HMS/SYZ mirror symmetry:
Recent progress and going forward

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I am reporting on the works of Tristan Collins, Yu-Wei Fan, Hansol Hong, Shinobu Hosono, An Huang, Adam Jacob, Atsushi Kanazawa, Yoosik Kim, Siu-Cheong Lau, Tsung-Ju Lee, Bong Lian, Yu-Shen Lin, Artan Sheshmani, Hiromichi Takagi, Chenglong Yu, Dingxin Zhang, Jingyu Zhao, Yang Zhou, Minxian Zhu, Xinwen Zhu, and myself.

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Outline

1. Deformed Hermitian–Yang–Mills equations and stability conditions
2. Weil–Petersson geometry on the space of stability conditions
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1. Deformed Hermitian–Yang–Mills equations and stability conditions
Deformed Hermitian–Yang–Mills

Calculations of Leung-Y.-Zaslow, and Marino-Minasian-Moore-Strominger give the equation for BPS D-branes on the $B$-model.

**Definition**

A holomorphic line bundle $L \to (X, \omega)$ is a BPS D-brane if it admits a hermitian metric $h$ solving the deformed Hermitian–Yang–Mills equation

$$\text{Im} \left( e^{-\sqrt{-1}{\hat{\theta}}} (\omega + F(h))^n \right) = 0.$$

for $\hat{\theta}$ constant.

We are also interested in solutions of this equation on proper subvarieties $V \subset X$, and so we consider the equation on general Kähler manifolds.
Deformed Hermitian–Yang–Mills

- It is conjectured that the solvability of the dHYM equation should be equivalent to the stability of $L$ in $D^bCoh(X)$ in the sense of Bridgeland.
- Necessary and sufficient analytic conditions for solvability were given by Collins–Jacob–Y. (arXiv:1508.01934) when $\hat{\theta}$ is sufficiently large.
- Recently, Collins–Y. gave a general construction of algebraic obstructions to dHYM using ideas from infinite dimensional GIT.
Deformed Hermitian–Yang–Mills

Theorem (Collins–Y.)

Suppose $L$ admits a solution of $dHYM$ with $\hat{\theta} \in ((n - 1) \frac{\pi}{2}, n \frac{\pi}{2})$. Let $\mathcal{J}_0 \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_{r-1} \subset \mathcal{J}_r = \mathcal{O}_X$ be a sequence of ideal sheaves, and let

$$\mu : \mathcal{X} \to X \times \{|t| \leq 1\}$$

be a log resolution of $\mathcal{I} := \mathcal{J}_0 + t\mathcal{J}_1 + \cdots + t^{r-1}\mathcal{J}_{r-1} + (t^r) \subset \mathcal{O}_X \otimes \mathbb{C}[t]$. Write $\mu^{-1}\mathcal{I} = \mathcal{O}_X(E)$. Then for all $\delta > 0$ sufficiently small, we have

$$Z_E := -E \cdot e^{-\sqrt{-1}\mu^*\omega} \text{ch}(\mu^*L - \delta E), \quad Z_X = -\int_X e^{-\sqrt{-1}\omega} \text{ch}(L)$$

satisfy $\text{Im}(Z_E) \geq 0$, $\text{Im}(Z_X) > 0$, and

$$\text{Im} \left( \frac{Z_E}{Z_X} \right) \geq 0.$$

Furthermore, we have equality if and only if $\mathcal{I} = (t^k)$ for some $k$. 

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In dimension 3, using Chern number inequalities for bundles with solutions of dHYM, like
\[ \omega^3 ch_3(L) < 3(\omega.ch_2(L))(\omega^2.ch_1(L)), \]

Collins–Y. explain how these inequalities give rise to Bridgeland-type stability conditions.

In work-in-progress Collins–Y. use semi-flat mirror symmetry to construct analogous obstructions and degenerations for Lagrangian sections in Landau–Ginzburg models mirror to toric Fano manifolds.

We make conjectures concerning necessary and sufficient conditions for existence of solutions to dHYM, related to the above algebraic obstructions.
To explore our conjectures, in work-in-progress, Jacob, along with graduate student Sheu, look at manifolds with Calabi symmetry. Here the dH YM equation reduces to an ODE with boundary conditions. Our conjectured stability, along with Chern number inequalities, prevent the flow lines of this ODE from becoming multi-valued functions, demonstrating existence.

This setup also allows us to study examples of singular solutions when existence is known to be obstructed.

In future work we hope to use our infinite dimensional GIT techniques, together with techniques from nonlinear PDE and algebraic geometry, to confirm our conjectures, and to expand our understanding of stability conditions, dH YM, and special Lagrangians.
2. Weil–Petersson geometry on the space of Bridgeland stability conditions
WP geometry on Stab - WP on complex moduli

- $Y = \text{Calabi–Yau manifold of } \dim_{\mathbb{C}} = n$.
- Complex moduli space $\mathcal{M}_{\text{cpx}}(Y)$ has a canonical Kähler metric, the Weil–Petersson metric introduced by Candelas–Hübsch–Schimmrigk:

$$\frac{-i}{2} \partial_z \bar{\partial}_z \log(i^2 \int_Y \Omega_z \wedge \bar{\Omega}_z).$$

- Fan–Kanazawa–Y. (arXiv:1708.02161) constructs the mirror of Weil–Petersson geometry. The mirror of complex moduli space is the “quantum-corrected” Kähler moduli space, which should be defined via Bridgeland stability conditions on the derived category of coherent sheaves on the mirror $X$ by a conjecture of Bridgeland.

