30 years of Floer homology  (of Lagragian submanifold)

Kenji Fukaya

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Ancient days  (300 years ago)

Hamilton formalism

\[ \gamma : S^1 \rightarrow T^* M \quad \mathcal{H}(\gamma) = \int_{S^1} H(\gamma(t))dt + \int_{S^1} \gamma^*(pdq) \]

Lagrange formalism

\[ \gamma : S^2 \rightarrow M \quad \mathcal{L}(\gamma) = \int_{S^1} L \left( \gamma, \frac{d\gamma}{dt} \right) + \frac{1}{2} \int_{S^1} \left( \frac{d\gamma}{dt} \right)^2 dt \]
Hamilton formalism \[ \mathcal{H}(\gamma) = \int_{S^1} H(\gamma(t)) dt + \int_{S^1} \gamma^*(pdq) \]

The Morse index of \( \mathcal{H} \) is \( \infty / 2 \)

Critical point of \( \mathcal{H} \) is detected by Floer homology

Differential equation \( \delta \mathcal{H} = 0 \) is 1st order.

\( p \) and \( q \) are symmetric. (super symmetry)

Lagrange formalism \[ \mathcal{L}(\gamma) = \int_{S^1} L\left(\gamma, \frac{d\gamma}{dt}\right) + \frac{1}{2} \int_{S^1} \left(\frac{d\gamma}{dt}\right)^2 dt \]

The Morse index of \( \mathcal{L} \) is finite.

Critical point of \( \mathcal{L} \) is detected by ordinary homology

Differential equation \( \delta \mathcal{L} = 0 \) is 2nd order.
Floer 1980’ (30 years ago)

∞/2 dimensional homology theories Floer homology via Morse theory.

Various version.

Symplectic Geometry
periodic Hamiltonian system
Lagrangian submanifolds

Contact homology
Embedded homology

Gauge theory
Instanton Floer homology (Yang-Mills)
Instanton Floer homology (Seiberg-Witten)
Heegard Floer homology
periodic Hamiltonian system (the simplest)

\[ H : S^1 \times X \rightarrow \mathbb{R} \quad X \text{ symplectic (compact)} \]

\[ HF(X, H) \]

(1) \( HF(X, H) \) detects periodic orbit of the periodic Hamiltonian system associated to \( H \).

(2) \( HF(X, H) \) is independent of \( H \).

(3) \( HF(X, H) \cong H(X) \) isomorphic to ordinary homology.

(4) \( HF(X, H) \) has ring structure isomorphic to quantum homolog.
Lagrangian Floer homology

\[ X \text{ symplectic (compact)} \quad L_1, L_2 \subset X \quad \text{Lagrangian submanifold.} \]

\[ HF(L_1, L_2) \]

(1) \( HF(L_1, L_2) \) detects \( \# L_1 \cap L_2 \)

(2) \( HF(L_1, L_2) \) is independent by Hamiltonian perturbation of \( L_i \).

(3) \( HF(L, L) \) may or may not be isomorphic to \( H(L) \)

(4) \( HF(L, L) \) has ring structure related to one of \( H(L) \).
Lagrangian Floer homology

Floer himself did the exact case \( \int_{D^2} u^* \omega = 0 \)

\[ u : (D^2, \partial D^2) \to (X, L) \]

1. \( HF(L_1, L_2) \) detects \#L_1 \cap L_2

2. \( HF(L_1, L_2) \) is independent by Hamiltonian perturbation of \( L_i \).

3. \( HF(L, L) \) is isomorphic to \( H(L) \)
Lagrangian Floer homology

Oh generalize it to the monotone case \( W(L) \in \mathbb{Q} \)

1. \( HF(L_1, L_2) \) defined if \( W(L_1) = W(L_2) \)
2. \( HF(L_1, L_2) \) detects \( \# L_1 \cap L_2 \)
3. \( HF(L_1, L_2) \) is independent by Hamiltonian perturbation of \( L_i \).
4. \( HF(L, L) \) may or may not be isomorphic to \( H(L) \).
5. \( HF(L, L) \) has ring structure related to one of \( H(L) \).
\[ X \text{ Einstein (const Ricci curvature)} \]
\[ c_1(X) = c[\omega_X] \]

\[ L \subset X \text{ Special (minimal + alpha)} \]
\[ \text{Maslov} = c \int_D^2 u^* \omega \]

monotone
Lagrangian Floer homology (F-Oh-Ohta-Ono, 2009)

