Toward Local-to-Global Proofs of Mirror Symmetry

Simons Collaboration Annual Meeting, 2018

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November 16, 2018

Maximal degeneration of CY manifolds remains the central motivating problem in Homological Mirror symmetry. All the tools are now in place to prove the conjecture in the subclass of *toroidal degenerations* studied by Gross-Siebert, and the plan is to provide a proof that can be applied to general degenerations as long as certain local properties hold. The general strategy can be technically implemented from two different perspectives:

Lagrangians fibres	Points of the mirror
Lagrangian sections	Line bundles on the mirror

I expect both approaches to be important for generalisations and applications, as the most well-adapted approach will depend on the situation.

Ambient toroidal mirror symmetry

The most well-studied class of mirror pairs are those arising for CY complete intersections in toric varieties (Batyrev-Borisov), by degenerating a generic such hypersurface to a singular hypersurface lying in the toric boundary.

In this setting, the most general HMS result we have is due to Sheridan-Smith, generalising earlier work of Sheridan on Fermat hypersurfaces in projective spaces. The key idea is to use the branched cover

$$\{\sum_{i=0}^{n} x_i^{n+1} = 0\} \mapsto \{\sum_{i=0}^{n} x_i = 0\}$$

which allows one to study the Fukaya category by pullback. This strategy uses global information in an essential way.

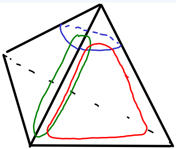
Abstract toroidal mirror symmetry

Gross and Siebert abstracted the construction of degenerations of complete intersection in toric varieties to a notion of *toroidal degeneration.* This is a family over the disc

$$\mathcal{X} \to D$$

with central fibre a union of toric varieties, so that the map is locally given by a toric monomial. We work with local coordinates $\mathbb{C}_x^{m+1} \times (\mathbb{C}_y^*)^n$ so that the fibre X over $\lambda \in D$ is given by

$$\prod_{i=0}^{m} x_i^{d_i} = \lambda \cdot f(y).$$



$x_1 x_2 x_3 = \lambda$

$x_1 x_2 = \lambda f(y)$

$x = \lambda f(y_1, y_2)$

$\deg f = 4$

$24 = 4 \times 6$

HMS for local models

In order for the mirror family to have normal crossings, Gross and Siebert specialise even further to the case

$$f(y) = 1 + \sum_{j=1}^{n} y_j^{e_j}.$$

The exponents correspond geometrically to taking covers, so we may set them equal to 1. Let $X_{m,n} \subset \mathbb{C}_x^{m+1} \times (\mathbb{C}_y^*)^n$ be the hypersurface

$$\prod_{i=0}^{m} x_i = 1 + \sum_{j=1}^{n} y_j.$$

Theorem (A-Sylvan, in preparation)

The hypersurfaces $X_{m,n}$ and $X_{n,m}$ are mirror.

I will explain two different formulations: one in terms of points, the other in terms of functions.

A cover by tori away from codimension 2

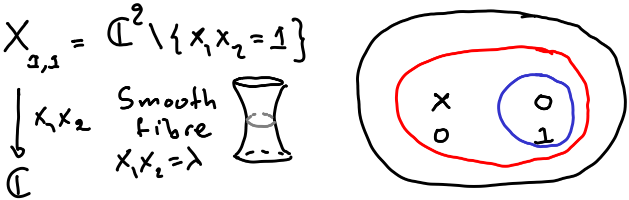
Consider the affine subset

$$X^j_{n,m} = X_{n,m} \setminus \bigcup_{k \neq j} \{x_k = 0\}.$$

Since the coordinates $\{x_k\}_{k \neq j}$ are invertible on $X^j_{n,m}$, this is $(\mathbb{C}^*)^{m+n}$.

Lemma

There is a singular Lagrangian torus fibration $X_{m,n} \to Q$ whose smooth fibres are mirror to $\bigcup_{j=0}^{m} X^j_{n,m}$.



Mirror theorem away from codimension 2

Going back to the study of degenerations, we see that the charts $X^j_{n,m}$ correspond to adjacencies between (toric) local models. Removing a neighbourhood of a codimension 2 subset yields $\mathring{X} \to \mathring{Q}$ which is a (smooth) Lagrangian torus fibration.
Considering rank-1 local systems on these fibres as objects of the Fukaya category, we obtain an analytic space $\mathring{Y}$. Given any Lagrangian in X, taking Floer homology with these fibres yields a coherent sheaf $\mathring{Y}$. If we could extend to a (non-singular) Lagrangian fibration on X, we could appeal to:

Theorem (A.'17)

If $X \to Q$ is a Lagrangian torus fibration, the Fukaya category of X embeds fully faithfully in a (twisted) category of sheaves on Y.

We know that we cannot apply this directly, because the family of Lagrangians acquires singularities.