- The strategy therefore is to first define the Weil–Petersson geometry on the space of Bridgeland stability conditions.
The key observation is that there is a categorical analogue of the Weil–Petersson potential function which can be defined on the space of Bridgeland stability conditions \( \text{Stab}(\mathcal{D}) \) for any Calabi–Yau category \( \mathcal{D} \).

Let \( \{A_i\} \) be a basis of \( H^n(Y; \mathbb{Z}) \), then

\[
\int_Y \Omega \wedge \overline{\Omega} = \sum_{i,j} \gamma^{i,j} \int_{A_i} \Omega \int_{A_j} \overline{\Omega},
\]

where \( (\gamma^{i,j}) = (A_i \cdot A_j)^{-1} \).

The period integrals of the holomorphic top form conjecturally are the central charges of a Bridgeland stability condition on the derived Fukaya category of \( Y \) (c.f. Douglas, Bridgeland, Joyce, Hosono–Lian–Y.’95, Hosono’05).
WP geometry on Stab - Definition

- \( \mathcal{D} = \) Calabi–Yau category of dimension \( n \).
- Let \( \{ E_i \} \) be a basis of the numerical Grothendieck group \( \mathcal{N}(\mathcal{D}) \) (the quotient of Grothendieck group of \( \mathcal{D} \) by the Euler pairing \( \chi \)).
- Define a bilinear form \( b : \text{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}) \otimes 2 \rightarrow \mathbb{C} \) by
  
  \[ Z_1 \otimes Z_2 \mapsto b(Z_1, Z_2) := \sum_{i,j} \chi^{i,j} Z_1(E_i)Z_2(E_j), \]

  where \( (\chi^{i,j}) := (\chi(E_i, E_j))^{-1} \).

- Let \( \text{Stab}^*(\mathcal{D}) \subset \text{Stab}(\mathcal{D}) \) be the subset of stability conditions whose central charge satisfies \( b(Z, Z) = 0 \) and \( i^{-n}b(Z, \overline{Z}) > 0 \).
- We define the Weil–Petersson metric on \( \text{Aut}(\mathcal{D}) \backslash \text{Stab}^*(\mathcal{D})/\mathbb{C} \) as
  
  \[ -\frac{i}{2} \partial_{\sigma} \overline{\partial}_{\sigma} \log(i^{-n}b(Z_{\sigma}, \overline{Z}_{\sigma})), \]

  where \( \sigma \in \text{Stab}^*(\mathcal{D}) \).
Theorem (Fan–Kanazawa–Y., 2017)

- Let $E$ be an elliptic curve and $\mathcal{D} = \mathcal{D}^b\text{Coh}(E)$. Then $\text{Aut}(\mathcal{D})\setminus\text{Stab}^*(\mathcal{D})/\mathbb{C} \cong \text{PSL}(2;\mathbb{Z})\setminus\mathbb{H}$, and our Weil–Petersson metric gives the Poincaré metric on $\mathbb{H}$.

- Let $A = E_\tau \times E_\tau$ be the self-product of a generic elliptic curve and $\mathcal{D} = \mathcal{D}^b\text{Coh}(A)$. Then $\text{Aut}(\mathcal{D})\setminus\text{Stab}^*(\mathcal{D})/\mathbb{C} \cong \text{Sp}(4;\mathbb{Z})\setminus\mathcal{H}_2$, and our Weil–Petersson metric gives the Bergman metric on $\text{Sp}(4;\mathbb{Z})\setminus\mathcal{H}_2$.

Conjecture (Fan–Kanazawa–Y., 2017)

- For $\dim(X) \geq 3$, there is an embedding of the stringy Kähler moduli space $i : \mathcal{M}_{\text{Kah}}(X) \hookrightarrow \text{Aut}(\mathcal{D})\setminus\text{Stab}^*(\mathcal{D})/\mathbb{C}$.

- Moreover, the pullback of our Weil–Petersson metric gives a non-degenerate Kähler metric on $\mathcal{M}_{\text{Kah}}(X)$. 
The existence of Bridgeland stability conditions on the derived category of Calabi–Yau threefolds is not known in general. Conjectural formulae of central charges such as

$$Z(E) \sim -\int_X e^{-\omega} ch(E) \sqrt{Td_X} + \cdots$$

(or replace $\sqrt{Td_X}$ by the Gamma class $\hat{\Gamma}_X$) need to be modified by certain “quantum corrections”.

We plan to use the non-degeneracy condition of Weil–Petersson metric, as well as the Picard–Fuchs equations that period integrals should satisfy, to characterize the stringy Kähler moduli space and find the correct quantum corrections.

We plan to relate our work with classical results on curvature properties of the Weil–Petersson metric on the complex moduli spaces and on the complexified Kähler cones.
Split attractor flow

- Split attractor flow has been studied by Denef and Moore in the theory of black holes, and by Kontsevich–Soibelman in the mathematical setting of CY3 category.

- The flow starts with a class $\Gamma$ and a stability condition $\sigma$ where there exists a $\sigma$-stable object of class $\Gamma$. The central charge of $\Gamma$ decreases along the flow.

- When the flow hits a wall of marginal stability, the class splits into classes that support stable objects, and the flow continues.

Let $X \rightarrow B$ be the Lefschetz fibration such that the Fukaya–Seidel category is the quiver category $A_2$.

There exists a holomorphic map $\psi : B \rightarrow Stab(A_2)$ such that for every $u \in B$, the stable objects (up to an overall auto-equivalence) in the Fukaya–Seidel category with respect to the stability condition $\varphi(u)$ can be represented as special Lagrangians.

Understanding this correspondence, as well as the wall-crossing phenomenon, is vital to the HMS/SYZ program.
3. Noncommutative mirror functor
**Noncommutative mirror functor - SYZ**

When $X$ admits Lagrangian torus fibration, its mirror $\tilde{X}$ is obtained by taking dual torus fibration.

