Extra twisted connected sums and their $\nu$-invariants

Sebastian Goette

Mathematisches Institut der Universität Freiburg

SCSHGAP Second Annual Meeting
September 13–14, 2018
Outline

- **$G_2$-geometry**
  Intro, properties, examples, questions
- **The $\nu$-invariant**
  Differential topology, definition of $\nu$, properties, first examples
- **Extra twisted connected sums**
  Construction, properties, problems
- **Computation of the $\nu$-invariant**
  Computations with $\eta$-invariants, examples, questions
Consider parallel translation along a spherical triangle
Consider parallel translation along a spherical triangle.
Consider parallel translation along a spherical triangle
Consider parallel translation along a spherical triangle
Consider parallel translation along a spherical triangle

A vector is rotated by an angle equal to the spherical area of the triangle (Gauß-Bonnet theorem)
Consider parallel translation along a spherical triangle

A vector is rotated by an angle equal to the spherical area of the triangle (Gauß-Bonnet theorem)

The subgroup of Aut($T_pS^2$) of all these parallel translations is the holonomy group

$$\text{Hol}(S^2, g^{\text{rd}}) \cong SO(2)$$
Consider parallel translation along a spherical triangle

A vector is rotated by an angle equal to the spherical area of the triangle (Gauß-Bonnet theorem)

The subgroup of \( \text{Aut}(T_p S^2) \) of all these parallel translations is the holonomy group

\[
\text{Hol}(S^2, g^{\text{md}}) \cong \text{SO}(2)
\]

Related: the spherical area formula

\[
A(\Delta) = \alpha + \beta + \gamma - \pi.
\]
Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are
Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

<table>
<thead>
<tr>
<th>Holonomy group</th>
<th>dim</th>
<th>ric</th>
<th>Structure</th>
<th>Parallel spinors</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(n)$</td>
<td>$n$</td>
<td></td>
<td></td>
<td></td>
<td>generic case</td>
</tr>
<tr>
<td>$U(k)$</td>
<td>$2k$</td>
<td>0</td>
<td>$J$</td>
<td></td>
<td>Kähler</td>
</tr>
<tr>
<td>$SU(k)$</td>
<td>$2k$</td>
<td>0</td>
<td>$J, \Omega$</td>
<td>2</td>
<td>Calabi-Yau</td>
</tr>
<tr>
<td>$Sp(\ell) \cdot Sp(1)$</td>
<td>$4\ell$</td>
<td>const</td>
<td>$\langle I, J, K \rangle$</td>
<td></td>
<td>Quat. Kähler</td>
</tr>
<tr>
<td>$Sp(\ell)$</td>
<td>$4\ell$</td>
<td>0</td>
<td>$I, J, K, \Omega$</td>
<td>$\ell + 1$</td>
<td>hyper Kähler</td>
</tr>
<tr>
<td>$G_2$</td>
<td>7</td>
<td>0</td>
<td>$\varphi \in \Omega^3$</td>
<td>1</td>
<td>exceptional</td>
</tr>
<tr>
<td>$Spin(7)$</td>
<td>8</td>
<td>0</td>
<td>$\psi \in \Omega^4$</td>
<td>1</td>
<td>exceptional</td>
</tr>
</tbody>
</table>
G₂-geometry—Berger’s list

Theorem (Berger)

The only possible holonomy groups of complete, simply connected Riemannian manifolds that are neither a product nor a symmetric space are

<table>
<thead>
<tr>
<th>Holonomy group</th>
<th>dim</th>
<th>ric</th>
<th>Structure</th>
<th>Parallel spinors</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO(n)</td>
<td>n</td>
<td></td>
<td></td>
<td></td>
<td>generic case</td>
</tr>
<tr>
<td>U(k)</td>
<td>2k</td>
<td>0</td>
<td>J</td>
<td></td>
<td>Kähler</td>
</tr>
<tr>
<td>SU(k)</td>
<td>2k</td>
<td>0</td>
<td>J, Ω</td>
<td>2</td>
<td>Calabi-Yau</td>
</tr>
<tr>
<td>Sp(ℓ) · Sp(1)</td>
<td>4ℓ</td>
<td>const</td>
<td>⟨I, J, K⟩</td>
<td></td>
<td>Quat. Kähler</td>
</tr>
<tr>
<td>Sp(ℓ)</td>
<td>4ℓ</td>
<td>0</td>
<td>I, J, K, Ω</td>
<td>ℓ + 1</td>
<td>hyper Kähler</td>
</tr>
<tr>
<td>G₂</td>
<td>7</td>
<td>0</td>
<td>φ ∈ Ω³</td>
<td>1</td>
<td>exceptional</td>
</tr>
<tr>
<td>Spin(7)</td>
<td>8</td>
<td>0</td>
<td>ψ ∈ Ω⁴</td>
<td>1</td>
<td>exceptional</td>
</tr>
</tbody>
</table>
Why consider $G_2$-manifolds?

---

Mathematical motivation

▶ Only special holonomy group for odd dimensional manifolds
▶ Only $G_2$ and Spin$(7)$ holonomy have no direct relation to algebraic geometry

Hence, new methods are needed

Physical motivation

▶ In string theory, spacetime takes the form $\mathbb{R}^3, 1 \times V$, where $V$ is Calabi-Yau
▶ In M-theory, spacetime takes the form $\mathbb{R}^3, 1 \times M$, where $M$ is a $G_2$-manifold

▶ Possible relations to other physical theories

Hence, many fruitful interactions possible
Why consider $G_2$-manifolds?

**Mathematical motivation**

- Only special holonomy group for odd dimensional manifolds
- Only $G_2$ and Spin(7) holonomy have no direct relation to algebraic geometry

Hence, new methods are needed
Why consider $G_2$-manifolds?

**Mathematical motivation**

- Only special holonomy group for odd dimensional manifolds
- Only $G_2$ and Spin(7) holonomy have no direct relation to algebraic geometry

Hence, new methods are needed

**Physical motivation**

- In string theory, spacetime takes the form $\mathbb{R}^{3,1} \times V$, where $V$ is Calabi-Yau
- In M-theory, spacetime takes the form $\mathbb{R}^{3,1} \times M$, where $M$ is a $G_2$-manifold
- Possible relations to other physical theories

Hence, many fruitful interactions possible
Characterisations of the Lie group $G_2$ give characterisations of $G_2$-manifolds

- $G_2 = \text{Aut}(\mathcal{O})$
  
  Hence $G_2$ preserves $\varphi = \langle \cdot \times \cdot, \cdot \rangle \in \Lambda^3 \text{Im} \mathcal{O} \cong \Lambda^3 \mathbb{R}^7$
Characterisations of the Lie group $G_2$ give characterisations of $G_2$-manifolds

- $G_2 = \text{Aut} (\mathcal{O})$
  
  Hence $G_2$ preserves $\varphi = \langle \cdot \times \cdot, \cdot \rangle \in \Lambda^3 \text{Im} \mathcal{O} \cong \Lambda^3 \mathbb{R}^7$

- The stabiliser of $\varphi$ in $\text{GL}(7, \mathbb{R})$ is $G_2$
  
  The $\text{GL}(7, \mathbb{R})$-orbit of $\varphi \in \Lambda^3 \mathbb{R}^7$ is open (not dense)
  
  Forms in this orbit are called positive
Characterisations of the Lie group $G_2$ give characterisations of $G_2$-manifolds

- $G_2 = \text{Aut}(\mathcal{O})$
  Hence $G_2$ preserves $\varphi = \langle \cdot \times \cdot , \cdot \rangle \in \Lambda^3 \text{Im} \mathcal{O} \cong \Lambda^3 \mathbb{R}^7$

- The stabiliser of $\varphi$ in $GL(7, \mathbb{R})$ is $G_2$
  The $GL(7, \mathbb{R})$-orbit of $\varphi \in \Lambda^3 \mathbb{R}^7$ is open (not dense)
  Forms in this orbit are called positive

- A positive 3-form on $M$ determines a $G_2$-structure and a metric $g_\varphi$
  Call $\varphi$ torsion free if $d\varphi = d^*_{g_\varphi} \varphi = 0$ (nonlinear condition)
  Then $(M, g_\varphi)$ has $\text{Hol}(M, g_\varphi) \subset G_2$ if and only if $\varphi$ is torsion-free
Characterisations of the Lie group $G_2$ give characterisations of $G_2$-manifolds

- $G_2 = \text{Aut}(\mathbb{O})$
  Hence $G_2$ preserves $\varphi = \langle \cdot \times \cdot , \cdot \rangle \in \Lambda^3 \text{Im} \mathbb{O} \cong \Lambda^3 \mathbb{R}^7$ and $1 \in \mathbb{R} \subset \mathbb{O}$

- The stabiliser of $\varphi$ in $\text{GL}(7, \mathbb{R})$ is $G_2$
  The $\text{GL}(7, \mathbb{R})$-orbit of $\varphi \in \Lambda^3 \mathbb{R}^7$ is open (not dense)
  Forms in this orbit are called positive

- A positive 3-form on $M$ determines a $G_2$-structure and a metric $g_{\varphi}$
  Call $\varphi$ torsion free if $d\varphi = d^{\ast}g_{\varphi} \varphi = 0$ (nonlinear condition)
  Then $(M, g_{\varphi})$ has $\text{Hol}(M, g_{\varphi}) \subset G_2$ if and only if $\varphi$ is torsion-free

- $G_2$ is compact, simply connected, so $G_2$-manifolds are Riemannian and spin
  The stabiliser of a nonzero spinor in $\text{Spin}(7)$ is $G_2$
  A Riemannian 7-manifold $(M, g)$ has $\text{Hol}(M, g) \subset G_2$
  if and only if it is spin and there exists a nonzero parallel spinor
Let $\varphi \in \Omega^3(M)$ be positive and closed.
There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by

$$B([\alpha], [\beta]) = ([\alpha] \bowtie [\beta] \bowtie [\varphi])[M] = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class.

---

$^1$ Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410.
Let $\varphi \in \Omega^3(M)$ be positive and closed.

There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by

$$B([\alpha], [\beta]) = ([\alpha] \smile [\beta] \smile \varphi)(M) = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class.

If $M$ is closed and $\text{Hol}(M, \varphi) \subset G_2$ then

- $M$ is oriented and spin and $b_3(M) = \dim H^3(M; \mathbb{R}) \geq 1$

---

$^1$ Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410
Let $\varphi \in \Omega^3(M)$ be positive and closed.

There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by:

$$B([\alpha], [\beta]) = ([\alpha] \smile [\beta] \smile [\varphi])[M] = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class.