Beyond the monotone case

(0.1) \( \text{HF}(L_1, L_2) \) may or may not be defined

(0.2) \( \mathcal{M}(L) \) moduli space of bounding cochains \( b \in \mathcal{M}(L) \)

(0.3) \( W : \mathcal{M}(L) \rightarrow \Lambda \) certain map to Novikov ring. (LG potential)

(0.4) \( \text{HF}(((L_1, b_1), (L_2, b_2))) \) is defined if \( b_i \in \mathcal{M}(L_i) \)

\[ W(b_1) = W(b_2) \]

(1) \( \text{HF}(((L_1, b_1), (L_2, b_2))) \) detects \( \#L_1 \cap L_2 \)

(2) \( \text{HF}(((L_1, b_1), (L_2, b_2))) \) is independent by Hamiltonian perturbation of \( L_i \).

(3) \( \text{HF}(((L, b), (L, b))) \) may or may not be isomorphic to \( H(L) \)

(4) Lagrangian Floer homologies has A infinity structure and defines an A infinity category (Fukaya category).
**Homological Mirror symmetry**  
Kontsevich (1994)

\[(X, \omega) \quad \overset{\sim}{\longleftrightarrow} \quad (X^\wedge, J)\]
Symplectic manifold \quad complex manifold

\[(L, b) \quad \overset{\sim}{\longleftrightarrow} \quad \mathcal{E}\]
Lagrangian submanifold and bounding cochain \quad holomorphic vector bundle and coherent sheaf \quad object of its derived category

\[HF((L_1, b_1), (L_2, b_2)) \cong \text{Ext}(\mathcal{E}_1, \mathcal{E}_2)\]
Floer homology \quad ext group

Isomorphisms preserves multiplicative structure  
and so becomes isomorphism of category

\((W = 0 \text{ for simplicity})\)
What happens in Lagrangian Floer homology in these 10 years 2009-2018 ?

(1) We now have many example of calculation of Lagrangian Floer homology.

(2) Based on calculation (mainly in exact or monotone case) Homological Mirror symmetry is proved in various cases.

(3) Much more detail and/or enhancement on foundation of definition etc.

(4) Various generalization and additional structures of the construction.
What happens in Lagrangian Floer homology in these 10 years 2009-2018?

(4) Various generalization and additional structures of the construction. (Some or many are in progress.)

(a) Bulk deformation = open-closed theory.

(b) Generalization to the immersed Lagrangian submanifolds.

(c) Categorification.

(d) Family of Floer homology.

(e) Generalization to non-compact $\left( X, \omega \right)$ and/or non-compact $L$

(f) Equivariant Lagrangian Floer homology

(g) Lift to loop space

(h) Floer homotopy type

(i) Including maps from more general bordered Riemann surface.
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I want to mainly talk on (4). Before I mention a bit on (3).
(3) Much more detail and/or enhancement on foundation of definition etc.

**Why foundation of Lagrangian Floer homology is lengthy?**

Lagrangian Floer theory is based on the moduli space of

\[ u : (D^2, \partial D^2) \to (X, L) \]

\[ M_k(L; \beta) \]

\[ \beta = [u] \]

\[ k = \text{number of points we fix on } \partial D^2 \]
\( \mathcal{M}_k(L; \beta) \) is much singular.

We need an intersection theory on \( \mathcal{M}_k(L; \beta) \)

\[ \text{ev} = (\text{ev}_1, \ldots, \text{ev}_k) \quad \mathcal{M}_k(L; \beta) \rightarrow L^k \]

\( P_1, \ldots, P_k \subset L \)

\[ \text{ev}_1^{-1}(P_1) \cap \cdots \cap \text{ev}_k^{-1}(P_k) \]
In algebraic geometry a similar intersection theory is studied in Gromov-Witten theory (moduli space of holomorphic spheres) using scheme and stack.

In symplectic geometry
1996
Theory of Virtual fundamental chain and cycle is started (F-Ono, Ruan, Tian-Li-Liu, Siebert) and developing.
Theory of Virtual fundamental chain and cycle is started and developing.

In 2018

FOOO

D. Joyce

Hofer-Wyssoski-Zehnder

Pardon
Theory of Virtual fundamental chain and cycle is started and developing.

