \begin{cases} 
W(\mathbb{Q}) & n = 0 \mod 4 \\
0 & \text{otherwise}
\end{cases}$$
Self-dual t–structures

A bounded $t$-structure $C^{\leq 0} \subset C$ is

- self-dual if $C^{\leq 0} = D(C^{\geq 0})$
- equivalently $C^{\heartsuit} = C^{\leq 0} \cap C^{\geq 0}$ abelian with exact duality

Theorem (Schürmann–W, cf Balmer)

Suppose $C$ has a self-dual $t$–structure with heart $C^{\heartsuit}$. Then

$$W_n(C) \cong \begin{cases} W_+(C^{\heartsuit}) & n = 0 \pmod{4} \\ W_-(C^{\heartsuit}) & n = 2 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

Example (Witt groups of field $Q$)

$$W_n(D^b(Q)) \cong \begin{cases} W(Q) & n = 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$
Theorem (Balmer)

Suppose $T \to C \to Q$ is a ‘short exact sequence’ of triangulated categories with duality. Then there is a long exact sequence

$$\cdots \to W_n(T) \to W_n(C) \to W_n(Q) \xrightarrow{\partial} W_{n-1}(T) \to \cdots$$

of Balmer–Witt groups.
Localisation Long Exact Sequence

Theorem (Balmer)
Suppose $T \rightarrow C \rightarrow Q$ is a ‘short exact sequence’ of triangulated categories with duality. Then there is a long exact sequence

$$\cdots \rightarrow W_n(T) \rightarrow W_n(C) \rightarrow W_n(Q) \xrightarrow{\partial} W_{n-1}(T) \rightarrow \cdots$$

of Balmer–Witt groups. The boundary is constructed using

$$
\begin{array}{ccc}
  c & \xrightarrow{\sigma} & Dc[-n] \\
  \| & & \| \\
  c & \xrightarrow{D\sigma[-n]} & Dc[-n] \\
  \| & & \| \\
  c & \xrightarrow{D\sigma[-n]} & D\text{Cone}(\sigma)[-n+1] \\
\end{array}
$$
Part II

Stratified Witt Groups
Stratified Witt groups

Suppose
- $X_S$ finite-dim locally conically stratified space
- $\frac{1}{2} \in R$ regular, Noetherian, commutative, finite Krull dim
- $D(X_S; R)$ the $S$-constructible derived category
- $D = \mathcal{H}om(-, D_X)$ the Verdier duality.

The stratified Witt groups are

$$W_n(X_S; R) = W_n(D(X_S; R))$$

There is a natural surjective homomorphism

$$W_n(X_S; R) \rightarrow \Omega^{SD}(X_S; R)$$
Properties of Stratified Witt Groups

Functoriality

Suppose \( f : X_S \to Y_T \) stratified. Then

- proper \( f \) induces \( W_n(X_S; R) \to W_n(Y_T; R) \) since \( f_*D \cong Df_* \)
- normally nonsingular \( f \) of codim \( d \) induces

\[
W_n(Y_T; R) \to W_{n-d}(X_S; R)
\]

since \( Df^* \cong f^*D[-d] \)
Properties of Stratified Witt Groups

Functoriality
Suppose \( f : X_S \to Y_T \) stratified. Then
- proper \( f \) induces \( W_n(X_S; R) \to W_n(Y_T; R) \) since \( f_* D \cong Df_* \).
- normally nonsingular \( f \) of codim \( d \) induces
  \[
  W_n(Y_T; R) \to W_{n-d}(X_S; R)
  \]
  since \( Df^* \cong f^* D[-d] \).

Geometric Localisation
If \( Z_S \) closed union of strata and \( U_S = X_S - Z_S \) there is a LES
\[
\to W_n(Z_S; R) \to W_n(X_S; R) \to W_n(U_S; R) \to W_{n-1}(Z_S; R) \to
\]
Example: complex projective spaces

Let $X_S = (\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \subset \mathbb{CP}^d)$ and $Q$ a field. Middle perversity t-structure is self-dual. Geometric localisation yields

\[
W_n(\mathbb{CP}^d; Q) \cong \begin{cases} 
W(Q) \oplus [d/2] + 1 & n \equiv 0 \mod 4 \\
W(Q) \oplus [d/2] & n \equiv 2 \mod 4 \\
0 & \text{otherwise}
\end{cases}
\]
Example: complex projective spaces