If $M$ is closed and $\text{Hol}(M, \varphi) \subset G_2$ then:

- $M$ is oriented and spin and $b_3(M) = \dim H^3(M; \mathbb{R}) \geq 1$
- $\text{Hol}(M, g) = G_2 \iff \# \pi_1(M) < \infty$

---

\(^1\) Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410
Let $\varphi \in \Omega^3(M)$ be positive and closed. There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by

$$B([\alpha], [\beta]) = ([\alpha] \smile [\beta] \smile [\varphi])[M] = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class. If $M$ is closed and $\text{Hol}(M, \varphi) \subset G_2$ then

- $M$ is oriented and spin and $b_3(M) = \dim H^3(M; \mathbb{R}) \geq 1$
- $\text{Hol}(M, g) = G_2 \iff \#\pi_1(M) < \infty$
- $\text{Hol}(M, g) = G_2 \implies B$ is negative definite

---

1 Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410
Let $\varphi \in \Omega^3(M)$ be positive and closed

There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by

$$B([\alpha], [\beta]) = ([\alpha] \sim [\beta] \sim [\varphi])[M] = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class

If $M$ is closed and $\text{Hol}(M, \varphi) \subset G_2$ then

- $M$ is oriented and spin and $b_3(M) = \dim H^3(M; \mathbb{R}) \geq 1$
- $\text{Hol}(M, g) = G_2 \iff \# \pi_1(M) < \infty$
- $\text{Hol}(M, g) = G_2 \implies B$ is negative definite
- $\text{Hol}(M, g) = G_2 \implies (p_1(TM) \sim [\varphi])[M] < 0$

---

1 Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410
Let $\varphi \in \Omega^3(M)$ be positive and closed. There exists a bilinear form $B$ on $H^2(M; \mathbb{R})$ given for closed forms $\alpha, \beta \in \Omega^2(M)$ by

$$B([\alpha], [\beta]) = ([\alpha] \lrcorner [\beta] \lrcorner [\varphi])[M] = \int_M \alpha \wedge \beta \wedge \varphi$$

Let $p_1(TM) \in H^4(M; \mathbb{R})$ denote the first Pontryagin class.

If $M$ is closed and $\text{Hol}(M, \varphi) \subset G_2$ then

- $M$ is oriented and spin and $b_3(M) = \dim H^3(M; \mathbb{R}) \geq 1$
- $\text{Hol}(M, g) = G_2 \iff \# \pi_1(M) < \infty$
- $\text{Hol}(M, g) = G_2 \implies B$ is negative definite
- $\text{Hol}(M, g) = G_2 \implies (p_1(TM) \lrcorner [\varphi])[M] < 0$

These are all known\(^1\) obstructions against holonomy $G_2$

---

\(^1\)Spiro Karigiannis pointed out after the talk that another requirement is almost formality, see recent work by Chan, Karigiannis and Tsang, arXiv:1801.06410
Let $M$ be a compact oriented spin 7-manifold and define

$$\mathcal{X} = \{ \varphi \in \Omega^3(M) \mid \varphi \text{ is positive and torsion free} \}$$
Let $M$ be a compact oriented spin 7-manifold and define

$$\mathcal{X} = \{ \varphi \in \Omega^3(M) \mid \varphi \text{ is positive and torsion free} \}$$

Let $\mathcal{D} \subset \text{Diff}(M)$ be the connected component of $\text{id}_M$. Then

$$M = \mathcal{X}/\mathcal{D}$$

is called the $G_2$-moduli space of $M$
Let $M$ be a compact oriented spin 7-manifold and define

$$\mathcal{X} = \{ \varphi \in \Omega^3(M) \mid \varphi \text{ is positive and torsion free} \}$$

Let $\mathcal{D} \subset \text{Diff}(M)$ be the connected component of $\text{id}_M$. Then

$$\mathcal{M} = \mathcal{X}/\mathcal{D}$$

is called the $G_2$-moduli space of $M$

**Theorem (Joyce)**

The $G_2$-moduli space is a manifold, and the map

$$\mathcal{M} \to H^3(M; \mathbb{R}) \quad \text{with} \quad [\varphi] \mapsto [\varphi]$$

is a local diffeomorphism
Let $M$ be a compact oriented spin 7-manifold and define

$$\mathcal{X} = \{ \varphi \in \Omega^3(M) \mid \varphi \text{ is positive and torsion free} \}$$

Let $\mathcal{D} \subset \text{Diff}(M)$ be the connected component of $\text{id}_M$. Then

$$\mathcal{M} = \mathcal{X}/\mathcal{D}$$

is called the $G_2$-moduli space of $M$

**Theorem (Joyce)**

The $G_2$-moduli space is a manifold, and the map

$$\mathcal{M} \longrightarrow H^3(M; \mathbb{R}) \quad \text{with} \quad [\varphi] \mapsto [\varphi]$$

is a local diffeomorphism

Not much is known about the global structure of $\mathcal{M}$
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first

- Bryant '87: first non-complete examples
- Bryant and Salamon '89: first complete examples
- Joyce '96: first closed examples

Joyce's construction: let a "rich enough" finite subgroup $\Gamma \subset G_2$ act on flat $T_7$ with "sufficiently many" fixpoints, preserving a parallel positive 3-form $\phi \in \Omega^3(T_7)$.

The stabilisers of fixpoints $p \in T_7$ are isomorphic to subgroups of $SU(2)$ or $SU(3)$.

By gluing in suitable bundles of noncompact Calabi-Yau manifolds in place of the singularities, Joyce constructs a desingularisation $M \to T_7/\Gamma$.

The closed $G_2$-structure on $M$ obtained by gluing is close to a torsion-free one.
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first

- Bryant '87: first non-complete examples
- Bryant and Salamon '89: first complete examples
- Joyce '96: first closed examples

Joyce’s construction: let a “rich enough” finite subgroup $\Gamma \subset G_2$ act on flat $T^7$ with “sufficiently many” fixpoints, preserving a parallel positive 3-form $\varphi \in \Omega^3(T^7)$
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first

- Bryant '87: first non-complete examples
- Bryant and Salamon '89: first complete examples
- Joyce '96: first closed examples

Joyce’s construction: let a “rich enough” finite subgroup $\Gamma \subset G_2$ act on flat $T^7$ with “sufficiently many” fixpoints, preserving a parallel positive 3-form $\varphi \in \Omega^3(T^7)$

The stabilisers of fixpoints $p \in T^7$ are isomorphic to subgroups of $SU(2)$ or $SU(3)$
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first

- Bryant ’87: first non-complete examples
- Bryant and Salamon ’89: first complete examples
- Joyce ’96: first closed examples

Joyce’s construction: let a “rich enough” finite subgroup $\Gamma \subset G_2$ act on flat $T^7$ with “sufficiently many” fixpoints, preserving a parallel positive 3-form $\varphi \in \Omega^3(T^7)$.

The stabilisers of fixpoints $p \in T^7$ are isomorphic to subgroups of $SU(2)$ or $SU(3)$.

By gluing in suitable bundles of noncompact Calabi-Yau manifolds in place of the singularities, Joyce constructs a desingularisation

$$M \rightarrow T^7/\Gamma$$
To construct subsets of the $G_2$-moduli space means to construct deformation families of $G_2$-manifolds first

- Bryant '87: first non-complete examples
- Bryant and Salamon '89: first complete examples
- Joyce '96: first closed examples

Joyce’s construction: let a “rich enough” finite subgroup $\Gamma \subset G_2$ act on flat $T^7$ with “sufficiently many” fixpoints, preserving a parallel positive 3-form $\varphi \in \Omega^3(T^7)$

The stabilisers of fixpoints $p \in T^7$ are isomorphic to subgroups of $SU(2)$ or $SU(3)$

By gluing in suitable bundles of noncompact Calabi-Yau manifolds in place of the singularities, Joyce constructs a desingularisation

$$M \longrightarrow T^7/\Gamma$$

The closed $G_2$-structure on $M$ obtained by gluing is close to a torsion-free one
Kovalev ’03, Corti-Haskins-Nordström-Pacini ’15: Twisted connected sums
Kovalev ’03, Corti-Haskins-Nordström-Pacini ’15: Twisted connected sums

Let $V_+, V_-\,$ be asymptotically cylindrical Calabi-Yau threefolds
Kovalev ’03, Corti-Haskins-Nordström-Pacini ’15: Twisted connected sums

Let $V_+, V_-\ldots$ be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces.
Kovalev ’03, Corti-Haskins-Nordström-Pacini ’15: Twisted connected sums

Let \( V_+ \), \( V_- \) be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to \( \Sigma_\pm \times S^1 \times \mathbb{R} \), where \( \Sigma_\pm \) are K3 surfaces. Glue \( V_- \times S^1 \) to \( V_+ \times S^1 \), flipping the circles.
Let $V_+, V_-$ be asymptotically cylindrical Calabi-Yau threefolds with ends asymptotic to $\Sigma_\pm \times S^1 \times \mathbb{R}$, where $\Sigma_\pm$ are K3 surfaces. Glue $V_- \times S^1$ to $V_+ \times S^1$, flipping the circles. The closed $G_2$-structure on $M$ obtained by gluing is close to a torsion-free one.
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known

- Only a few compact examples are known—only $\sim 10^8$ deformation families

- Known compact examples represent points close to the boundary of the moduli space—the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems/questions

- Find more invariants for/obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
  How far can one deform a given $G_2$-metric?
- Construct $G_2$-metrics with prescribed singularities

Singularities allow massless particles to appear in $M$-theory
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular? How far can one deform a given $G_2$-metric?
- Construct $G_2$-metrics with prescribed singularities
  Singularities allow massless particles to appear in $M$-theory
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
- How far can one deform a given $G_2$-metric?
- Construct $G_2$-metrics with prescribed singularities

Singularities allow massless particles to appear in $M$-theory
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics

Singularities allow massless particles to appear in $M$-theory
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
  How far can one deform a given $G_2$-metric?
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
  How far can one deform a given $G_2$-metric?
- Construct $G_2$-metrics with prescribed singularities

Singularities allow massless particles to appear in $M$-theory
Let us summarise first

- Only a few obstructions against $G_2$-holonomy are known
- Only a few compact examples are known—only $\sim 10^8$ deformation families
- Known compact examples represent points close to the boundary of the moduli space—
  the $G_2$-metric is locally close to metrics with holonomy groups $SU(2)$ or $SU(3)$

Important open problems / questions

- Find more invariants for / obstructions against $G_2$-metrics
- Construct $G_2$-metrics far away from the boundary of the moduli space
- How can families of $G_2$-metrics become singular?
  How far can one deform a given $G_2$-metric?
- Construct $G_2$-metrics with prescribed singularities
  Singularities allow massless particles to appear in $M$-theory
We want to describe $G_2$-manifolds using differential topology

**Definition**

A $G_2$-structure on a seven-manifold $M$ is a reduction of the $GL(7, \mathbb{R})$-frame bundle to a bundle with structure group $G_2$.
We want to describe $G_2$-manifolds using differential topology

**Definition**

A $G_2$-structure on a seven-manifold $M$ is a reduction of the $GL(7,\mathbb{R})$-frame bundle to a bundle with structure group $G_2$

**Equivalent descriptions**

- Positive three form $\varphi$ on $M$
We want to describe $G_2$-manifolds using differential topology.