The Family Floer functor

The essential idea behind the Family Floer functor is to construct a "big brane" on each Lagrangian fibre, which corresponds to the structure sheaf of a piece of the mirror. When the fibres are tori, this is the local system associated to the group ring:

$$\mathbb{C}[\pi_1 T^n] \cong \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}].$$

This works well to recover mirror symmetry between $T^* T^n$ and $(\mathbb{C}^*)^n$, but we have to complete both sides when considering domains. Since $X_{1,1}$ is self-mirror, the Floer theory of the symplectic side should be controlled by the ring

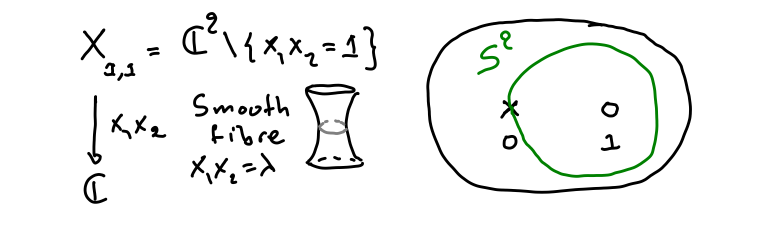
$$\mathbb{C}[x, y, \frac{1}{xy - 1}].$$

Theorem (Dimitroglou–Rizell, Ekholm, and Tonkonog, ++)

There is a brane on the Lagrangian skeleton of $X_{1,1}$, with self-Floer homology as above.

The skeleton of $X_{1,1}$

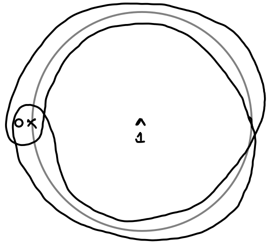
This skeleton is the singular fibre of a Lagrangian fibration $X_{1,1} \to \mathbb{R}^2$.



At this stage, a proof of HMS in dimension 2, for a space that has a (singular) Lagrangian torus fibration, follows from applying the same technique as in the non-singular case (injectivity on morphisms is straightforward, surjectivity requires some additional computations of Floer groups of "big branes".)

An immersed Lagrangian in $X_{m,n}$

In higher dimensions, this construction yields a very singular Lagrangian. Consider a nearby immersed circle that misses the singularity, and let $L_{m,n}$ be the transport of a "skeleton" of the fibre:



Theorem (in preparation)

There is a brane supported on $L_{m,n}$ whose Floer theory recovers the ring of regular functions on $X_{n,m}$ (i.e. the space of (simple) objects supported on $L_{m,n}$ is $X_{n,m}$.)

Motivation

If X is the smooth fibre of a toric degeneration, then the interesting part of the topology of X comes from the singularities of the 0-fibre. The (interior) of top-dimensional strata is just a copy of $(\mathbb{C}^*)^n$.

If we consider instead a general degeneration, we should expect that the interior of top-dimensional strata to contribute to the Floer theory. We need a more flexible approach to handle this situation.

The basic idea is that we want to describe our global Fukaya category as being built from restriction maps to domains.

Theorem (A-Seidel '08)

An inclusion $X' \subset X$ of Liouville domains induces a functor on Fukaya categories.

In the last few years, work of Lee, Groman, Venkatesh, and Varolgunes has made it possible to consider such constructions away from the exact setting, by defining *local Floer cohomology groups* associated to subsets of arbitrary symplectic manifolds.

Varolgunes descent

The idea of local Floer cohomology is to associate to each subset $U \subset X$ a chain complex $CF_U^*(X)$, constructed from Hamiltonian dynamics in U, and holomorphic curves in X.
Given an open cover $\mathcal{U}$ of a symplectic manifold X, Varolgunes constructed a map

$$HF^*(X) \to \check{H}^*(\mathcal{U}; CF^*)$$

for Hamiltonian Floer homology groups. He proved:

Theorem (Varolgunes '18)

If $X \to B$ is a map with coisotropic fibres, and $\mathcal{U}$ is induced from a cover of B, then the above map is an isomorphism.

A similar result should hold for the Floer homology of Lagrangian submanifolds in X.

A conjectural criterion for local-to-global equivalence

The construction local Floer homology should give rise to a category $\mathcal{F}_U(X)$ for each subset U of a symplectic manifold X, with objects Lagrangians in X, and morphisms constructed from dynamics in U. There should again be a local-to-global functor

$$\mathcal{F}(X) \to \check{C}^*(\mathcal{U}, \mathcal{F}). \tag{1}$$

Conjecture (A-Groman-Varolgunes)

If $\mathcal{U}$ is induced from a map with coisotropic fibres, Equation (1) is an equivalence.

We expect the full package of Floer theory to apply in this context. In particular, we should have a open-closed map

$$HH_* \mathcal{F}_U(X) \to HF^*_U(X), \tag{2}$$

and a local version of the generation criterion should hold.

Application to degenerations

Associated to each normal crossing degeneration of a symplectic manifold X is a simplicial complex B with a cell of dimension k for each codimension k stratum of the singular fibre.
There is a map $X \to B$ with coisotropic fibres, which is canonical up to contractible choice. The decomposition of B into (neighbourhoods) of cells gives rise to a cover of X by domains which are essentially (fibre) products of conic bundles over the manifolds associated to the (interior) of the strata.

Problem

Compute the local-to-global functor associated to maximal degenerations.

The above approach reduces the problem to local computations. In the case of toroidal degenerations, we can now compute all local models, before deformation.

Tools for computation

The strategy outlined above gives a very abstract "mirror theorem" where the mirror is a gluing of non-commutative spaces. We need to understand when they are commutative:

Theorem (Fukaya-Oh-Ohta-Ono '09)

If L is fixed by an anti-symplectic involution, then the Floer homology algebra of L is commutative.

I expect that the same result holds for local Floer cohomology with respect to invariant subsets. As long as we can identify a cover of B which is mirror to an affine cover, we can prove the desired result.

Warning

Because mirrors of Calabi-Yau manifolds may be non-commutative, we should expect obstructions to produce such a cover. One of the goals of this approach is thus to understand, from the symplectic point of view, local obstructions to commutativity of the mirror.