**SYZ**: construct a mirror space as a complexified deformation space of a Lagrangian torus.

**Difficulty**: singular fibers cause wall-crossing and scattering.

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*Strominger–Y.–Zaslow’96* When $X$ admits Lagrangian torus fibration, its mirror $\tilde{X}$ is obtained by taking dual torus fibration.

**SYZ**: construct a mirror space as a complexified deformation space of a Lagrangian torus.

**Difficulty**: singular fibers cause wall-crossing and scattering.
Seidel is the first one who found the importance of Lagrangian immersion and used it to derive HMS. Sheridan extended his method to prove HMS for the Fermat quintic.

Cho–Hong–Lau (JDG, 2017) constructed a local mirror functor for a Lagrangian immersion $\mathbb{L}$.

Quantum-corrected deformations and smoothings of $\mathbb{L}$ give a local chart

$$U_\mathbb{L} = \{b \in CF^1(\mathbb{L}, \mathbb{L}) : m_0(1) + m_1(b) + m_2(b, b) + \cdots = 0\}.$$ 

Want: glue the local charts and combine the local functors to derive HMS.
Noncommutative mirror functor - gluing construction 1

- SYZ in the past mostly focus on smooth torus fibers.
- Resulting mirrors have **missing points**.
- Key idea: **glue in charts attached to singular fibers** to complete mirrors.
- Typical singular SYZ fiber: \((\text{immersed } \mathbb{S}^2) \times T\). Hong–Kim–Lau (arXiv:1805.11738) developed a method to glue in its local chart.
- Application: partial flags. Mirrors of flag varieties constructed by Hori–Vafa have mysterious missing critical points. They do not match with closed-string mirror symmetry.
- We find back these missing points and produces a complete mirror for certain flag manifolds.
Aim: develop a local-to-global approach for mirror construction.

Main tool for gluing the charts $U_{\mathbb{L}_i}$: isomorphisms between $\mathbb{L}_i$ as objects in the Fukaya category

$$\alpha \in \text{Mor}(\mathbb{L}_1, \mathbb{L}_2), \beta \in \text{Mor}(\mathbb{L}_2, \mathbb{L}_1)$$

with $d\alpha = 0$, $d\beta = 0$, $\alpha \cdot \beta = 1_{\mathbb{L}_2}$, $\beta \cdot \alpha = 1_{\mathbb{L}_1}$.

Solving these equations give the required gluing maps.

The method works well even when the SYZ base is a singular simplicial complex rather than a smooth manifold.

Cho–Hong–Lau (arXiv:1810.02045) carried this out in dimension one. They will promote this to higher dimensions in the very near future.

Lee found another gluing method via neck-stretching by Hamiltonians. Our method is good for constructing the mirror variety.
We will also study **immersed** $S^n$ in the very near future.

They play a crucial role in compactifying mirror varieties in higher dimensions.

The Maurer–Cartan space of $\mathbb{L}$ produces a local chart that are compatibly glued with other charts.

**Immersed $S^n$ can be obtained by Seidel’s double suspension of an immersed circle.** Essentially gluing data can be obtained by our previous calculation in dimension 1.

Cho–Hong–Kim–Lau are applying this gluing method to a class of partial flag varieties to construct their complete mirrors.
Noncommutative mirror functor - appearance of NC mirror

- Cho–Hong–Lau (to appear in Memo. AMS): Local mirror charts in general are noncommutative.
- $\mathbb{L} = \bigcup_i \mathbb{L}_i$ produces a quiver $Q$.

\begin{center}
\includegraphics[width=0.5\textwidth]{quiver_diagram.png}
\end{center}

a_1, b_1 \in CF^1(\mathbb{L}_0, \mathbb{L}_1) \\
b_2 \in CF^1(\mathbb{L}_1, \mathbb{L}_0)

- (Local) mirror chart for $\mathbb{L}$ is given by the (noncommutative) quiver algebra $\mathcal{A} := \Gamma Q/R$ where $R$ consists of Maurer–Cartan relations.
- We applied to deformation quantizations [Kontsevich].
- Hong–Kim–Lau will lift the construction to local Calabi–Yau manifolds and their deformation quantizations.

Take two 3-spheres \( \mathbb{L}_0, \mathbb{L}_1 \) in the deformed conifold. They give the mirror:

\[
\mathcal{A} = \Gamma Q/\langle \partial \Phi \rangle \quad \text{with} \quad Q = v_0 \circ v_1, \quad \Phi = xyzw - wzyx.
\]

\( \mathcal{A} \): noncommutative crepant resolution of the conifold.

On \( \mathcal{A} \), Atiyah flop switches \( v_0 \) and \( v_1 \).

Corresponding action on deformed conifold switches the roles of \( \mathbb{L}_0 \) and \( \mathbb{L}_1 \).

Fan–Hong–Lau–Y. found a symplectomorphism realizing this, and interpreted the corresponding operation on the Fukaya category in terms of stability condition.
Future work: investigate relations between noncommutative geometry and wall-crossing.

SYZ fibration produce scattering diagrams which are encoding the quantum corrections.

General singular fibers produce non-commutative mirrors.

Different ways of interactions of incoming rays in the scattering diagram are expected to be simultaneously encoded in the noncommutative cloud.

Hong–Kim–Lau’s current project: understand this relation more precisely.