**Definition**

A $G_2$-structure on a seven-manifold $M$ is a reduction of the $GL(7, \mathbb{R})$-frame bundle to a bundle with structure group $G_2$.

**Equivalent descriptions**

- Positive three form $\varphi$ on $M$
- Riemannian metric, spin structure, and a unit spinor (up to sign)
The $\nu$-invariant—$G_2$-structures

We want to describe $G_2$-manifolds using differential topology.

**Definition**

A $G_2$-structure on a seven-manifold $M$ is a reduction of the $GL(7, \mathbb{R})$-frame bundle to a bundle with structure group $G_2$.

**Equivalent descriptions**

- Positive three form $\varphi$ on $M$
- Riemannian metric, spin structure, and a unit spinor (up to sign)

**Idea.** Use nowhere vanishing spinors to describe and distinguish $G_2$-structures.
When we (Diarmuid Crowley, Johannes Nordström and myself) started our project, we asked the following questions
When we (Diarmuid Crowley, Johannes Nordström and myself) started our project, we asked the following questions:

Can different constructions of $G_2$-holonomy metrics

- lead to the same closed 7-manifold up to diffeomorphism?
When we (Diarmuid Crowley, Johannes Nordström and myself) started our project, we asked the following questions:

Can different constructions of $G_2$-holonomy metrics lead to the same closed 7-manifold up to diffeomorphism?

- if so, are the underlying $G_2$-structures the same up to homotopy and spin diffeomorphism?
When we (Diarmuid Crowley, Johannes Nordström and myself) started our project, we asked the following questions

Can different constructions of $G_2$-holonomy metrics

- lead to the same closed 7-manifold up to diffeomorphism?
- if so, are the underlying $G_2$-structures the same up to homotopy and spin diffeomorphism?
- if so, do the two metrics belong to the same connected component of the $G_2$-moduli space?
When we (Diarmuid Crowley, Johannes Nordström and myself) started our project, we asked the following questions:

Can different constructions of $G_2$-holonomy metrics

- lead to the same closed 7-manifold up to diffeomorphism?
- if so, are the underlying $G_2$-structures the same up to homotopy and spin diffeomorphism?
- if so, do the two metrics belong to the same connected component of the $G_2$-moduli space?

Diarmuid Crowley and Johannes Nordström also asked

- Are there pairs of $G_2$-manifolds that are homeomorphic but not diffeomorphic?
The $\nu$-invariant—Overview of invariants

Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
The $\nu$-invariant—Overview of invariants

Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
Let $s$ be a nowhere vanishing spinor on $M$, defining a $G_2$-structure
Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
Let $s$ be a nowhere vanishing spinor on $M$, defining a $G_2$-structure
Relevant differential topological invariants

\[
\mu(M) \in \mathbb{Z}/\gcd(28, \tilde{d}/4) \quad \text{generalised Eells-Kuiper invariant}
\]
The $\nu$-invariant—Overview of invariants

Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
Let $s$ be a nowhere vanishing spinor on $M$, defining a $G_2$-structure

Relevant differential topological invariants

\[ \mu(M) \in \mathbb{Z}/\text{gcd}(28, \tilde{d}/4) \quad \text{generalised Eells-Kuiper invariant} \]
\[ \xi(M, s) \in \mathbb{Z}/3\tilde{d} \]
Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$
torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
Let $s$ be a nowhere vanishing spinor on $M$, defining a $G_2$-structure

Relevant differential topological invariants

$$
\mu(M) \in \mathbb{Z}/\text{gcd}(28, \tilde{d}/4) \quad \text{generalised Eells-Kuiper invariant}
$$

$$
\xi(M, s) \in \mathbb{Z}/3\tilde{d}
$$

$$
\nu(M, s) \in \mathbb{Z}/48 \quad \text{see below}
$$
Assume that $M$ is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free
Let $d \in \mathbb{N}$ be the divisibility of $p_1(TM)$, then $4 \mid d$ (Wu)
Let $\tilde{d} = \text{lcm}(4, d/2)$
Let $s$ be a nowhere vanishing spinor on $M$, defining a $G_2$-structure

Relevant differential topological invariants

\[ \mu(M) \in \mathbb{Z}/\gcd(28, \tilde{d}/4) \quad \text{generalised Eells-Kuiper invariant} \]
\[ \xi(M, s) \in \mathbb{Z}/3\tilde{d} \]
\[ \nu(M, s) \in \mathbb{Z}/48 \quad \text{see below} \]

Important relations

\[ \xi(M, s) \equiv 7\nu(M, s) \quad \text{mod 12} \]
\[ \frac{\xi(M, s) - 7\nu(M, s)}{12} \equiv \mu(M) \quad \text{mod } \gcd(28, \tilde{d}/4) \]
Let $\sigma_0, \sigma_1$ be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(S^+(M \times [0, 1]))$ A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

$$\text{rk} S^+(M \times [0, 1]) = 8 = \dim(M \times [0, 1])$$
Let $\sigma_0, \sigma_1$ be two nowhere vanishing spinors. Extend to $\tilde{\sigma} \in \Gamma(S^+(M \times [0, 1]))$

A generic $\tilde{\sigma}$ will have nondegenerate isolated zeros because

$$\text{rk } S^+(M \times [0, 1]) = 8 = \dim(M \times [0, 1])$$

Orient $S^+(M \times [0, 1])$ and count with signs

$$\Delta \nu(M; \sigma_0, \sigma_1) = 2 \cdot \# \tilde{\sigma}^{-1}(0) = 2 \cdot \sum_{p \in \tilde{\sigma}^{-1}(0)} \text{sign}(d_p \tilde{\sigma})$$
Let $\sigma_0, \sigma_1$ be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(S^+(M \times [0, 1]))$

A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

$$\text{rk } S^+(M \times [0, 1]) = 8 = \dim(M \times [0, 1])$$

Orient $S^+(M \times [0, 1])$ and count with signs

$$\Delta \nu(M; \sigma_0, \sigma_1) = 2 \cdot \# \bar{\sigma}^{-1}(0) = 2 \cdot \sum_{p \in \bar{\sigma}^{-1}(0)} \text{sign}(d_p \bar{\sigma})$$

**Theorem (Crowley-Nordström)**

Let $F: M \to M$ be a spin diffeomorphism, then

$$\Delta \nu(M; \sigma, F^* \sigma) \in 48\mathbb{Z}$$
The $\nu$-invariant—Comparing $G_2$-structures

Let $\sigma_0$, $\sigma_1$ be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(S^+(M \times [0, 1]))$
A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

$$\text{rk} \ S^+(M \times [0, 1]) = 8 = \dim(M \times [0, 1])$$

Orient $S^+(M \times [0, 1])$ and count with signs

$$\Delta \nu(M; \sigma_0, \sigma_1) = 2 \cdot \# \bar{\sigma}^{-1}(0) = 2 \cdot \sum_{p \in \bar{\sigma}^{-1}(0)} \text{sign}(d_p \bar{\sigma})$$

Theorem (Crowley-Nordström)

Let $F: M \to M$ be a spin diffeomorphism, then

$$\Delta \nu(M; \sigma, F^* \sigma) \in 48\mathbb{Z}$$

Can we write $\Delta \nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$?
The $\nu$-invariant—Cobordism definition

Idea. If $M$ is spin, then $M$ is the spin boundary of some compact 8-manifold $W$
Extend $\sigma$ to $\bar{\sigma} \in \Gamma(S^+W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on $W$—not well-defined yet!
The $\nu$-invariant—Cobordism definition

Idea. If $M$ is spin, then $M$ is the spin boundary of some compact 8-manifold $W$
Extend $\sigma$ to $\bar{\sigma} \in \Gamma(S^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on $W$—not well-defined yet!

- $\chi(W)$—Euler characteristic of $W$
- $\text{sign}(W)$—signature of $W$

Definition (Crowley-Nordström)
Assume that $M = \partial W$ with $W$ spin, compact. Define

$$\nu(M, \sigma) = \chi(W) - 3 \text{sign}(W) - 2\#\bar{\sigma}^{-1}(0) \mod 48$$
The $\nu$-invariant—Cobordism definition

Idea. If $M$ is spin, then $M$ is the spin boundary of some compact 8-manifold $W$. Extend $\sigma$ to $\bar{\sigma} \in \Gamma(S^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on $W$—not well-defined yet!

- $\chi(W)$—Euler characteristic of $W$
- $\text{sign}(W)$—signature of $W$

Definition (Crowley-Nordström)

Assume that $M = \partial W$ with $W$ spin, compact. Define

$$\nu(M, \sigma) = \chi(W) - 3 \text{sign}(W) - 2\#\bar{\sigma}^{-1}(0) \mod 48$$

Theorem (Crowley-Nordström)

$$\Delta \nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$$
The $\nu$-invariant—Cobordism definition

**Idea.** If $M$ is spin, then $M$ is the spin boundary of some compact 8-manifold $W$. Extend $\sigma$ to $\bar{\sigma} \in \Gamma(S^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on $W$—not well-defined yet!