In 2018

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<tr>
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<td>F-Oh-Ohta-Ono</td>
<td>(generalization of) Manifold theory</td>
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What happens in Lagrangian Floer homology in these 10 years 2009-2018?

(4) Various generalization and additional structures of the construction. (Some or many are in progress.)

(a) Bulk deformation \( \Rightarrow \) open-closed theory.

(b) Generalization to the immersed Lagrangian submanifolds.

(c) Categorification.

(d) Family of Floer homology.

(e) Generalization to non-compact \((X, \omega)\) and/or non-compact \(L\)

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(i) Including maps from more general bordered Riemann surface.
(4) Various generalization and additional structures of the construction. (Some or many are in progress.)

(a) Bulk deformation `= open-closed theory.

\[ \mathcal{F}(X, \omega) \] A infinity category

Object \( (L, b) \) Lagrangian submanifold

+ bounding cochain

Morphism \( HF((L_1, b_1), (L_2, b_2)) \)

\[ q : HQ(X) \to HH^*(\mathcal{F}(X, \omega), \mathcal{F}(X, \omega)) \] closed open map (ring homo)

\[ p : HH_*(\mathcal{F}(X, \omega)) \to H_*(X) \] open closed map
\( \mathbf{q} : HQ(X) \to HH^*(\mathcal{F}(X, \omega), \mathcal{F}(X, \omega)) \)

Floer homology is deformed by the parameter in \( H(X) \)

\( \mathbf{p} : HH_*(\mathcal{F}(X, \omega)) \to H_*(X) \) is compatible with Poincare duality
Why Bulk deformation?

(1) There are lots of Lagrangian submanifolds for which usual Floer homology is trivial but Floer homology becomes nontrivial after bulk deformation.

(eg. infinitely many such example in $S^2 \times S^2$ (FOOO).)

(2) $q : HQ(X) \to HH^*(\mathcal{F}(X, \omega), \mathcal{F}(X, \omega))$ can be used to show classical Mirror symmetry from Homological Mirror symmetry.

Kontsevitch 94, realized by Ganatra, Perutz, Sheridan

(3) $p : HH_*(\mathcal{F}(X, \omega)) \to H_*(X)$ can be used to calculate $\mathcal{F}(X, \omega)$

$\mathcal{C} \subset \mathcal{F}(X, \omega)$ $HH(\mathcal{C}) \to H(X)$ hits $1_X$ implies $\mathcal{C}$ is a generator

(Abouzaid, AFOOO)

$p$ has application to symplectic geometry. (P. Albers, Biran-Cornea)
(b) Generalization to the immersed Lagrangian submanifolds.

Akaho-Joyce

\[ i : L \to X \quad i^*(\omega) = 0 \quad \dim L = \frac{1}{2} \dim X \]

\( L \)

\[ i(p) = i(q) \]

\[ H(L) \quad \to \quad HF(L, L) \quad \text{deformation by holomorphic disk} \]

\[ HF(L) \oplus \bigoplus_{(p, q): i(p) = i(q), p \neq q} \Lambda[(p, q)] \quad \to \quad HF(L, L) \quad \text{immersed case} \]

\[ \quad \text{deformation by holomorphic disk} \]
Why generalization to the immersed Lagrangian submanifolds?

(1) Embedded Lagrangian submanifold is hard to find, immersed one is easier to construct.

(2) P. Seidel, ….., Nick Sheridan, …. found certain nice immersed Lagrangian submanifold calculate its Floer homology and proved HMS.

(3) To obtain functoriality of Lagrangian Floer theory it is important to include immersed case.
Do we need to include more general singular Lagrangian submanifolds?

**Conjecture**  If $L$ is a strata wise smooth Lagrangian submanifold and suppose we have perverse sheaf on it such that Poincare duality holds then $L$ has Floer homology?
Gromov-Witten invariant does not have functoriality.

\[ f : X \rightarrow Y \quad \xrightarrow{\text{"morphism"}} \quad f^* : QH(Y) \rightarrow QH(X) \]
(c) Categorification.

Gromov-Witten invariant does not have functoriality.

\[ f : X \to Y \quad \text{and} \quad f^* : \mathbb{QH}(Y) \to \mathbb{QH}(X) \]
(c) Categorification.

Lagrangian Floer theory has functoriality.