Gluing between noncommutative charts of singular fibers and commutative torus charts will produce a new class of geometries. The future is promising.
4. A new construction of Calabi–Yau mirror pairs: Cyclic covers and fractional complete intersections
Cyclic covers

Construction: For \( m \in \mathbb{Z}_{>0} \) and data \((Z, L, \sigma)\) with

- \( Z \): a smooth semi-Fano projective toric variety,
- \( L \): a line bundle on \( Z \), and
- \( \sigma \): a holomorphic section of \( L^m \),

we construct

- Locally: \( \text{Spec} B \to \text{Spec} A \subset Z \), where
  \[
  B = \text{normalization of } A[t]/(t^m - f_\sigma) \quad \text{with}
  \]
  \[
  t: \text{coordinate on } L \quad \text{and} \quad f_\sigma = \sigma|_{\text{Spec} A}.
  \]

- Gluing.

We get an \( m \)-fold cyclic cover \( X_\sigma \) over \( Z \).
Smooth case

Remark:

1. \( \{ \sigma = 0 \} \) is smooth \( \iff \) \( X_\sigma \) is smooth.

2. \( X_\sigma \) is Calabi–Yau \( \iff \) \( L^{m-1} \cong \omega_{Z}^{-1} \).

[Lee–Lian–Y.’18]:

- \( X_\sigma \hookrightarrow \mathbb{P}(L \oplus \mathbb{C}Z) \) and \( X_\sigma \cap Z_\infty = \emptyset \). \( Z_\infty \): infinity divisor.
  - Note: \( X_\sigma \) may be not anti-canonical in \( \mathbb{P}(L \oplus \mathbb{C}Z) \).
  - By contracting \( Z_\infty \), we get \( \mathbb{P}(L \oplus \mathbb{C}Z) \to Z' \).
  - \( Z' \) is a semi-Fano toric variety and \( X_\sigma \) is anti-canonical in \( Z' \).

For \( X_\sigma \) smooth, its mirror \( X_\sigma^\vee \) is constructed via \( Z' \).
Singular case

Question:

What if branched locus \( \{ \sigma = 0 \} \) is singular?

Example: \( Z = \mathbb{P}^2 \)

- Geometric setup:
  \[
  \mathcal{M} := \{ \text{6-hyperplane arrangement } \mathcal{A} = (\sigma_i)_{i=1}^6 \text{ in } \mathbb{P}^2 \}.
  \]
  \[
  \sigma_{\mathcal{A}} := \prod_{\sigma_i \in \mathcal{A}} \sigma_i, \ \mathcal{A} \in \mathcal{M}.
  \]
  \[
  X_{\sigma_{\mathcal{A}}} = \text{Calabi–Yau 2-fold cover over } \mathbb{P}^2 \text{ branched along } \mathcal{A}.
  \]
- Moving \( \mathcal{A} \Rightarrow \) Get a (singular) family \( \mathcal{X}' \to \mathcal{M} \).
- Blowing up \( \Rightarrow \) Get a smooth family \( \mathcal{X} \to \mathcal{M} \) of K3 surfaces.
Previous results on $\mathcal{X} \to \mathcal{M}$ [Matsumoto–Sasaki–Yoshida’92]:

1. Construct period integrals on 4-dimensional GIT quotient $GL_3(\mathbb{C}) \backslash M(3, 6)/\mathbb{C}^* \,^6$.

2. Locally period integrals satisfy the Aomoto–Gel’fand system $E(3, 6)$.

3. They compared the Baily–Borel–Satake (BBS) compactification of the period domain with the GIT quotient using modular forms on the Siegel upper half space $\mathbb{H}_2$.

4. In general, there exist LCSLs on moduli but there exist no LCSLs on the BBS compactification.
Recent progress on singular cases


**New idea:** To do mirror symmetry, construct a new toric compactification for the moduli space.

1. (Abelian gauge fixing) $GL_3$-action on $\mathbb{P}^2$ allows us to arrange 3 of 6 hyperplanes. We can assume $\sigma_4 = x$, $\sigma_5 = y$ and $\sigma_6 = z$. Then $GL_3 \times (\mathbb{C}^*)^6$-action reduces to abelian $T = (\mathbb{C}^*)^5$-action.
2. Fix a reference fiber $X_{\sigma A_0}$. On $\{z = 1\}$, the periods become

   \[ \Pi_\gamma(A) := \int_{\gamma} \frac{1}{\sigma_1^{1/2} \sigma_2^{1/2} \sigma_3^{1/2}} \frac{dx \wedge dy}{xy}, \quad \gamma \in H_2(X_{\sigma A_0}, \mathbb{Z}) \]

   and parameterized by $V \times V \times V$, $V := H^0(\mathbb{P}^2, O(1))^\vee$.
3. $(\mathbb{C}^*)^2 \subset \mathbb{P}^2$ gives natural integral structures on $V$.
4. Instead of integers, the exponents of $\sigma_i$’s are fractional.
Recent progress on singular cases (conti.)

- Periods can be studied by GKZ (Gel’fand, Kapranov and Zelevinski) hypergeometric systems.
- For hypersurfaces or complete intersections, periods satisfy a GKZ system with an integral exponent.

**Upshot:**

(A) Periods $\Pi_\gamma(A)$ satisfy a GKZ $A$-system with exponent $\beta$:

$$
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0
\end{bmatrix},
\quad
\beta = \begin{bmatrix}
-1/2 \\
-1/2 \\
-1/2 \\
0 \\
0
\end{bmatrix}.
$$

(B) Cyclic covers have a **fractional complete intersection (FCI)** structure.
Recent progress on singular cases (conti.)

Results: From the integral structure $A$, one gets

- The secondary fan (toric) compactification of the moduli space.
- Exactly 6 LCSLs; all are isomorphic under an $S_6$-action. These LCSLs only appear in the new compactification.
- A cohomological-valued $B$-series giving explicit power series for period integrals.
- The mirror family is given by 2-fold covers over $Bl_{\{p_1,p_2,p_3\}}\mathbb{P}^2$, where $p_i$ are $(\mathbb{C}^*)^2$ fixed points.
- The mirror map $\mu$ generalizes elliptic $\lambda$-function for 2-fold covers over $\mathbb{P}^1$ (the Legendre family: $y^2 = x(x - 1)(x - \lambda)$).
Recent progress on singular cases (conti.)