Let \( X_S = (\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \subset \mathbb{CP}^d) \) and \( Q \) a field. Middle perversity t–structure is self-dual. Geometric localisation yields

\[
W_n(\mathbb{CP}^d; Q) \cong \begin{cases} 
W(Q)^{\oplus \lceil d/2 \rceil} + 1 & n \equiv 0 \pmod{4} \\
W(Q)^{\oplus \lfloor d/2 \rfloor} & n \equiv 2 \pmod{4} \\
0 & \text{otherwise}
\end{cases}
\]

‘Assembly’ LES associated to \( p: X_S \to \text{pt} \) yields

\[
W_n(\ker p_*) \cong \begin{cases} 
W(Q)^{\oplus \lfloor d/2 \rfloor} & n \equiv 0 \pmod{4} \\
W(Q)^{\oplus \lceil d/2 \rceil} & n \equiv 2 \pmod{4} \\
0 & \text{otherwise}
\end{cases}
\]

This implies existence of indecomposable self-dual middle perverse sheaves with vanishing cohomology.
Example: the circle

Let \( X_S = \{ \text{pt} \subset S^1 \} \) and \( Q \) a field. Geometric localisation yields

\[
W_n(X_S; Q) \cong \begin{cases} 
W(Q) & n = 0, 1 \mod 4 \\
0 & \text{otherwise}
\end{cases}
\]
Example: the circle

Let $X_S = (\text{pt} \subset S^1)$ and $Q$ a field. Geometric localisation yields

$$W_n(X_S; Q) \cong \begin{cases} W(Q) & n = 0, 1 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

The coherent-constructible correspondence

$$D(X_S; \mathbb{C}) \cong D^b\text{Coh}(\mathbb{CP}^1)$$

implies (in agreement with a theorem of Walter)

$$W^n(\mathbb{CP}^1) \cong \begin{cases} W(\mathbb{C}) & n = 0, 1 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

Challenge

Compute the coherent Witt groups of complex toric varieties using the coherent-constructible correspondence.
Part III

Witt Groups of Perverse Sheaves
Witt Groups of Perverse Sheaves

Suppose $X_S$ has only even codim strata and $Q$ a field. Then

$$W_d(X_S; Q) \cong W (^m \text{Perv}(X_S; Q))$$

Moreover the geometric localisation LES splits and

$$W (^m \text{Perv}(X_S; Q)) \cong W_\pm (^m \text{Perv}(Z_S; Q)) \oplus W (^m \text{Perv}(U_S; Q))$$

for closed $\iota: Z_S \hookrightarrow X_S$ with complement $j: U_S \hookrightarrow X_S$. 
Witt Groups of Perverse Sheaves

Suppose $X_S$ has only even codim strata and $Q$ a field. Then

$$W_d(X_S; Q) \cong W(\mathcal{m}\text{Perv}(X_S; Q))$$

Moreover the geometric localisation LES splits and

$$W(\mathcal{m}\text{Perv}(X_S; Q)) \cong W(\mathcal{m}\text{Perv}(Z_S; Q)) \oplus W(\mathcal{m}\text{Perv}(U_S; Q))$$

for closed $i: Z_S \hookrightarrow X_S$ with complement $j: U_S \hookrightarrow X_S$. By induction

$$W_d(X_S; Q) \cong \bigoplus_{S \subset X} W_{\epsilon(S)}(\text{Loc}(S)) \quad (1)$$

where $\epsilon(S) = (-1)^{\text{codim } S/2}$. 
Witt Groups of Perverse Sheaves

Suppose $X_S$ has only even codim strata and $Q$ a field. Then

$$W_d(X_S; Q) \cong W (^m \text{Perv}(X_S; Q))$$

Moreover the geometric localisation LES splits and

$$W (^m \text{Perv}(X_S; Q)) \cong W_\pm (^m \text{Perv}(Z_S; Q)) \oplus W (^m \text{Perv}(U_S; Q))$$

for closed $\imath: Z_S \hookrightarrow X_S$ with complement $j: U_S \hookrightarrow X_S$. By induction

$$W_d(X_S; Q) \cong \bigoplus_{S \subset X} W_{\epsilon(S)} (\text{Loc}(S)) \quad (1)$$

where $\epsilon(S) = (-1)^{\text{codim } S/2}$.