\((X, \omega_X)\) \hspace{1cm} \text{Weinstein}

symplectic manifolds

\((Y, \omega_Y)\)

\(\mathcal{L} \subset (X \times Y, -\omega_X \oplus \omega_Y)\) \hspace{1cm} \text{Lagrangian submanifold,}

is the morphism from \((X, \omega_X)\) to \((Y, \omega_Y)\)

modify a bit

\((\mathcal{L}, b)\) \hspace{1cm} \mathcal{L} \subset (X \times Y, -\omega_X \oplus \omega_Y) \hspace{1cm} \text{immersed Lagrangian submanifold}

\[ b \in H(L) \oplus \bigoplus_{p,q: i(p)=i(q), p \neq q} \Lambda[(p, q)] \]

\(\text{bounding cochain.}\)

is the morphism from \((X, \omega_X)\) to \((Y, \omega_Y)\)

unobstructed immersed Weinstein category
(c) Categorification.

Theorem (F, arXiv:1808.06106)

\((X, \omega_X) \to \mathcal{F}(X, \omega)\)

is enhanced to the 2 functor

\{ unobstructed immersed Weinstein category \}

\{ all A infinity category \}

Earlier works  Wehrheim-Woodward, Mau-Wehrheim-Woodward

The case all the Lagrangian submanifolds are embedded and monotone.

On going work  Bottmann-Wehrheim

Similar conclusion but in more analytic way.
(c) Categorification.

\[(X, \omega_X) \xrightarrow{\mathcal{F}(X, \omega)} (X, \omega)\]

\[\mathcal{L} \subset X \times Y \quad \xrightarrow{L \times_X \mathcal{L} \to Y} \quad \text{this is generically immersed}\]

Corollary

\[\begin{array}{c}
L \subset X \\
\mathcal{L} \subset X \times Y
\end{array} \quad \left\{ \begin{array}{l}
\text{have bounding chain} \quad b \quad \mathcal{L} \subset X \times Y \\
\text{have bounding chain} \quad b
\end{array} \right. \]

This gives a way to construct Lagrangian immersion with \( b \).
Conjecture. \( \mathcal{L} \subset (X \times Y, -\omega_X \oplus \omega_Y) \) immersed Lagrangian submanifold with \( b \)

\[
\begin{align*}
HH_*(\mathcal{F}(X, \omega)) & \to HH_*(\mathcal{F}(Y, \omega)) \\
\downarrow p & \\
H_*(X) & \to H_*(Y)
\end{align*}
\]

Does it imply certain functoriality of Gromov-Witten theory?
(d) Family of Floer homology.

\[ L \subset X \quad \text{Lagrangian submanifold} \]

\[ \mathcal{M}(L) \quad \text{the set of } b \text{ up to gauge equivalence.} \]

the condition for \( b \) is Maurer Cartan

\[ \sum_k m_k(b, \ldots, b) = 0 \]

\[ \{ L(u) \mid u \in B \} \quad \text{family of Lagrangian submanifolds parametrized by certain space } B \]

\[ \mathcal{M} = \bigcup_{u \in B} \mathcal{M}(L(u)) \times \{ u \} \]

rigid analytic space (cf Kontsevich-Soibelman)

example family of fibers of SYZ fibration.
(d) Family of Floer homology.

$$\mathcal{M} = \bigcup_{u \in B} \mathcal{M}(L(u)) \times \{u\}$$

rigid analytic space (cf Kontsevich-Soibelman)

example family of fibers of SYZ fibration.

Other \((L', b')\) is given.

Family Floer homology Abouzaid …… (F)

$$HF(L(u), b), (L', b')) \quad (b, u) \in \mathcal{M}$$

define object of derived category of coherent sheaf on \(\mathcal{M}\)

can extend to the point \(L(u)\) becomes singular.
Many people working on it. So I will be very brief on the part various people are working.

The case \((X, \omega)\) is convex at infinity. Two methods are used:

- Use Hamiltonian \(H : X \to \mathbb{R}\) which goes to infinity at the end.
  - Absolute case: Symplectic homology
  - Floer-Hofer-Cielibak
  - Bourgeois-Oancha

- \(L\) nonimpact and is product type near infinity
  - Wrapped Floer homology
  - Abouzaid-Seidel ………

- Do not use Hamiltonian but study Reeb orbit (arc) at infinity
  - Eliashberg-Givental-Hofer
  - E.Bao-Honda
  - Pardon
  - Ishikawa

The case with \(L\) is still under construction.