- $\chi(W)$—Euler characteristic of $W$
- $\text{sign}(W)$—signature of $W$

**Definition (Crowley-Nordström)**

Assume that $M = \partial W$ with $W$ spin, compact. Define

$$\nu(M, \sigma) = \chi(W) - 3 \text{sign}(W) - 2 \#\bar{\sigma}^{-1}(0) \mod 48$$

**Theorem (Crowley-Nordström)**

$$\Delta \nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$$

**Problem.** Given $M$, how to determine $W$ with $M = \partial W$?
Idea. Use the APS-index theorem and Mathai-Quillen currents
The $\nu$-invariant—Analytic description

Idea. Use the APS-index theorem and Mathai-Quillen currents

- $\psi(\nabla^{SM}, g^{SM})$—Mathai-Quillen form in $\Omega^\bullet(SM)$
- $D_M$—spin Dirac operator on $\Gamma(SM)$
- $B_M$—odd signature operator $*d \pm d*$ on $\Omega^{ev}(M)$
- $h$—dimension of the kernel
- $\eta$—Atiyah-Patodi-Singer $\eta$-invariant
The $\nu$-invariant—Analytic description

Idea. Use the APS-index theorem and Mathai-Quillen currents

- $\psi(\nabla^{SM}, g^{SM})$—Mathai-Quillen form in $\Omega^*(SM)$
- $D_M$—spin Dirac operator on $\Gamma(SM)$
- $B_M$—odd signature operator $\ast d \pm d\ast$ on $\Omega^{\text{ev}}(M)$
- $h$—dimension of the kernel
- $\eta$—Atiyah-Patodi-Singer $\eta$-invariant

Theorem (Crowley-G-Nordström)

$$\nu(M, \sigma) = 2 \int_M \sigma^* \psi(\nabla^{SM}, g^{SM}) - 24(\eta + h)(D_M) + 3\eta(B_M) \in \mathbb{Z}/48$$

Proof.

Use

$$2e(\nabla^{S^+W}) = e(\nabla) + 48\hat{A}(\nabla)[8] - 3L(\nabla)[8] \in \Omega^8(W)$$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^* \psi(\nabla^{SM}, g^{SM}) = 0$
In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^* \psi(\nabla^S M, g^S M) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^* \psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the $G_2$-moduli space $\mathcal{M}$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^*\psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the $G_2$-moduli space $\mathcal{M}$

Definition (Crowley-G-Nordström)

Let $(M, g)$ be a compact manifold with $\text{Hol}(M, g) \subset G_2$. Put

$$\bar{\nu}(M, g) = 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z}$$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^* \psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the $G_2$-moduli space $\mathcal{M}$

Definition (Crowley-G-Nordström)

Let $(M, g)$ be a compact manifold with $\text{Hol}(M, g) \subset G_2$. Put

$$\bar{\nu}(M, g) = 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z}$$

- $\nu(M, \sigma) \equiv \bar{\nu}(M, g) - 24(1 + b_1(M)) \mod 48$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^* \psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the $G_2$-moduli space $\mathcal{M}$

Definition (Crowley-G-Nordström)

Let $(M, g)$ be a compact manifold with $\text{Hol}(M, g) \subset G_2$. Put

$$\tilde{\nu}(M, g) = 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z}$$

- $\nu(M, \sigma) \equiv \tilde{\nu}(M, g) - 24(1 + b_1(M)) \mod 48$
- $\tilde{\nu}(M, g)$ is locally constant on $\mathcal{M}$
The $\nu$-invariant—The extended $\nu$-invariant

In the case of $G_2$-holonomy, things simplify

- $\sigma$ is parallel, so $\sigma^*\psi(\nabla^{SM}, g^{SM}) = 0$
- Harmonic spinors are parallel, so $h(D_M) = 1 + b_1(M)$
- $\eta(D_M) \in \mathbb{R}$ is smooth on the $G_2$-moduli space $\mathcal{M}$

Definition (Crowley-G-Nordström)

Let $(M, g)$ be a compact manifold with $\text{Hol}(M, g) \subset G_2$. Put

$$\bar{\nu}(M, g) = 3\eta(B_M) - 24\eta(D_M) \in \mathbb{Z}$$

- $\nu(M, \sigma) \equiv \bar{\nu}(M, g) - 24(1 + b_1(M)) \mod 48$
- $\bar{\nu}(M, g)$ is locally constant on $\mathcal{M}$
- $\bar{\nu}(M, g) = 0$ if $M$ admits an orientation reversing isometry
What about the known examples by Joyce and Kovalev?

- \( \bar{\nu}(M, g) = 0 \) for all twisted connected sums
  (but \( \xi(M, s) \neq 0 \) is possible—Wallis, arXiv:1808.09443)

Question. Is \( \bar{\nu}(M, g) = 0 \) whenever \( \text{Hol}(M, g) = G_2 \)?

- If yes, then \( \bar{\nu}(M, g) \neq 0 \) or \( \nu(M, \sigma) \neq 24 \) is a new obstruction against \( G_2 \)-holonomy
- If no, then \( \bar{\nu}(M, g) \) is a non-trivial new invariant

Answer. We will construct examples with \( \bar{\nu}(M, g) \neq 0 \)
Using \( \bar{\nu}(M, g) \), we will show that for some particular \( M \), the \( G_2 \)-moduli space \( M \) has several connected components
The \( \nu \)-invariant—First examples

What about the known examples by Joyce and Kovalev?

- \( \bar{\nu}(M, g) = 0 \) for all twisted connected sums
  (but \( \xi(M, s) \neq 0 \) is possible—Wallis, arXiv:1808.09443)
- \( \bar{\nu}(M, g) = 0 \) for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries
The $\nu$-invariant—First examples

What about the known examples by Joyce and Kovalev?

- $\bar{\nu}(M, g) = 0$ for all twisted connected sums
  (but $\xi(M, s) \neq 0$ is possible—Wallis, arXiv:1808.09443)
- $\bar{\nu}(M, g) = 0$ for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries

Question. Is $\bar{\nu}(M, g) = 0$ whenever $\text{Hol}(M, g) = G_2$?
What about the known examples by Joyce and Kovalev?

- \( \bar{\nu}(M, g) = 0 \) for all twisted connected sums
  (but \( \xi(M, s) \neq 0 \) is possible—Wallis, arXiv:1808.09443)
- \( \bar{\nu}(M, g) = 0 \) for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries

Question. Is \( \bar{\nu}(M, g) = 0 \) whenever \( \text{Hol}(M, g) = G_2 \)?

- If yes, then \( \bar{\nu}(M, g) \neq 0 \) or \( \nu(M, \sigma) \neq 24 \) is a new obstruction against \( G_2 \)-holonomy

Answer. We will construct examples with \( \bar{\nu}(M, g) \neq 0 \) using \( \bar{\nu}(M, g) \), we will show that for some particular \( M \), the \( G_2 \)-moduli space \( M \) has several connected components
What about the known examples by Joyce and Kovalev?

- \( \bar{\nu}(M, g) = 0 \) for all twisted connected sums (but \( \xi(M, s) \neq 0 \) is possible—Wallis, arXiv:1808.09443)
- \( \bar{\nu}(M, g) = 0 \) for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries

**Question.** Is \( \bar{\nu}(M, g) = 0 \) whenever \( \text{Hol}(M, g) = G_2 \)?

- If **yes**, then \( \bar{\nu}(M, g) \neq 0 \) or \( \nu(M, \sigma) \neq 24 \) is a new obstruction against \( G_2 \)-holonomy
- If **no**, then \( \bar{\nu}(M, g) \) is a non-trivial new invariant
What about the known examples by Joyce and Kovalev?

- $\bar{\nu}(M, g) = 0$ for all twisted connected sums
  (but $\xi(M, s) \neq 0$ is possible—Wallis, arXiv:1808.09443)
- $\bar{\nu}(M, g) = 0$ for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries

Question. Is $\bar{\nu}(M, g) = 0$ whenever $\text{Hol}(M, g) = G_2$?

- If yes, then $\bar{\nu}(M, g) \neq 0$ or $\nu(M, \sigma) \neq 24$ is a new obstruction against $G_2$-holonomy
- If no, then $\bar{\nu}(M, g)$ is a non-trivial new invariant

Answer. We will construct examples with $\bar{\nu}(M, g) \neq 0$
The $\nu$-invariant—First examples

What about the known examples by Joyce and Kovalev?

- $\bar{\nu}(M, g) = 0$ for all twisted connected sums
  (but $\xi(M, s) \neq 0$ is possible—Wallis, arXiv:1808.09443)
- $\bar{\nu}(M, g) = 0$ for some of Joyce’s examples—some are twisted connected sums, some admit orientation reversing isometries

Question. Is $\bar{\nu}(M, g) = 0$ whenever $\text{Hol}(M, g) = G_2$?

- If yes, then $\bar{\nu}(M, g) \neq 0$ or $\nu(M, \sigma) \neq 24$ is a new obstruction against $G_2$-holonomy
- If no, then $\bar{\nu}(M, g)$ is a non-trivial new invariant

Answer. We will construct examples with $\bar{\nu}(M, g) \neq 0$
Using $\bar{\nu}(M, g)$, we will show that for some particular $M$, the $G_2$-moduli space $\mathcal{M}$ has several connected components
Recall twisted connected sums

\[ V_- - \times - S^1_{-\text{int}} - \times - S^1_{-,\text{ext}} \]

\[ \Sigma_- - S^1_{-,\text{int}} - \times - S^1_{-,\text{ext}} \]

\[ V_+ - \times - S^1_{+,\text{int}} - \times - S^1_{+,\text{ext}} \]

Gluing of tori at angle \( \vartheta = \frac{\pi}{2} \) between exterior circles
**Extra twisted connected sums**

Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S_{\pm,\text{ext}}^1$.

The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise.