Let $\overline{M}$ be our toric compactification.

**Conjecture (modular identities):** The mirror map $\mu = (\lambda_1, \ldots, \lambda_4)$ on a chart of $\overline{M}$ for the 2-fold cover family over $\mathbb{P}^2$ is given by

$$
\begin{align*}
\lambda_1 &= \frac{\Theta_3^2 + \Theta_9^2 - \omega_0^2}{\omega_0^2 - \Theta_7^2}, \\
\lambda_2 &= \frac{\Theta_3^2 + \Theta_9^2 - \omega_0^2}{\omega_0^2 - \Theta_9^2}, \\
\lambda_3 &= \frac{(\omega_0^2 - \Theta_7^2)(\omega_0^2 - \Theta_9^2)}{\omega_0^2(\Theta_4^2 + \Theta_9^2 - \omega_0^2)}, \\
\lambda_4 &= \frac{\Theta_4^2 + \Theta_9^2 - \omega_0^2}{\Theta_4^2 + \Theta_9^2 - \omega_0^2}.
\end{align*}
$$

- $\overline{M}$ is a blow up of the BBS compactification.
- $\Theta_i$ are weight two modular forms (w.r.t. certain arithmetic group $\Gamma$) on $\mathbb{H}_2$ corresponding to 10 different even spin structures,
- $\omega_0$ is the (normalized) holomorphic period at LCSL.

Numerical check by computers to high degree.
Fractional complete intersections (FCI)

Key points:

- ‘Abelian gauge fixing’ replaces $\text{GL}_3(\mathbb{C}) \times (\mathbb{C}^*)^6$ by $T \simeq (\mathbb{C}^*)^5$ gives ‘integral structure’.
- ‘Integral structure’ gives rise to ‘FCI structure’ for periods, and toric compactification for moduli space.

‘Calabi–Yau arising from cyclic covers’ are interpreted as ‘FCI’s.

This gives a new class of examples for mirror symmetry.
Generalizations

Result (To appear in [Hosono–Lee–Lian–Y.–Zhang’18]):
The mirror of a cyclic cover Calabi–Yau over $Z$ is given by a cyclic cover over $Z^\vee$, where $Z^\vee$ is the Batyrev–Borisov mirror for a suitable nef-partition in $Z$.

Ideas:

- Construct toric compactifications by abelian gauge fixings.
- Study GKZ hypergeometric systems with fractional exponents.
- Find the LCSLs on the toric compactification.
- Construct the cohomological-valued $B$-series and the mirror map explicitly.
On-going work

On-going work for general bases $Z$ (e.g. $Z = G/P$).

- Toric compactifications by abelian gauge fixings.
- General existence of LCSLs.
- Cohomological-valued $B$-series for periods.
- Compute central charges for Bridgeland–Douglas stability conditions (BDSC), and study the BDSC moduli space.
- Mirror maps and relations to modular forms ($\dim Z = 2$).
- New examples to test the HMS conjecture.
5. Progress on tautological systems – B model geometry
Tautological systems

Geometric situation.

- \( \pi : Y \to B \): \( d \)-dimensional compact complex manifolds.
- \( Y_b := \pi^{-1}(b), \ b \in B \).
- \( \omega \in \Gamma(B, R^0 \pi_* \Omega^d_{Y/B}) \).

**Definition.** The period sheaf \( \Pi(\omega) \) is the sheaf generated by

\[
\int_{\gamma} \omega.
\]

Here \( \gamma \in H_d(Y_{b_0}, \mathbb{Z}) \) for some fixed \( b_0 \).

- \( \Pi(\omega) \) determines variation of Hodge structure on B-model.
Tautological systems

- To describe $\Pi(\omega)$, we must solve the explicit Riemann-Hilbert problem: to find an explicit complete D-module for $\Pi(\omega)$.
- i.e. construct an explicit system $\tau$ of linear partial differential equations such that

$$\text{Sol}(\tau) \supseteq \Pi(\omega).$$

In the case “=” holds, we say that $\tau$ is complete.

Example:
- GKZ systems on toric varieties.
- Extended GKZ systems on toric varieties.

These systems in general are not complete.
Tautological systems

Geometric setup

- $G$ and $H$: complex Lie groups.
- $M$: smooth $(G \times H)$-manifold such that $X := M/H$ is a projective $G$-manifold.
- $L$: $G$-equivariant base point free line bundle on $X).

- Introduce the tautological system $\tau$.
- Derive a global Poincaré–Leray residue formula on $X$.
- $\text{Sol}(\tau) \supset \Pi(\omega)$ for periods of hyperplane sections of $L$.

Remark:
When $X$ is toric, $G = \text{maximal torus}$, and $L = -K_X$, $\tau = \text{GKZ}$. 
Completeness of $\tau$

Question:

What’s the solution to $\tau$? How to describe them?


- When $X = \mathbb{P}^n$, $G = \text{SL}_{n+1}$, and $L = -K_X$, $\tau$ is complete.

Conjecture: (Holonomic rank conjecture [BHLSY’14])

- $G$: semi-simple Lie group.
- $X$: $n$-dimensional projective homogeneous $G$-manifold.
- $L = -K_X$.
- $f \in H^0(X, L)$ and $U_f := X \setminus \{f = 0\}$.

$$\dim \mathbb{C} \text{Sol}(\tau)_f = \dim \mathbb{C} H^n_{dR}(U_f).$$

[Huang–Lian–Zhu’16 (arXiv:1303.2560)] solved the conjecture by showing

$$\text{Sol}(\tau)_f \cong H^n_{dR}(U_f).$$
Completeness of $\tau$ (conti.)