These two are expected to be equivalent but are not proved to be so.
I want to say something in the case \((X, \omega)\) is not convex at infinity. Joint with A. Daemi.

\[
(X, \omega)
\]

\(D \subset X\) smooth divisor (real codimension 2 submanifold, exist \(J\) in a neighborhood of \(D\) such that \(D\) is a complex submanifold.)

\(D \subset X\) normal crossing divisor (a finite union of real codimension 2 submanifolds intersecting transversally, exist \(J\) in a neighborhood of \(D\) such that irreducible components of \(D\) are complex submanifolds.

study Lagrangian Floer theory in \(X \setminus D\)

(e) Generalization to non-compact \((X, \omega)\) and/or non-compact \(L\)

arXiv:1808.08915 and 1809.03409
(e) Generalization to non-compact \((X, \omega)\) and/or non-compact \(L\)

Study Lagrangian Floer theory in \(X \setminus D\)

\[ L \subset X \setminus D \quad \text{compact} \quad A \infty \text{ structure on } H(L) \]

smooth divisor case: Mostly written

normal crossing divisor OK but is not written yet

**Conjecture** to be worked out

\[ L \subset X \setminus D \quad \text{non-compact} \quad L \cap D_k \quad \text{codimension } k \text{ corner of } L \]

\[ D_k \quad \text{intersection of } k \text{ irreducible components.} \]

expected to be coincides with Wrapped Floer homology in convex case.
conjecture to be worked out

$$L \subset X \setminus D$$ non-compact $$L \cap D_k$$ codimension $k$ corner of $L$ intersection of $k$$D_k$ irreducible components.

expected to be coincides with Wrapped Floer homology in convex case.

conjecture to be worked out

Relative Gromov-Witten theory gives a ring structure of certain group

$$H(X \setminus D) \oplus H(D)[q, q^{-1}] \oplus \ldots$$

Exists open closed and closed open maps,
(f) Equivariant Lagrangian Floer homology

\((X, \omega)\) \hspace{1cm} \text{with } G \text{ (compact Lie group) action;}

Work in progress

\[ \mathcal{F}(X, \omega, G) \]

\(G\) invariant Lagrangian submanifold

A infinity category \hspace{1cm} \text{object } \hspace{1cm} L \hspace{1cm} \text{and } \hspace{1cm} b

bounding chain in \(H_G(L)\)

Morphism \(H_G(L_1, L_2)\)

Exist a work by K. Hendricks, R. Lipshitz, and S. Sarkar based on higher `homotopy’

Exist a work by Cho …. in case \(G\) is finite.

Should be equal to \(\mathcal{F}(X//G, \omega)\) \hspace{1cm} \text{if } G \text{ action is Hamiltonian and } X//G \text{ is smooth}

Woodward-Xu \hspace{1cm} \text{based on gauged sigma model} \hspace{1cm} \text{arXiv:1806.06717}

F \hspace{1cm} \text{based on Lagrangian correspondence (to be written)}
(f) Equivariant Lagrangian Floer homology

Why equivariant Lagrangian Floer homology?

relation between $\mathcal{F}(X, \omega, G)$ and $\mathcal{F}(X//G, \bar{\omega})$

appears in Hori-Vafa `proof’ of Mirror symmetry

Corollary

$$\dim G = \frac{1}{2} \dim X$$

$HF(L, L)$ is well defined.

$L \cong G$ (via $G$ action).

When $X//G$ is singular we can use $\mathcal{F}(X, \omega, G)$ in place of $\mathcal{F}(X//G, \bar{\omega})$
(g) Lift to loop space

\[ M_k(L; \beta) \]

\[ u : (D^2, \partial D^2) \to (X, L) \]

\[ \beta = [u] \]

\[ k = \text{number of points we fix on } \partial D^2 \]

Evaluation map \[ M_k(L; \beta) \to L^k \]

We may consider evaluation map \[ M_1(L; \beta) \to \Omega(L) \]

\[ u \mapsto u|_{\partial D^2} \]
Consider evaluation map \( \mathcal{M}_1(L; \beta) \to \Omega(L) \quad u \mapsto u|_{\partial D^2} \)

Usual Floer homology \( HF(L,L) \) deforms ordinary homology of \( L \).