Lemma (Schürmann-W)

Each $[\sigma] \in W_d(X_S; Q)$ has decomposition $[\sigma] = [\imath_* \imath!^* \sigma] + [j! j^* \sigma]$ which corresponds to above if $\sigma$ anisotropic or $j!*$ exact.
The Topological Decomposition Theorem

**Theorem (Schürmann-W, cf Cappell–Shaneson)**

Let $j_S: S \hookrightarrow \overline{S}$ and $\iota_S: \overline{S} \hookrightarrow X$. Suppose either $\sigma$ is anisotropic or $j_S!*$ exact for all $S$. Then

$$[\sigma] = \sum_{S \in S} [j_S!*j_S^*\iota_S^!*\sigma]$$
The Topological Decomposition Theorem

**Theorem (Schürmann-W, cf Cappell–Shaneson)**

Let \( j_S : S \hookrightarrow \overline{S} \) and \( \iota_S : \overline{S} \hookrightarrow X \). Suppose either \( \sigma \) is anisotropic or \( j_S! * \) exact for all \( S \). Then

\[
[\sigma] = \sum_{S \in S} [j_S! * j_S^* \iota_S^! * \sigma]
\]

**Remarks**

- Intermediate extensions exact if eg \( \pi_1 S \) finite for each \( S \in S \) and \( m\overline{H}^{(\dim L - 1)/2}(L; \mathcal{E}) = 0 \) for each relative link \( L \) and \( \mathcal{E} \).
- Counterexamples can be constructed using quiver descriptions.
- There is a stratum-local algorithm for constructing anisotropic representatives.
- Perverse sheaves underlying pure algebraic Hodge modules carry anisotropic forms coming from polarisations.
Part IV

Constructible Witt Groups and $L$-classes
Constructible Witt groups

For a filtered family $\mathcal{S}$ of locally conical stratifications of $X$

$$D(X_\mathcal{S}; R) = \text{colim}_{S \in \mathcal{S}} D(X_S; R)$$

and the $\mathcal{S}$-constructible Witt groups are

$$W_n(X_\mathcal{S}; R) = W_n(D(X_\mathcal{S}; R)) \cong \text{colim}_{S \in \mathcal{S}} W_n(X_S; R)$$
Constructible Witt groups

For a filtered family $\mathcal{S}$ of locally conical stratifications of $X$

$$D(X_\mathcal{S}; R) = \text{colim}_{\mathcal{S} \in \mathcal{S}} D(X_S; R)$$

and the $\mathcal{S}$-constructible Witt groups are

$$W_n(X_\mathcal{S}; R) = W_n(D(X_\mathcal{S}; R)) \cong \text{colim}_{\mathcal{S} \in \mathcal{S}} W_n(X_S; R)$$

Theorem (W)

For compact PL spaces $X$ the PL-constructible Witt groups are Ranicki’s free symmetric $L$-groups: $W_n(X_{PL}; R) \cong H_n(X; \mathbb{L} \cdot (R))$
Rational coefficients and Witt spaces

Suppose $X_S$ cpt oriented $d$-dim PL Witt space and

$$m(S) = \lfloor \text{codim } S/2 \rfloor$$

is the middle perversity. The intersection form

$$m\mathcal{IC}^\bullet(X; \mathbb{Q}) \to D^m\mathcal{IC}^\bullet(X; \mathbb{Q})$$

yields class $[X] \in W_d(X_S; \mathbb{Q})$ with images

L-class $L(X) \in W_d(X_{PL}; \mathbb{Q}) \cong H_d(X; \mathbb{L} \cdot (\mathbb{Q}))$

Witt class $w(X) \in W_d(\text{pt}; \mathbb{Q})$
Rational coefficients and Witt spaces

Suppose \( X_S \) cpt oriented \( d \)-dim PL Witt space and

\[ m(S) = \lfloor \text{codim } S / 2 \rfloor \]

is the middle perversity. The intersection form

\[ mIC^\bullet(X; \mathbb{Q}) \rightarrow D^mIC^\bullet(X; \mathbb{Q}) \]

yields class \([X] \in W_d(X_S; \mathbb{Q})\) with images

- **L-class** \( L(X) \in W_d(X_{PL}; \mathbb{Q}) \cong H_d(X; \mathbb{L} \cdot (\mathbb{Q})) \)
- **Witt class** \( w(X) \in W_d(\text{pt}; \mathbb{Q}) \)