- $X$: smooth projective Fano toric variety.
- $G$: maximal torus.
- $L = -K_X$.
- $D_\infty$: union of all toric divisors.
- $f \in H^0(X, L)$ and $U_f = X \setminus \{f = 0\}$.

\[
\text{Sol}(\tau)_f \cong \text{Sol}(GKZ)_f \cong H_{dR}^n(U_f, U_f \cap D_\infty).
\]

**Ingredients** for [HLZ’16, HLYZ’16]: Riemann–Hilbert (RH) correspondence.


**Ingredients**: RH and Dwork cohomology.
Applications to B-model: Hyperplane conjecture

[Hosono–Lian–Y.’96] For smooth projective Fano toric variety $X$

- Existence of LCSLs on the compactified moduli (via 2nd fan) of the CY family in $X$.
- An explicit construction of the unique power series solution $B(\vec{a})$ near LCSLs. $\vec{a} =$ coordinates on moduli.

**Conjecture ([HLY'96]):**
Near LCSLs, there is a cohomological-valued $B$-series $B(\vec{a})$ s.t.

$$\{\text{period integrals}\} = \{\text{coefficients of } B(\vec{a}) \cup [-K_X]\}.$$

[Lian–M. Zhu’16 (arXiv:1610.07125)] proved the hyperplane conjecture for $\mathbb{P}^n$.

**Ingredients:**
- Extended GKZ(= tautological system) is complete.
Applications to arithmetic geometry: mod $p$ aspects of periods

Geometric setup:

- $X$: Fano toric variety or $G/P$ of dim $n$ over $\mathbb{Z}$.
- $\pi : \mathcal{Y} \to B$: universal family of CY/general type hypersurfaces.
- $\text{Frob}_p$: Frobenius action on $R^{n-1} \pi_*(\mathcal{O}_{\mathcal{Y}}) \pmod{p}$.
- $\text{HW}_p$: Hasse–Witt matrix: the matrix of this Frobenius action under a suitable choice of basis. (It is a number in the case of CY hypersurfaces)

[Huang–Lian–Y.–Yu’18 (arXiv:1801.01189)]: An appropriate limit of a rescalling of $\text{HW}_p$ as $p \to \infty$ recovers the unique holomorphic period of the family over the complex numbers, at the rank 1 point (LCSL candidate) $s_\infty$. Conversely, $\text{HW}_p$ is equal to an appropriate truncation of Taylor series expansion of the complex period at $s_\infty \pmod{p}$. 
Remark:

1. When $X$ is toric, this explains part of the work by Candelas, de la Ossa, Rodriguez–Villegas in 2000, regarding Calabi–Yau varieties over finite fields.

2. [Huang–Lian–Y.–Yu’18] There is a $p$-adic extension of this result, that proves and extends a recent conjecture of M. Vlasenko regarding higher Hasse–Witt matrices – an explicit formula for the Frobenius action on the unit root part of the crystalline cohomology of the hypersurface, which may also be expressed in terms of certain truncations of the holomorphic period at the LCSL point.
6. SYZ fibrations on log Calabi–Yau
Ricci-Flat Metrics on Log Calabi–Yau Manifolds

- $Y$ = projective manifold of dimension $n$ with

- $D = s^{-1}(0)$ irreducible effective anticanonical divisor, where $s \in H^0(Y, -K_Y)$ and no curves disjoint from $D$ can be realized as linear combination of curves support in $D$.

Then $Y \setminus D$ admits a non-vanishing holomorphic volume form $\Omega$

**Theorem (Tian–Y. ’90)**

*There exists a complete Ricci-flat metric $\omega_{Ric}$ on $X = Y \setminus D$ with asymptotics near $D*

$$\omega_{Ric} \sim \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial}(- \log \| s \|)^{(n+1)/n}.$$

Question: Is there a SYZ fibration on $X$?
Lagrangian Mean Curvature Flow

- Let $L$ be a graded Lagrangian submanifold in $X$. Then the phase $\theta : L \to \mathbb{R}$ is the function such that

$$\Omega|_L = e^{i\theta} \text{vol}_L.$$

$L$ is a special Lagrangian if $\theta$ is a constant.

- The mean curvature $\vec{H} = J \nabla \theta$ and the mean curvature flow is given by evolving family of immersions $F_t : L \to X$ with

$$\frac{\partial}{\partial t} F_t = \vec{H}.$$

- (Smoczyk) Maslov zero Lagrangian condition is preserved under mean curvature flow in Kähler–Einstein manifolds.
Lagrangian Mean Curvature Flow continued

- Lagrangian mean curvature flow also preserves the Hamiltonian isotopy class.
- (Thomas–Y. ’01) There exists at most one special Lagrangians in a given Hamiltonian isotopy class. Thomas–Y. proposed to use mean curvature flow to find the unique special Lagrangian representative analogue of stable bundles in the B-side.
- (Neves ’07) Example of Lagrangian mean curvature flow can develop finite time singularities in dimension four.
- (Joyce ’14) Proposed to use Lagrangian mean curvature flow to study the Bridgeland stability conditions on Fukaya categories.
Theorem (Collins–Jacob–Lin)

For $Y$ rational elliptic surface with $D$ be a smooth fibre, there exists a special Lagrangian fibration on $Y \setminus D$.