The actions on $S_{\pm,\text{int}}^1$ and $S_{\pm,\text{ext}}^1$ have to be free.
Extra twisted connected sums

\[ \Gamma_- \curvearrowright \begin{array}{c}
S_{-,\text{ext}} \\
\times \\
\Sigma_-
\end{array} \times \\
\begin{array}{c}
V_-
\times \\
S_{-,\text{int}}
\end{array} \]

\[ \Gamma_+ \curvearrowright \begin{array}{c}
S_{+,\text{int}} \\
\times \\
\Sigma_+
\end{array} \times \\
\begin{array}{c}
V_+ \\
\times \\
S_{+,\text{ext}}
\end{array} \]

Assume that \( \Gamma_\pm \cong \mathbb{Z}/k_\pm \) acts both on \( V_\pm \) and on \( S_{\pm,\text{ext}} \)

The induced action on \( \partial V_\pm \) has to fix \( \Sigma_\pm \) pointwise

The actions on \( S_{\pm,\text{int}} \) and \( S_{\pm,\text{ext}} \) have to be free

Then \( (S_{\pm,\text{int}} \times S_{\pm,\text{ext}})/\Gamma_\pm \) is again a flat 2-torus
Extra twisted connected sums

Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S^1_{\pm,\text{ext}}$

The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise

The actions on $S^1_{\pm,\text{int}}$ and $S^1_{\pm,\text{ext}}$ have to be free

Then $(S^1_{\pm,\text{int}} \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ is again a flat 2-torus

If both the tori and the K3 surfaces are isometric, we can glue $M_\pm = (V_\pm \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ at various angles $\vartheta$
Extra twisted connected sums

Extra twisted connected sums

\[ \Gamma_- \sim \mathbb{Z}/2 \]

Modified gluing of tori at angle \( \vartheta = \frac{3}{4} \pi \)

\[ \Gamma_- \sim \mathbb{Z}/2 \]

\[ \Gamma_+ \sim \{0\} \]
Extra twisted connected sums—Construction

Extra twisted connected sums

\[ V^- \times S^1_- \times S^1_-, \text{int} \]
\[ \Sigma^- \times S^1_- \times S^1_-, \text{int} \]
\[ \Sigma^+ \times S^1_+ \times S^1_+, \text{int} \]
\[ V^+ \]
\[ \Gamma_- \sim \mathbb{Z}/2 \]
\[ \Gamma_+ \sim \mathbb{Z}/2 \]

Modified gluing of tori at angle \( \vartheta = \frac{2}{3} \pi \)
Extra twisted connected sums—Construction

Extra twisted connected sums

\[ S_{-\text{, ext}} \times \Sigma - \times S_{+\text{, int}} \]

\[ \Gamma_- \quad \Gamma_+ \]

Modified gluing of tori at angle \( \vartheta = \arccos \left( \frac{1}{\sqrt{6}} \right) \)

\[ S_{-\text{, ext}} \quad S_{+\text{, int}} \]

\[ \Gamma_- \cong \mathbb{Z}/3 \]

\[ \Gamma_+ \cong \mathbb{Z}/4 \]
The CY structures on $V_\pm$ induce $G_2$-structures on $M_\pm$ and on $\Sigma_\pm \times T_\pm^2 \times \mathbb{R}$

Then we need an isomorphism of $G_2$-manifolds $\Sigma_+ \times T_+^2 \times \mathbb{R} \xrightarrow{\sim} \Sigma_- \times T_-^2 \times \mathbb{R}$
The CY structures on $V_{\pm}$ induce $G_2$-structures on $M_{\pm}$ and on $\Sigma_{\pm} \times T_{\pm}^2 \times \mathbb{R}$.

Then we need an isomorphism of $G_2$-manifolds $\Sigma_+ \times T_+^2 \times \mathbb{R} \sim \Sigma_- \times T_-^2 \times \mathbb{R}$.

Let $u_\pm, v_\pm$ be coordinates on the interior and exterior circles, respectively.

$$u_- = -u_+ \cos \vartheta + v_+ \sin \vartheta \quad \text{and} \quad v_- = u_+ \sin \vartheta + v_+ \cos \vartheta$$

Let $t = t_- = -t_+$ be the coordinate in the cylinder direction.
Extra twisted connected sums—The $G_2$-structure

The CY structures on $V_\pm$ induce $G_2$-structures on $M_\pm$ and on $\Sigma_\pm \times T^2_\pm \times \mathbb{R}$

Then we need an isomorphism of $G_2$-manifolds $\Sigma_+ \times T^2_+ \times \mathbb{R} \xrightarrow{\cong} \Sigma_- \times T^2_- \times \mathbb{R}$

Let $u_\pm, v_\pm$ be coordinates on the interior and exterior circles, respectively

$$u_- = -u_+ \cos \vartheta + v_+ \sin \vartheta \quad \text{and} \quad v_- = u_+ \sin \vartheta + v_+ \cos \vartheta$$

Let $t = t_- = -t_+$ be the coordinate in the cylinder direction

Let $\omega_1^\pm, \omega_2^\pm, \omega_3^\pm \in \Omega^2,+(\Sigma)$ be hyperkähler triples, $\omega_1^\pm$ is the Kähler form from $V_\pm$
The CY structures on $V_{\pm}$ induce $G_2$-structures on $M_{\pm}$ and on $\Sigma_{\pm} \times T^2_{\pm} \times \mathbb{R}$.

Then we need an isomorphism of $G_2$-manifolds $\Sigma_+ \times T^2_+ \times \mathbb{R} \xrightarrow{\sim} \Sigma_- \times T^2_- \times \mathbb{R}$.

Let $u_\pm, v_\pm$ be coordinates on the interior and exterior circles, respectively

$$u_- = -u_+ \cos \vartheta + v_+ \sin \vartheta \quad \text{and} \quad v_- = u_+ \sin \vartheta + v_+ \cos \vartheta$$

Let $t = t_- = -t_+$ be the coordinate in the cylinder direction.

Let $\omega_1^\pm, \omega_2^\pm, \omega_3^\pm \in \Omega^2_+(\Sigma)$ be hyperkähler triples, $\omega_1^\pm$ is the Kähler form from $V_{\pm}$

$$\varphi = dv_\pm \wedge \omega_1^\pm + du_\pm \wedge \omega_2^\pm + dt_\pm \wedge \omega_3^\pm + dt_\pm \wedge du_\pm \wedge dv_\pm$$

$$\omega_1^- = \cos \vartheta \omega_1^+ + \sin \vartheta \omega_2^+ \quad \omega_2^- = \sin \vartheta \omega_1^+ - \cos \vartheta \omega_2^+ \quad \omega_3^- = -\omega_3^+$$

Both the torus matching and the K3 matching depend on the gluing angle $\vartheta$. 
Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $V_\pm \times S^1_{\pm, \text{ext}}$

A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$
Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $V_\pm \times S^1_{\text{ext}}$

A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$
- A gluing matrix $G = \begin{pmatrix} m & p \\ n & q \end{pmatrix}$
  with $\det G = -k_+k_-$ and $mq \leq 0$, $np \geq 0$

satisfying some extra conditions (only finitely many choices for $\varepsilon_+$, $G$ possible)
Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $V_\pm \times S^1_{\pm,\text{ext}}$

A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)\times$ if $k_+ > 1$
- A gluing matrix $G = (\begin{pmatrix} m & p \\ n & q \end{pmatrix})$
  with $\det G = -k_+k_-$ and $mq \leq 0$, $np \geq 0$

satisfying some extra conditions (only finitely many choices for $\varepsilon_+$, $G$ possible)

From $G$, recover

- The aspect ratios $s_+ = \frac{\ell(S^1_{+\text{,ext}})}{\ell(S^1_{+\text{,int}})} = \sqrt{-\frac{mq}{mp}}$ and $s_- = \frac{\ell(S^1_{-\text{,ext}})}{\ell(S^1_{-\text{,int}})} = \sqrt{-\frac{mn}{pq}}$
- The gluing angle $\vartheta = \arg(ms_+ + \text{i}n) \in (-\pi, \pi]$
- The fundamental group $\pi_1(M) \cong \mathbb{Z}/p$
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_\pm = \text{im}(H^2(V_\pm) \to H^2(\Sigma)) \subset L = H^2(\Sigma) \cong E_8^2 \oplus U^3$$

The pair $(N_+, N_-)$ of sublattices of $L$ (up to $\text{Aut}(L)$) is called a configuration
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_\pm = \text{im}(H^2(V_\pm) \to H^2(\Sigma)) \subset L = H^2(\Sigma) \cong E_8^2 \oplus U^3$$

The pair $(N_+, N_-)$ of sublattices of $L$ (up to $\text{Aut}(L)$) is called a configuration.

We get positive orthonormal triples $[\omega_1^\pm] \in N_{\pm, \mathbb{R}}$ and $[\omega_2^\pm], [\omega_3^\pm] \in N_{\pm, \mathbb{R}} \subset L_{\mathbb{R}}$.
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_\pm = \text{im}(H^2(V_\pm) \to H^2(\Sigma)) \quad \subset \quad L = H^2(\Sigma) \cong E_8^2 \oplus U^3$$

The pair $(N_+, N_-)$ of sublattices of $L$ (up to $\text{Aut}(L)$) is called a configuration.

We get positive orthonormal triples $[\omega^{\pm}_1] \in N_{\pm,R}$ and $[\omega^{\pm}_2], [\omega^{\pm}_3] \in N^\perp_{\pm,R} \subset L_R$.

Let $A_{N_\pm}$ denote the reflections of $L_R \cong \mathbb{R}^{3,19}$ in $N_{\pm,R}$. Because

$$[\omega^{-}_1] = \cos \vartheta [\omega^{+}_1] + \sin \vartheta [\omega^{+}_2] \quad \text{and} \quad [\omega^{-}_2] = \sin \vartheta [\omega^{+}_1] - \cos \vartheta [\omega^{+}_2] \quad (*)$$

the classes $[\omega^{\pm}_1], [\omega^{\pm}_2]$ lie in the subspace $L_{2\vartheta} \subset L_R$ that $A_{N_+}A_{N_-}$ rotates through $2\vartheta$. 
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_{\pm} = \text{im}(H^2(V_{\pm}) \to H^2(\Sigma)) \subset L = H^2(\Sigma) \cong E_8^2 \oplus U^3$$

The pair $(N_+, N_-)$ of sublattices of $L$ (up to $\text{Aut}(L)$) is called a configuration.

We get positive orthonormal triples $[\omega_1^{\pm}] \in N_{\pm, \mathbb{R}}$ and $[\omega_2^{\pm}], [\omega_3^{\pm}] \in N_{\pm, \mathbb{R}}^+ \subset L_{\mathbb{R}}$. Let $A_{N_{\pm}}$ denote the reflections of $L_{\mathbb{R}} \cong \mathbb{R}^{3,19}$ in $N_{\pm, \mathbb{R}}$. Because

$$[\omega_1^-] = \cos \vartheta [\omega_1^+] + \sin \vartheta [\omega_2^+] \quad \text{and} \quad [\omega_2^-] = \sin \vartheta [\omega_1^+] - \cos \vartheta [\omega_2^+] \quad (*)$$

the classes $[\omega_1^{\pm}], [\omega_2^{\pm}]$ lie in the subspace $L_{2\vartheta} \subset L_{\mathbb{R}}$ that $A_{N_+} A_{N_-}$ rotates through $2\vartheta$.

**Matching Problem.** To construct $M$, find $N_+, N_- \subset L$ and positive classes $[\omega_1^{\pm}] \in L_{2\vartheta} \cap N_{\pm, \mathbb{R}}$ and $[\omega_2^{\pm}] \in L_{2\vartheta} \cap N_{\pm, \mathbb{R}}^+$ of length 1 satisfying (*)
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_\pm = \text{im}(H^2(V_\pm) \to H^2(\Sigma)) \subset L = H^2(\Sigma) \cong E_8^2 \oplus U^3$$

The pair $(N_+, N_-)$ of sublattices of $L$ (up to Aut($L$)) is called a configuration. We get positive orthonormal triples $[\omega^\pm_1] \in N_{\pm,\mathbb{R}}$ and $[\omega^\pm_2], [\omega^\pm_3] \in N_{\pm,\mathbb{R}} \subset L_{\mathbb{R}}$.