**Theorem (W, cf Siegel)**

*There are natural maps\( \Omega_n^{Witt}(X) \rightarrow W_n(X_{PL}; \mathbb{Q}) \) inducing*

\[ W_n(X_{PL}; \mathbb{Q}) \cong \text{colim}_{k \rightarrow \infty} \Omega_n^{Witt}(X) \]

*where we stabilise by products with \( \mathbb{CP}^2 \).*
Let $X_S$ be a $d$-dimensional pseudomanifold and $Q$ a field. A $Q$-orientation determines $[X^{\text{reg}}] \in W_d(X^{\text{reg}}; Q)$.

Let $S$ be a closed stratum and $Y_S = X_S - S$. Assume have extended $[X^{\text{reg}}]$ to

$$[Y] = [\mathcal{E} \to D\mathcal{E}] \in W_d(Y_S; Q)$$

Extension of $[Y]$ to $W_d(X_S; Q)$ is governed by LES

$$W_d(S; Q) \to W_d(X_S; Q) \to W_d(Y_S; Q) \xrightarrow{\partial} W_{d-1}(S; Q)$$
Witt and L-classes of non-Witt spaces

\textbf{codim} S \textit{even} \quad W_{d-1}(S; Q) = 0 \text{ so } [Y] \text{ extends to } W_d(X_S; Q).

Choose extension \( m_j!_* \mathcal{E} \to D^{m_j!}_* \mathcal{E}. \)

\textbf{codim} S \textit{odd} \quad W_d(S; Q) = 0 \text{ so unique extension exists}

\[ \iff \partial[Y] = 0 \]
\[ \iff \text{Cone}(m_j!_* \mathcal{E} \to D^{m_j!}_* \mathcal{E}) \text{ has Lagrangian} \]

When so, isotropically reduce \( m_j!_* \mathcal{E} \to D^{m_j!}_* \mathcal{E} \) to obtain extension.
Witt and L-classes of non-Witt spaces

\textbf{codim }S \text{ even} \quad W_{d-1}(S; Q) = 0 \text{ so } [Y] \text{ extends to } W_d(X_S; Q).
Choose extension \( m_{j!*}\mathcal{E} \to D^{m_{j!*}}\mathcal{E} \).

\textbf{codim }S \text{ odd} \quad W_d(S; Q) = 0 \text{ so unique extension exists}

\[ \iff \partial[Y] = 0 \]
\[ \iff \text{Cone}(m_{j!*}\mathcal{E} \to D^{m_{j!*}}\mathcal{E}) \text{ has Lagrangian} \]

When so, isotropically reduce \( m_{j!*}\mathcal{E} \to D^{m_{j!*}}\mathcal{E} \) to obtain extension.

\textbf{Theorem (cf Banagl)}

\begin{itemize}
    \item \([X^{\text{reg}}]\) extends to \( W_d(X_S; Q) \iff \text{SD}(X_S; Q) \neq \emptyset \)
    \item \( L(X) \) and \( w(X) \) independent of representative in \( \text{SD}(X_S; Q) \)
\end{itemize}
Goresky–Siegel’s Peripheral Invariant

- \( R \) Dedekind domain with field of fractions \( Q \)
- Assume have \([X] \in W_d(X_S; Q)\)

Considering algebraic localisation LES

\[
\rightarrow W_d(X_S; R) \rightarrow W_d(X_S; Q) \xrightarrow{\partial} W_{d-1}(X_S; R^{\text{tor}}) \rightarrow
\]

see \( \partial[X] \) is obstruction to lifting \([X]\) to \( W_d(X_S; R) \). When \( X \) compact and \( d = 4m \)

\[
\begin{align*}
W_d(X_S; Q) & \xrightarrow{\partial} W_{d-1}(X_S; R^{\text{tor}}) \\
p_* & \downarrow \quad p_* \\
W(Q) & \xrightarrow{\partial} \bigoplus_{p \text{ prime}} W(R/p)
\end{align*}
\]

commutes so that \( p_* \partial[X] = \partial w(X) \) is Goresky–Siegel’s peripheral invariant — the global obstruction to lifting \([X]\) to \( W_d(X_S; R) \).
Part V