- This gives an affirmative answer to a conjecture of Auroux ’07.
- The key ingredient is to use mean curvature flow to flow the ansatz fibration to an SYZ fibration.
- The higher dimension case remains open since not much is known about the behavior of the Ricci-flat metrics.
- We will further investigate the more general log Calabi–Yau surfaces and the implication in enumerative geometry/mirror symmetry.
7. Descendant Gromov–Witten invariants and oscillatory integrals
Towards a quantum corrected SYZ mirror via family Floer theory

- **Fukaya** proved that family Floer homology of the Lagrangian torus fiber $L_u$ of the SYZ fibration varies analytically when $u$ varies in the base.

- **Abouzaid, Fukaya** and **Tu** separately provided methods of gluing *Maurer–Cartan spaces* associated to different Lagrangian torus fibers to construct the mirror.

- Transition maps between local charts of the mirror are given by the *wall-crossing maps* contributed by holomorphic disks.

- The enumerative information of holomorphic disks plays a crucial role in obtaining the quantum corrections of the mirror.
Lin used family Floer homology to define the open Gromov–Witten invariants \( \tilde{\Omega}(\gamma; u) \) which count holomorphic disks in class \( \gamma \in H_2(X, L_u) \), and proved that it satisfies the Kontsevich–Soibelman wall-crossing formula.

Lin proved a correspondence theorem on K3 surfaces, which is analogous to Mikhalkin and Nishinou–Siebert’s results on toric surfaces:

\[
\tilde{\Omega}(\gamma; u) = \tilde{\Omega}^{\text{trop}}(\gamma; u),
\]

where \( \tilde{\Omega}^{\text{trop}}(\gamma; u) \) is the weighted count of tropical disks which are adiabatic limits of the projections of holomorphic disks in the complex affine base.

Lin also generalized this when the singular fibers of the SYZ fibration are of type I\(_n\), II, III, IV other than I\(_1\) nodal fibers.
Hong–Lin–Zhao studied the bulk-deformed potentials of toric Fano surfaces $X$, which is a formal deformation of the mirror Landau–Ginzburg superpotential considered by Hori–Vafa.

Unlike the Hori–Vafa potential, the bulk-deformed potential $W^b$ defines a discontinuous function over the SYZ base. The loci where $W^b$ becomes discontinuous are called the walls.

The expressions of $W^b$ in chambers separated by the walls are related by the wall-crossing transformations (i.e. certain birational maps between complex tori).

The order-by-order formula of $W^b$ can be obtained by generalizing Fukaya’s work on pseudoisotopies in family Floer theory in the bulk-deformed case.
A tropical holomorphic correspondence

- **Hong–Lin–Zhao** established a correspondence between open Gromov–Witten invariants with point constraints and that of weighted count of tropical disks.
- Unlike the tropical predictions, the walls of $W^b$ are *not* straight lines in complex affine base of the SYZ fibration.
- **Hong–Lin–Zhao** proved that after passing to the tropical limit defined by Mikhalkin, the holomorphic walls deformation retract to the tropical one. So the combinatorics of the scattering diagrams encoding the wall structures can still be applied.
- This recovers the tropical computations by Gross for $\mathbb{P}^2$. 
Descendant GW invariants and oscillatory integrals

- Fukaya–Oh–Ohta–Ono proved the closed-string isomorphism between the big quantum cohomology and Jacobian ring of the bulk-deformed potential for toric varieties

\[ (QH^*(X), \ast_b) \cong Jac(W^b), \quad b \in H^*(X), \]

as Frobenius algebras. It is natural to study the flat coordinates associated to the Frobenius manifold structures.

- Following ideas of Tonkonog, Hong–Lin–Zhao proved that:

**Theorem (Hong–Lin–Zhao)**

**There is a relationship between oscillatory integrals of \( W^b \) and descendant Gromov-Witten invariants**

\[
\frac{1}{(2\pi i)^2} \int_{T^2} e^{W^b / \hbar} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} = 1 + \sum_{m \geq 2} \sum_{k \geq 0} \frac{1}{k!} \langle \psi^{m-2}(PD[pt]), b, \ldots, b \rangle \hbar^{-m}. 
\]
The relationship with GKZ systems for toric varieties: proposed future research

- This oscillatory integral is monodromy invariant and gives the first flat coordinate on both A and B sides by the theorem of Hong–Lin–Zhao.
- Non-monodromy invariant periods are expected to be related to open GW invariants with descendants with boundary on certain Lagrangian branes in $X$.
- Hong–Lin–Zhao aim to generalize the previous theorem for all toric varieties to match descendant open GW invariants with solutions to the GKZ-type systems, which are of the form

$$\int_{\Gamma} e^{\frac{W_q}{\hbar}} \Omega_q.$$

Here $\Gamma$ is a Lefschetz thimble of $W_q$ and $\Omega_q$ is the relative holomorphic volume form for the family of LG potentials $W_q$ as $q$ varies the (complexified) Kähler moduli space.
8. Higher-genus Gromov–Witten invariants and BCOV
Higher-genus GW invariants and BCOV

[Bershadsky–Cecotti–Ooguri–Vafa’93,’94]

Let $X$, $\check{X}$ be a mirror pair of Calabi–Yau 3-folds.

- There is a B-model “topological string amplitude”
  \[
  \mathcal{F}_g^B(z, \bar{z}) \in \Gamma_{\mathcal{C}}(\mathcal{M}, \mathcal{L}^{(2-2g)})
  \]
  satisfying the holomorphic anomaly equations (HAE), where $\mathcal{L} =$ Hodge bundle on the complex moduli $\mathcal{M}$ of $\check{X}$.