Let $A_{N_\pm}$ denote the reflections of $L_{\mathbb{R}} \cong \mathbb{R}^{3,19}$ in $N_{\pm,\mathbb{R}}$. Because

$$[\omega^-_1] = \cos \vartheta [\omega^+_1] + \sin \vartheta [\omega^+_2] \quad \text{and} \quad [\omega^-_2] = \sin \vartheta [\omega^+_1] - \cos \vartheta [\omega^+_2] \quad (*)$$

the classes $[\omega^\pm_1], [\omega^\pm_2]$ lie in the subspace $L_{2\vartheta} \subset L_{\mathbb{R}}$ that $A_{N_+} A_{N_-}$ rotates through $2\vartheta$.

**Matching Problem.** To construct $M$, find $N_+, N_- \subset L$ and positive classes $[\omega^\pm_1] \in L_{2\vartheta} \cap N_{\pm,\mathbb{R}}$ and $[\omega^\pm_2] \in L_{2\vartheta} \cap N_{\pm,\mathbb{R}}$ of length 1 satisfying $(*)$.

Then find $V_\pm$ with $\mathbb{Z}/k_\pm$-actions and $\Sigma$ that “realise” $N_\pm \subset L$ and $[\omega^\pm_1], \ldots, [\omega^\pm_3]$.
If $M$ is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let
\[
N_\pm = \text{im}(H^2(V_\pm) \to H^2(\Sigma)) \subset L = H^2(\Sigma) \cong E_8^2 \oplus U^3
\]
The pair $(N_+, N_-)$ of sublattices of $L$ (up to $\text{Aut}(L)$) is called a configuration.

We get positive orthonormal triples $[\omega_1^\pm] \in N_{\pm,R}$ and $[\omega_2^\pm], [\omega_3^\pm] \in N_{\pm,R}^+ \subset L_R$.

Let $A_{N_\pm}$ denote the reflections of $L_R \cong \mathbb{R}^{3,19}$ in $N_{\pm,R}$. Because
\[
[\omega_1^-] = \cos \vartheta [\omega_1^+] + \sin \vartheta [\omega_2^+] \quad \text{and} \quad [\omega_2^-] = \sin \vartheta [\omega_1^+] - \cos \vartheta [\omega_2^+]
\]
the classes $[\omega_1^\pm], [\omega_2^\pm]$ lie in the subspace $L_{2\vartheta} \subset L_R$ that $A_{N_+} A_{N_-}$ rotates through $2\vartheta$.

Matching Problem. To construct $M$, find $N_+, N_- \subset L$ and positive classes $[\omega_1^\pm] \in L_{2\vartheta} \cap N_{\pm,R}$ and $[\omega_2^\pm] \in L_{2\vartheta} \cap N_{\pm,R}^+$ of length 1 satisfying (*)

Then find $V_\pm$ with $\mathbb{Z}/k_\pm$-actions and $\Sigma$ that “realise” $N_\pm \subset L$ and $[\omega_1^\pm], \ldots, [\omega_3^\pm]$

All this works for “easy” $V_\pm$ if $\text{rk} N_+ = \text{rk} N_-$ and $N_{\pm,R} \subset L_{2\vartheta}$. 
To construct an extra twisted connected sum

- Pick deformation types $V_\pm$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_\pm$-actions (can be constructed from weak Fano threefolds $Z_\pm$)
To construct an extra twisted connected sum

- Pick deformation types $V_\pm$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_\pm$-actions (can be constructed from weak Fano threefolds $Z_\pm$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_\pm \subset L$ induced by $V_\pm$
To construct an extra twisted connected sum

- Pick deformation types $V_{\pm}$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_{\pm}$-actions (can be constructed from weak Fano threefolds $Z_{\pm}$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_\pm \subset L$ induced by $V_{\pm}$
- Determine hyperkähler triples with gluing angle $\vartheta$ induced by actual varieties $V_{\pm}$ (requires knowledge of moduli spaces of $\Sigma_{\pm}$ and $Z_{\pm}$)
To construct an extra twisted connected sum

- Pick deformation types $V_\pm$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_\pm$-actions (can be constructed from weak Fano threefolds $Z_\pm$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_\pm \subset L$ induced by $V_\pm$
- Determine hyperkähler triples with gluing angle $\vartheta$ induced by actual varieties $V_\pm$ (requires knowledge of moduli spaces of $\Sigma_\pm$ and $Z_\pm$)
- Find $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$ and a gluing matrix $G$ with gluing angle $\vartheta$
To construct an extra twisted connected sum

- Pick deformation types \( V_\pm \) of asymptotically cylindrical Calabi-Yau threefolds with \( \mathbb{Z}/k_\pm \)-actions (can be constructed from weak Fano threefolds \( Z_\pm \))
- Find a configuration \((N_+, N_-)\) for the sublattices \( N_\pm \subset L \) induced by \( V_\pm \)
- Determine hyperkähler triples with gluing angle \( \vartheta \) induced by actual varieties \( V_\pm \) (requires knowledge of moduli spaces of \( \Sigma_\pm \) and \( Z_\pm \))
- Find \( \varepsilon_+ \in (\mathbb{Z}/k_+)^\times \) if \( k_+ > 1 \) and a gluing matrix \( G \) with gluing angle \( \vartheta \)
- Determine the asymptotically cylindrical Calabi-Yau metrics on \( V_\pm \)
To construct an extra twisted connected sum

- Pick deformation types $V_\pm$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_\pm$-actions (can be constructed from weak Fano threefolds $Z_\pm$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_\pm \subset L$ induced by $V_\pm$
- Determine hyperkähler triples with gluing angle $\vartheta$ induced by actual varieties $V_\pm$ (requires knowledge of moduli spaces of $\Sigma_\pm$ and $Z_\pm$)
- Find $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$ and a gluing matrix $G$ with gluing angle $\vartheta$
- Determine the asymptotically cylindrical Calabi-Yau metrics on $V_\pm$
- Glue $(V_+ \times S^1_{\xi_+})/\Gamma_+$ to $(V_- \times S^1_{\xi_-})/\Gamma_-$ as specified by the data above
To construct an extra twisted connected sum

- Pick deformation types $V_{\pm}$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_{\pm}$-actions (can be constructed from weak Fano threefolds $Z_{\pm}$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_{\pm} \subset L$ induced by $V_{\pm}$
- Determine hyperkähler triples with gluing angle $\vartheta$ induced by actual varieties $V_{\pm}$ (requires knowledge of moduli spaces of $\Sigma_{\pm}$ and $Z_{\pm}$)
- Find $\varepsilon_+ \in (\mathbb{Z}/k_+)\times$ if $k_+ > 1$ and a gluing matrix $G$ with gluing angle $\vartheta$
- Determine the asymptotically cylindrical Calabi-Yau metrics on $V_{\pm}$
- Glue $(V_+ \times S^1_{\xi_+})/\Gamma_+$ to $(V_- \times S^1_{\xi_-})/\Gamma_-$ as specified by the data above
- Find a torsion-free $G_2$-structure close to the $G_2$-structure obtained by gluing
To construct an extra twisted connected sum

- Pick deformation types $V_\pm$ of asymptotically cylindrical Calabi-Yau threefolds with $\mathbb{Z}/k_\pm$-actions (can be constructed from weak Fano threefolds $Z_\pm$)
- Find a configuration $(N_+, N_-)$ for the sublattices $N_\pm \subset L$ induced by $V_\pm$
- Determine hyperkähler triples with gluing angle $\vartheta$ induced by actual varieties $V_\pm$ (requires knowledge of moduli spaces of $\Sigma_\pm$ and $Z_\pm$)
- Find $\epsilon_+ \in (\mathbb{Z}/k_+) \times$ if $k_+ > 1$ and a gluing matrix $G$ with gluing angle $\vartheta$
- Determine the asymptotically cylindrical Calabi-Yau metrics on $V_\pm$
- Glue $(V_+ \times S^1_{\xi_+})/\Gamma_+$ to $(V_- \times S^1_{\xi_-})/\Gamma_-$ as specified by the data above
- Find a torsion-free $G_2$-structure close to the $G_2$-structure obtained by gluing

In the following, we will only need the configuration $(N_+, N_-)$, the gluing matrix $G$, and the number $\epsilon_+$
Computation of the $\nu$-invariant—The gluing formula

With respect to modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_{\pm}, g) = 3\eta_{APS}(B_{M_{\pm}}; L_{B,\pm}) - 24\eta_{APS}(D_{M_{\pm}}; L_{D,\pm}) \in \mathbb{R}$$
Computation of the $\nu$-invariant—The gluing formula

With respect to modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_{\pm}, g) = 3\eta_{\text{APS}}(B_{M_{\pm}}; L_{B,\pm}) - 24\eta_{\text{APS}}(D_{M_{\pm}}; L_{D,\pm}) \in \mathbb{R}$$

The isometry $A_{N_+}A_{N_-}$ respects the decomposition $H^{2,+}(\Sigma) \oplus H^{2,-}(\Sigma)$

Let $\alpha_1^+, \ldots, \alpha_3^+, \alpha_1^-, \ldots, \alpha_{19}^- \in (-\pi, \pi]$ be the angles through which $A_{N_+}A_{N_-} \otimes \mathbb{C}$ rotates $H^{2,+}(\Sigma, \mathbb{C})$ and $H^{2,-}(\Sigma, \mathbb{C})$, respectively
The gluing formula

With respect to modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_{\pm}, g) = 3\eta_{APS}(B_{M_{\pm}}; L_{B, \pm}) - 24\eta_{APS}(D_{M_{\pm}}; L_{D, \pm}) \in \mathbb{R}$$

The isometry $A_{N_+}A_{N_-}$ respects the decomposition $H^{2, +}(\Sigma) \oplus H^{2, -}(\Sigma)$

Let $\alpha_1^+, \alpha_2^+, \alpha_3^-, \ldots, \alpha_1^- \in (-\pi, \pi]$ be the angles through which $A_{N_+}A_{N_-} \otimes \mathbb{C}$

rotates $H^{2, +}(\Sigma, \mathbb{C})$ and $H^{2, -}(\Sigma, \mathbb{C})$, respectively.