This version of Floer homology deforms ordinary homology of the loop space \( H(\Omega L) \)

Multiplicative structure deforms String topology of Chas-Sullivan

Proposed by F around 2007
(g) Lift to loop space

This version of Floer homology deforms ordinary homology of the loop space $H(\Omega L)$

Multiplicative structure deforms String topology of Chas-Sullivan

Why it is good?

Floer homology deforming ordinary homology of the loop space $H(\Omega L)$ exists without assuming the existence of bounding cochain $b$. 
Why it is good?

Floer homology deforming ordinary homology of the loop space \( H(\Omega L) \) exists without assuming the existence of bounding cochain \( b \).

**Theorem** (Irie (using the proposal by F)) arXiv:1801.04633

\[ L \subset C^3 \]  
embedded Lagrangian submanifold

oriented and irreducible

\[ L \cong S^1 \times \Sigma_g \]
Theorem (Irie)

\[ L \subset \mathbb{C}^3 \quad \text{embedded Lagrangian submanifold oriented and irreducible} \quad \rightarrow \quad L \cong S^1 \times \Sigma_g \]

Some related open problem.

- \( M_1, M_2 \) aspherical 3 manifolds \( \rightarrow \) \( M_1 \# M_2 \) connected sum
  
  can not be a Lagrangian submanifold of \( \mathbb{C}^3 \)

  hope studying string topology of \( M_1 \# M_2 \) will prove it.

- \( L \subset \mathbb{C}P^3 \) embedded Lagrangian submanifold oriented and irreducible

  hope \( S^1 \) equivariant version of loop space \( HF \) will prove it.

(related work by Viterbo)
(h) Floer homotopy type

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<th>Invariant</th>
<th>Dimension</th>
<th>Description</th>
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<td>Seiberg-Witten invariant</td>
<td>dim = 4</td>
<td>number</td>
</tr>
<tr>
<td></td>
<td>Bauer-Furuta invariant</td>
<td></td>
<td>an element of certain equivariant framed cobordism group</td>
</tr>
<tr>
<td></td>
<td>dim = 3</td>
<td></td>
<td>SW Floer homology</td>
</tr>
<tr>
<td>Yang-Mills theory</td>
<td>Donaldson theory</td>
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(h) Floer homotopy type

- Seiberg-Witten theory
  
  Seiberg-Witten invariant  Bauer-Furuta invariant
  
  number  an element of certain equivariant framed cobordism group
  
  dim = 4
  dim = 3  SW Floer homology  Spectrum  (Manolescu)

- Yang-Mills theory  Donaldson theory

  number

  dim = 4
  dim = 3  Instanton Floer homology  ?
How about Symplectic Floer theory?

There is a bubble so worse than SW theory.

However it is better than Yang-Mills theory since we have perfect obstruction theory for pseudo-holomorphic curve.

Conjecture (Abouzaid ???)

If the corresponding Fredholm complex of pseudo holomorphic curve has an orientation in general homology theory \( h \),
then \( h \) Floer homology theory is defined.

\( h \) some general homology theory (or spectrum).
Discussion: Conjecture (Abouzaid ???)

If the corresponding Fredholm complex of pseudo holomorphic curve has an orientation in general homology theory $h$, then $h$ Floer homology theory is defined.

\[ \begin{align*}
s &\quad \mathcal{E} \\
U &\quad \Gamma \\
s^{-1}(0)/\Gamma &\quad \mathcal{M}
\end{align*} \]

\[ \begin{align*}
\times \text{id} &\quad \mathcal{E} \times \mathbb{R}^m \\
U \times \mathbb{R}^m &\quad \Gamma \\
s^{-1}(0)/\Gamma &\quad \mathcal{M}
\end{align*} \]

Kuranishi model is well defined up to stabilization (related to stable homotopy theory.)
\[ N_{\varphi_{21}(U_{21})}U_2 \cong \mathcal{E}_2/\mathcal{E}_1 \]

preserve orientation in general homology theory \( h \)

\[ \varphi_{21} : U_{21} \subset U_2 \rightarrow U_1 \quad \text{coordinate change of Kuranishi structure} \]

\[ \bar{\varphi}_{21} : \mathcal{E}_2|_{U_{21}} \rightarrow E_1 \]
(i) Including maps from more general bordered Riemann surface.

No more time remains at this stage I think.