- Via the mirror map, the “holomorphic limit” of the A-model counterpart of $\mathcal{F}_g^B(z, \bar{z})$ is the generating function
  \[
  F_g(Q) := \sum_{d \geq 0} N_{g,d} Q^d
  \]
  of the Gromov–Witten invariants of $X$ of genus $g$. 
Higher-genus GW invariants and BCOV

- A-model predictions on Gromov–Witten invariants from BCOV HAE:
  - (1) The Feynman rule:
    \[ F_g = \text{Feynman sum involving } \{ F_{g'} \}_{g' < g} \]
    + finite amount of initial data (holomorphic ambiguity).
  - (2) Polynomiality [Yamaguchi–Y.’04] (for the quintic 3-fold):
    For all \( 2g - 2 + n > 0 \), we have
    \[
    \frac{(5 - 5X)^{g-1}}{\omega_0^{2g-2}} D^n F_g \left( q e^{\omega_1/\omega_0} \right) \in \mathbb{Q}[A_1, B_1, B_2, B_3, X]
    \]
    where \( D = q \frac{d}{dq} \) and \( A_1, B_1, B_2, B_3, X, \omega_0, \omega_1 \in \mathbb{Q}[[q]] \) are explicit power series deduced from the B-model special Kähler geometry.

- These A-model predictions have recently been verified for the quintic 3-fold by Chang–Guo–Li–Li using mixed spin p-fields.
We plan on generalizing this to other mirror manifolds, with higher dimensional moduli, e.g. complete intersections in product of weighted projective spaces.

We plan on understanding the global mirror symmetry: the B-model $\mathcal{F}_g^B(z, \bar{z})$ has asymptotic behavior at various points; and the A-model proof of CGLL relates various A-model theories.

We plan on establishing the mathematical formulation of higher-genus mirror symmetry, e.g. finding the objects on the A-side corresponding to the anti-holomorphic part of $\mathcal{F}_g^B(z, \bar{z})$, and fixing the holomorphic ambiguity.
9. Dualities between DT theories
Dualities between DT theories

DT invariants of $X$ are defined by integrating 1 against the DT virtual fundamental class:

$$ DT^X_{n, \beta} := \int_{[\text{Hilb}^n_\beta(X); \mathbb{Z}]^\text{vir}} 1 \in \mathbb{Z}. $$

Behrend proved that the DT invariant $DT^X_{n, \beta}$ is a weighted Euler characteristic:

$$ DT^X_{n, \beta}(X) = e(\text{Hilb}^n_\beta(X), \nu) = \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(k)) $$

where $\nu : \text{Hilb}^n_\beta(X) \to \mathbb{Z}$ is a constructible function, known as the Behrend function, and $e$ is the topological Euler characteristic. The RHS of the above equation is defined even when $X$ is a non-compact Calabi–Yau 3-fold.
The DT/SW correspondence was first conjectured using Gauge theory reduction approach, by Gukov–Liu–Sheshmani–Y. (GLSY). It relates the DT theory of sheaves with 2 dimensional support in a non-compact local surface threefold $X : L \to S, L \in Pic(S)$ over a smooth projective surface $S$, to the Seiberg–Witten (SW) invariants of the surface $S$ and invariants of “nested Hilbert scheme” on $S$.

- In a joint work Gholampour–Sheshmani–Y., we proved GLSY conjecture.
- We proved that nested Hilbert scheme invariants can specialize to Poincaré invariants of Dürr–Kabanov–Okonek and the stable pair invariants of Kool–Thomas.
We showed modular property of invariants of nested Hilbert scheme of points.

We showed that when $L \cong K_S$ the local DT invariants equal Vafa–Witten invariants (defined and studied mathematically by Tanaka–Thomas).

Our proof relates the Vafa–Witten invariants to SW invariants of surface + correction terms governed by invariants of nested Hilbert schemes.

Vafa–Witten invariants exhibit modularity property, hence the proof showed how to compute SW invariants in terms of modular forms in some cases.
Dualities between DT theories - DT/SW duality on 4-folds

There are several proposals (Cao–Leung, Joyce–Borisov) to construct DT invariants for Calabi–Yau 4 folds, despite their moduli spaces not having well-defined deformation-obstruction theories.

In a joint work Diaconescu–Sheshmani–Y., we studied DT theory of sheaves with 2-dimensional support on a smooth projective surface $S$ in a non-compact local surface Calabi–Yau 4 fold $X : \mathcal{O}_S(-D) \oplus K_S(D) \to S$ where $K_S$ is nef canonical bundle of $S$ and $D \subset S$ is an effective divisor.

- We proved using derived category techniques that universal truncated Atiyah class of codimension 2 sheaves in $X$ equals the universal truncated Atiyah class of such sheaves as codimension 1 sheaves in a 3 fold $Y := K_S(D) \to S$.
- We showed that DT invariants of $X$ reduce to DT invariants of $Y$ in this case.
- We showed modular property of invariants of such torsion sheaf invariants when $S$ is an elliptic K3 surface and $D$ is multiple of elliptic fiber class.
Dualities between DT theories - Future research

- Diaconescu–Sheshmani–Y. study DT theory of K3-fibered 4-folds. Here the sheaves have set-theoretic support on K3-fibers and it is expected that their partition function exhibits interesting modular behavior. In particular a new notion of Noether–Lefschetz numbers for K3-fibered 4-folds is expected to arise with modularity property.

- We study mathematical foundations for Kapustin–Witten theory of surfaces. These can be captured as invariants of torsion sheaves with support on a smooth projective surface $S$, embedded in an ambient noncompact 4-fold given as $X := T^*S \to S$. In order to compute the invariants DSY aim at constructing a degeneration procedure to relate invariants of $X$ to invariants of $X' := L_1 \oplus L_2 \to S$, where $L_2 < 0$ as above.

- The foundations of degeneration technique have not been studied for torsion sheaves on 4-folds in this case, and technology of derived algebraic geometry is being used for such construction.
Thank you!!