Assume $\vartheta \in (0, \pi)$, put $\rho = \pi - 2\vartheta \in (-\pi, \pi)$, and define

$$m_{\rho}(L; N_+, N_-) = \text{sign} \rho \left( \# \{ j \mid \alpha_j^- \in \{ \pi - |\rho|, \pi \} \} - 1 + 2 \# \{ j \mid \alpha_j^- \in (\pi - |\rho|, \pi) \} \right)$$

Put sign 0 = 0, then $m_{\rho}(L; N_+, N_-) = 0$ for ordinary twisted connected sums.
Computation of the $\nu$-invariant—The gluing formula

With respect to modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_\pm, g) = 3\eta_{APS}(B_{M_\pm}; L_{B,\pm}) - 24\eta_{APS}(D_{M_\pm}; L_{D,\pm}) \in \mathbb{R}$$

The isometry $A_{N_+} A_{N_-}$ respects the decomposition $H^{2, +}(\Sigma) \oplus H^{2, -}(\Sigma)$

Let $\alpha_1^+, \ldots, \alpha_3^+, \alpha_1^-, \ldots, \alpha_{19}^- \in (-\pi, \pi]$ be the angles through which $A_{N_+} A_{N_-} \otimes \mathbb{C}$

rotsates $H^{2, +}(\Sigma, \mathbb{C})$ and $H^{2, -}(\Sigma, \mathbb{C})$, respectively

Assume $\vartheta \in (0, \pi)$, put $\rho = \pi - 2\vartheta \in (-\pi, \pi)$, and define

$$m_\rho(L; N_+, N_-) = \text{sign } \rho \left( \# \{ j \mid \alpha_j^- \in \{ \pi - |\rho|, \pi \} \} - 1 + 2 \# \{ j \mid \alpha_j^- \in (\pi - |\rho|, \pi) \} \right)$$

Put sign $0 = 0$, then $m_\rho(L; N_+, N_-) = 0$ for ordinary twisted connected sums

From the gluing formulas for $\eta$-invariants by Bunke, Kirk-Lesch and others, we get

**Theorem (Crowley-G-Nordström)**

$$\bar{\nu}(M, g) = \bar{\nu}(M_+, g) + \bar{\nu}(M_-, g) - \frac{72}{\pi} \rho + 3m_\rho(L; N_+, N_-)$$
If $\Gamma_\pm \cong \{0\}$ or $\mathbb{Z}/2$ then $\bar{\nu}(M_\pm, g) = 0$
If $\Gamma \cong \{0\}$ or $\mathbb{Z}/2$ then $\bar{\nu}(M \pm, g) = 0$

**Example (Crowley-G-Nordström)**

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{97}, \quad \text{div} \ p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\bar{\nu}(M, g_1) = 0, \quad \bar{\nu}(M, g_2) = 36, \quad \bar{\nu}(M, g_3) = -36$$
If $\Gamma_{\pm} \cong \{0\}$ or $\mathbb{Z}/2$ then $\bar{\nu}(M_{\pm}, g) = 0$

**Example (Crowley-G-Nordström)**

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{97}, \quad \text{div} \ p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\bar{\nu}(M, g_1) = 0, \quad \bar{\nu}(M, g_2) = 36, \quad \bar{\nu}(M, g_3) = -36$$

Hence, the $G_2$-moduli space of $M$ is disconnected.
Computation of the $\nu$-invariant—Examples I

If $\Gamma_\pm \cong \{0\}$ or $\mathbb{Z}/2$ then $\bar{\nu}(M_\pm, g) = 0$

Example (Crowley-G-Nordström)

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{97}, \quad \text{div} \ p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\bar{\nu}(M, g_1) = 0, \quad \bar{\nu}(M, g_2) = 36, \quad \bar{\nu}(M, g_3) = -36$$

Hence, the $G_2$-moduli space of $M$ is disconnected

The metric $g_1$ comes from a rectangular twisted connected sum
For $g_2, g_3$, take $\Gamma_+ \cong \mathbb{Z}/2, \Gamma_- \cong \{0\}$ and $\vartheta = \frac{\pi}{4}$
Example (Crowley-G-Nordström)

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{109}, \quad \text{div} \, p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\tilde{\nu}(M, g_1) = 0, \quad \tilde{\nu}(M, g_2) = 48, \quad \tilde{\nu}(M, g_3) = -48$$

In particular, the $\nu$-invariants agree and the $G_2$-structures are homotopic
Example (Crowley-G-Nordström)

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{109}, \quad \text{div} \ p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\bar{\nu}(M, g_1) = 0, \quad \bar{\nu}(M, g_2) = 48, \quad \bar{\nu}(M, g_3) = -48$$

In particular, the $\nu$-invariants agree and the $G_2$-structures are homotopic.

Hence, one homotopy class of $G_2$-structures can give rise to several connected components of the $G_2$-moduli space.
Example (Crowley-G-Nordström)

There exists a spin 7-manifold $M$ with

$$H^1(M) \cong H^2(M) \cong 0, \quad H^4(M) \cong \mathbb{Z}^{109}, \quad \text{div} \ p_1(TM) = 4$$

admitting three different $G_2$-holonomy metrics $g_1, g_2, g_3$ with

$$\bar{\nu}(M, g_1) = 0, \quad \bar{\nu}(M, g_2) = 48, \quad \bar{\nu}(M, g_3) = -48$$

In particular, the $\nu$-invariants agree and the $G_2$-structures are homotopic.

Hence, one homotopy class of $G_2$-structures can give rise to several connected components of the $G_2$-moduli space.

The metric $g_1$ comes from a rectangular twisted connected sum.

For $g_2, g_3$, take $\Gamma_+ \cong \Gamma_- \cong \mathbb{Z}/2$ and $\psi = \frac{\pi}{6}$.
Let \((M, g)\) be an extra twisted connected sum with \(\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}\) where \(k_+, k_- \in \{1, 2\}\) and \(\vartheta \in \{\pm \frac{\pi}{6}, \pm \frac{\pi}{4}, \pm \frac{\pi}{3}, \frac{\pi}{2}\}\)

In this case, 3 divides \(\bar{\nu}(M, g) = -72 \frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-)\)
Let \((M, g)\) be an extra twisted connected sum with \(\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}\) where \(k_{+}, k_{-} \in \{1, 2\}\) and \(\vartheta \in \{\pm \frac{\pi}{6}, \pm \frac{\pi}{4}, \pm \frac{\pi}{3}, \frac{\pi}{2}\}\)

In this case, 3 divides \(\bar{\nu}(M, g) = -72 \frac{\rho}{\pi} + 3m_{\rho}(L; N_{+}, N_{-})\)

Note that the \(G_{2}\)-bordism group in dimension 7 is \(\Omega_{G_{2}}^{7} \cong \mathbb{Z}/3\)

The \(\nu\)-invariant mod 3 is an isomorphism \(\nu: \Omega_{G_{2}}^{7} \to \mathbb{Z}/3\)

Hence \((M, \sigma)\) is \(G_{2}\)-nullbordant if and only if \(3 \mid \nu(M, \sigma)\)
Let \((M, g)\) be an extra twisted connected sum with \(\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}\) where \(k_+, k_- \in \{1, 2\}\) and \(\vartheta \in \{\pm \frac{\pi}{6}, \pm \frac{\pi}{4}, \pm \frac{\pi}{3}, \frac{\pi}{2}\}\).

In this case, 3 divides \(\bar{\nu}(M, g) = -72\frac{\rho}{\pi} + 3m_{\rho}(L; N_+, N_-)\)

Note that the \(G_2\)-bordism group in dimension 7 is \(\Omega^7_{G_2} \cong \mathbb{Z}/3\)

The \(\nu\)-invariant mod 3 is an isomorphism \(\nu: \Omega^7_{G_2} \to \mathbb{Z}/3\)

Hence \((M, \sigma)\) is \(G_2\)-nullbordant if and only if \(3 \mid \nu(M, \sigma)\)

**Question.** Is the \(G_2\)-bordism class of \((M, \sigma)\) an obstruction against holonomy \(G_2\)?
Let \((M, g)\) be an extra twisted connected sum with \(\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}\)
where \(k_+, k_- \in \{1, 2\}\) and \(\vartheta \in \{\pm \frac{\pi}{6}, \pm \frac{\pi}{4}, \pm \frac{\pi}{3}, \frac{\pi}{2}\}\)
In this case, 3 divides \(\bar{\nu}(M, g) = -72 \frac{p}{\pi} + 3m_{\rho}(L; N_+, N_-)\)

Note that the \(G_2\)-bordism group in dimension 7 is \(\Omega^7_{G_2} \cong \mathbb{Z}/3\)
The \(\nu\)-invariant mod 3 is an isomorphism \(\nu: \Omega^7_{G_2} \rightarrow \mathbb{Z}/3\)
Hence \((M, \sigma)\) is \(G_2\)-nullbordant if and only if \(3 \mid \nu(M, \sigma)\)

**Question.** Is the \(G_2\)-bordism class of \((M, \sigma)\) an obstruction against holonomy \(G_2\)?

**Answer.** No, there are examples with \(3 \nmid \nu(M, \sigma)\).

This is our motivation to consider more complicated extra twisted connected sums.
Computation of the $\nu$-invariant—$G_2$-Bordism

Let $(M, g)$ be an extra twisted connected sum with $\Gamma_{\pm} \cong \mathbb{Z}/k_{\pm}$ where $k_+, k_- \in \{1, 2\}$ and $\vartheta \in \{\pm \frac{\pi}{6}, \pm \frac{\pi}{4}, \pm \frac{\pi}{3}, \frac{\pi}{2}\}$

In this case, 3 divides $\bar{\nu}(M, g) = -72 \frac{\rho}{\pi} + 3m_{\rho}(L; N_+, N_-)$

Note that the $G_2$-bordism group in dimension 7 is $\Omega^7_{G_2} \cong \mathbb{Z}/3$

The $\nu$-invariant mod 3 is an isomorphism $\nu: \Omega^7_{G_2} \to \mathbb{Z}/3$

Hence $(M, \sigma)$ is $G_2$-nullbordant if and only if $3 \mid \nu(M, \sigma)$

Question. Is the $G_2$-bordism class of $(M, \sigma)$ an obstruction against holonomy $G_2$?

Answer. No, there are examples with $3 \nmid \nu(M, \sigma)$.