Tilted-dual t–structures
Tilted-dual t–structures

A t–structure $C^{\leq 0}$ is tilted-dual if $C^{\leq 0}[1] \subset D(C^{\geq 0}) \subset C^{\leq 0}$. Then

- $T^\heartsuit = C^\heartsuit[1] \cap DC^\heartsuit$ torsion theory in $DC^\heartsuit$
- $F^\heartsuit = C^\heartsuit \cap DC^\heartsuit$ torsion-free theory in $DC^\heartsuit$
A t–structure $C^{\leq 0}$ is tilted-dual if $C^{\leq 0}[1] \subset D(C^{\geq 0}) \subset C^{\leq 0}$. Then

- $T^\heartsuit = C^\heartsuit[1] \cap D C^\heartsuit$ torsion theory in $D C^\heartsuit$
- $F^\heartsuit = C^\heartsuit \cap D C^\heartsuit$ torsion-free theory in $D C^\heartsuit$

Theorem (W)

Suppose $T^\heartsuit$ is hereditary. Then

- $T^\heartsuit$ is abelian with exact duality $D(\_)[1]$
- $Q^\heartsuit = D C^\heartsuit / T^\heartsuit$ is abelian with exact duality $D$
- there is an exact sequence

$$0 \to W_0(C) \to W(Q^\heartsuit) \xrightarrow{\partial} W(T^\heartsuit) \to W_{-1}(C) \to 0$$

and similarly with all indices incremented by 2.

Moreover $W(F^\heartsuit) \to W_0(C)$. 
Examples of tilted-dual t–structures

Suppose $R$ Dedekind domain with field of fractions $Q$.

**Example (Classical case)**

Dual of standard t–structure on $D^b(R)$ is hereditary tilted-dual:

$$0 \to W(R) \to W(Q) \xrightarrow{\partial} \bigoplus_{p \text{ prime}} W(R/p) \to W_{-1}(R) \to 0$$

**Example ($X_S$ $d$-dim conically stratified space)**

Heart $^n\# \Perv(X_S; R)$ glued from

- $\langle \Loc(S)^{tf}[-n(S)], \Loc(S)^t[-n(s) - 1]\rangle$ for codim $S$ even
- $\Loc(S)[-n(S)]$ for codim $S$ odd

is tilted-dual, but not in general hereditary tilted-dual.
Even-codimension stratifications

If all strata have even codimension then

\[ T^\bowtie = {}^n\text{Perv}(X_S; R^{\text{tor}}) \subset {}^n\text{Perv}(X_S; R) = DC^\bowtie \]

is hereditary and there is an exact sequence

\[
0 \to W_d(X_S; R) \to W({}^n\text{Perv}(X_S; Q)) \\
\to W({}^n\text{Perv}(X_S; R^{\text{tor}})) \to W_{d-1}(X_S; R) \to 0
\]

and similarly with all indices incremented by 2.

Corollary

\([X] \in W_d(X_S; Q)\) lifts \textit{uniquely} to \(W_d(X_S; R) \iff \partial[X] = 0.\)
Stratified Spaces with Parity Separation

Assume
- $R = Q$ so that $n^\# \text{Perv}(X_S; R) = n\text{Perv}(X_S; R)$
- the union $X_S^o$ of odd codim strata is closed.

Let $j: X_S^e \hookrightarrow X_S$. Then

$$T^\heartsuit = \ker j^* \subset m\text{Perv}(X_S; Q) = DC^\heartsuit$$

is hereditary, $Q^\heartsuit \cong m\text{Perv}(X_S^e; Q)$ and there is an exact sequence

$$0 \to W_d(X_S; Q) \to W(m\text{Perv}(X_S^e; Q)) \to W(\ker j^*) \to W_{d-1}(X_S; Q) \to 0$$

Moreover $F^\heartsuit = EP(X_S; Q)$ and $W(EP(X_S; Q)) \to W_d(X_S; Q)$.

Example ($X_S = (\partial M \subset M)$ is manifold with boundary)

Witt groups $W_d(X_S; Q)$ determined by $\pi_1 \partial M \to \pi_1 M$. In particular, if it is an isomorphism then $W_d(X_S; Q) = 0$ for all $d$. 