This is our motivation to consider more complicated extra twisted connected sums

Note. Recall that with $k_+ \geq 3$ or $k_- \geq 3$, we can get gluing angles $\vartheta \notin \mathbb{Q} \pi$

Because $\bar{\nu}(M, g) \in \mathbb{Z}$, expect $\bar{\nu}(M_{\pm}, g) \neq 0$ if $k_{\pm} > 2$
Let $M_\pm = V_\pm \times S^1_{\pm,\text{ext}}$, rescale $S^1_{\pm,\text{ext}}$ by $a > 0$ to get $M_{\pm,a}$.

The limit $a \to 0$ is called \textit{adiabatic limit}. 

We will focus on examples without isolated fixpoints.
Computation of the $\nu$-invariant—Adiabatic limits

Let $M_\pm = V_\pm \times S^1_{\pm,\text{ext}}$, rescale $S^1_{\pm,\text{ext}}$ by $a > 0$ to get $M_{\pm,a}$

The limit $a \to 0$ is called \textit{adiabatic limit}

From the adiabatic limit theorems of Bismut-Cheeger, Dai, G, we deduce

**Theorem (G-Nordström)**

Let $\gamma \in \Gamma_\pm$ be the generator that acts by $\frac{2\pi}{k_{\pm}}$ on $S^1_{\pm,\text{ext}}$

Let $V_{\pm,j}$ be the set of isolated fixpoints of $\gamma^j$ on $V_\pm$

Let $e^{i\alpha_{j,1}(p)}$, $e^{i\alpha_{j,2}(p)}$, $e^{i\alpha_{j,3}(p)}$ be the eigenvalues of $\gamma^j$ on $T_p V_\pm$ with $\alpha_{j,1}(p) + \alpha_{j,2}(p) + \alpha_{j,3}(p) = 0$. Then

$$\lim_{a \to 0} \bar{\nu}(M_{\pm,a}) = \frac{3}{k_{\pm}} \sum_{j=1}^{k_{\pm}-1} \cot \frac{\pi j}{k_{\pm}} \sum_{p \in V_{\pm,j}} \frac{\cos \frac{\alpha_{j,1}(p)}{2}}{\sin \frac{\alpha_{j,1}(p)}{2}} \frac{\cos \frac{\alpha_{j,2}(p)}{2}}{\sin \frac{\alpha_{j,2}(p)}{2}} \frac{\cos \frac{\alpha_{j,3}(p)}{2}}{\sin \frac{\alpha_{j,3}(p)}{2}} - 1 \in \mathbb{Q}$$

We will focus on examples without isolated fixpoints
Let $M_\pm = V_\pm \times S^1_{\pm,ext}$, rescale $S^1_{\pm,ext}$ by $a > 0$ to get $M_\pm, a$

The limit $a \to 0$ is called **adiabatic limit**

From the adiabatic limit theorems of Bismut-Cheeger, Dai, G, we deduce

**Theorem (G-Nordström)**

*Let $\gamma \in \Gamma_\pm$ be the generator that acts by $\frac{2\pi}{k_\pm}$ on $S^1_{\pm,ext}$

Let $V_{\pm,j}$ be the set of isolated fixpoints of $\gamma^j$ on $V_\pm$

Let $e^{i\alpha_{j,1}(p)}$, $e^{i\alpha_{j,2}(p)}$, $e^{i\alpha_{j,3}(p)}$ be the eigenvalues of $\gamma^j$ on $T_p V_\pm$ with $\alpha_{j,1}(p) + \alpha_{j,2}(p) + \alpha_{j,3}(p) = 0$. Then

$$\lim_{a \to 0} \bar{\nu}(M_\pm, a) = \frac{3}{k_\pm} \sum_{j=1}^{k_\pm - 1} \cot \frac{\pi j}{k_\pm} \sum_{p \in V_{\pm,j}} \cos \frac{\alpha_{j,1}(p)}{2} \cos \frac{\alpha_{j,2}(p)}{2} \cos \frac{\alpha_{j,3}(p)}{2} - 1 \in \mathbb{Q}$$

We will focus on examples without isolated fixpoints
By theorems of Bismut-Cheeger, Dai-Freed, the variational formula for the \( \eta \)-invariant of a Dirac type operator on a manifold with boundary consists of

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an \( \eta \)-form on the boundary

Theorem (G-Nordström)

\[
\overline{\nu}(M^\pm) - \lim_{a \to 0} \overline{\nu}(M^\pm, a) = F_{k^\pm, \epsilon^\pm}(s^\pm) = 288 \int_{s^\pm} \overline{\eta}(A)
\]

Bismut-Cheeger also give a formula for \( \overline{\eta}(A) \) as a sum over \( \mathbb{Z}_2 \) depending on \( a \).
Computation of the \( \nu \)-invariant—Variational formula

By theorems of Bismut-Cheeger, Dai-Freed, the variational formula for the \( \eta \)-invariant of a Dirac type operator on a manifold with boundary consists of:

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an \( \eta \)-form on the boundary

The interior contribution vanishes because \( M_{\pm,a} \) is locally a product
By theorems of Bismut-Cheeger, Dai-Freed, the variational formula for the $\eta$-invariant of a Dirac type operator on a manifold with boundary consists of

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an $\eta$-form on the boundary

The interior contribution vanishes because $M_{\pm,a}$ is locally a product.

The boundary contribution can be expressed in terms of the $\eta$-form $\tilde{\eta}(A)$ of the family of tori $(S^1_{\pm,int} \times aS^1_{\pm,ext})/\Gamma_{\pm}$ for $a \in (0, s_{\pm})$

**Theorem (G-Nordström)**

$$\bar{\nu}(M_{\pm}) - \lim_{a \to 0} \bar{\nu}(M_{\pm,a}) = F_{k_{\pm},\varepsilon_{\pm}}(s_{\pm}) = 288 \int_{0}^{s_{\pm}} \tilde{\eta}(A)$$

Bismut-Cheeger also give a formula for $\tilde{\eta}(A)$ as a sum over $\mathbb{Z}_2$ depending on $a$. 
By theorems of Bismut-Cheeger, Dai-Freed, the variational formula for the $\eta$-invariant of a Dirac type operator a manifold with boundary consists of

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an $\eta$-form on the boundary

The interior contribution vanishes because $M_{\pm,a}$ is locally a product.

The boundary contribution can be expressed in terms of the $\eta$-form $\tilde{\eta}(A)$ of the family of tori $(S^1_{\pm,\text{int}} \times aS^1_{\pm,\text{ext}})/\Gamma_{\pm}$ for $a \in (0, s_{\pm})$

**Theorem (G-Nordström)**

\[ \bar{\nu}(M_\pm) - \lim_{a \to 0} \bar{\nu}(M_{\pm,a}) = F_{k_\pm,\epsilon_\pm}(s_\pm) = 288 \int_0^{s_\pm} \tilde{\eta}(A) \]

Bismut-Cheeger also give a formula for $\tilde{\eta}(A)$ as a sum over $\mathbb{Z}^2$ depending on $a$. 

Computation of the $\nu$-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$

Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $SL(2, \mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi} dA_{\text{hyp}}$ by Bismut's family index theorem
Computation of the $\nu$-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}$

Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $SL(2, \mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi} dA_{\text{hyp}}$ by Bismut's family index theorem.

Idea. Use hyperbolic geometry to compute $F_{k-,\epsilon-}(s-) + F_{k+,\epsilon+}(s+)$
Represent the torus \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) by \( \tau \in \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \)

Then \( \tilde{\eta} \in \Omega^1(\mathcal{H}) \) is \( SL(2, \mathbb{Z}) \)-invariant and satisfies \( d\tilde{\eta} = -\frac{1}{4\pi} dA_{\text{hyp}} \) by Bismut's family index theorem

**Idea.** Use hyperbolic geometry to compute \( F_{k_-, \varepsilon_-}(s_-) + F_{k_+, \varepsilon_+}(s_+) \)

Adiabatic limits—geodesic rays

\[
\begin{align*}
\text{Cusps—families of adiabatic limits} \\
\text{Their contribution can be computed using a formula by Bunke and Ma}
\end{align*}
\]
Computation of the $\nu$-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$

Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $\text{SL}(2, \mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi} dA_{\text{hyp}}$ by Bismut's family index theorem

**Idea.** Use hyperbolic geometry to compute $F_{k-,\varepsilon-}(s-) + F_{k+,\varepsilon+}(s+)$

Adiabatic limits—geodesic rays

$\tilde{\eta}(A) = 0$ along families of rectangular ($k = 1$) or rhombic ($k = 2$) tori

\[ s_- \to 0 \]

\[ s_+ \to 0 \]
Computation of the $\nu$-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$

Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $\text{SL}(2, \mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi} dA_{\text{hyp}}$ by Bismut's family index theorem

Idea. Use hyperbolic geometry to compute $F_{k-,\varepsilon-}(s-) + F_{k+,\varepsilon+}(s+)$

Adiabatic limits—geodesic rays $\tilde{\eta}(\Lambda) = 0$ along families of rectangular ($k = 1$) or rhombic ($k = 2$) tori

Cusps—families of adiabatic limits

Their contribution can be computed using a formula by Bunke and Ma
Computation of the $\nu$-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ by $\tau \in \mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$
Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $SL(2, \mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi} dA_{hyp}$ by Bismut's family index theorem

Idea. Use hyperbolic geometry to compute $F_{k_-,\varepsilon_-}(s_-) + F_{k_+,\varepsilon_+}(s_+)$

Adiabatic limits—geodesic rays
$\tilde{\eta}(A) = 0$ along families of rectangular $(k = 1)$ or rhombic $(k = 2)$ tori
Cusps—families of adiabatic limits
Their contribution can be computed using a formula by Bunke and Ma

Compute $F_{k_-,\varepsilon_-}(s_-) + F_{k_+,\varepsilon_+}(s_+)$ using Stokes’ theorem from the cusp contributions and the hyperbolic area formula
The angle $2\vartheta$ at the finite corner cancels $-72\frac{\rho}{\pi}$ in the gluing formula
The logarithm of the Dedekind $\eta$-function is given by

$$L(\tau) = \frac{\pi i}{12} - \sum_{n=1}^{\infty} \sum_{d|n} d^{-1} e^{2\pi i n \tau}$$

Theorem (G-Nordström-Zagier)

There exists a constant $c_{k,\epsilon} \in \mathbb{Q}$ such that

$$F_{k,\epsilon}(s) = \frac{144}{\pi} \left( iL\left( \frac{s + i \epsilon}{k} \right) - iL\left( \frac{s - i \epsilon}{k} \right) + c_{k,\epsilon} \right)$$
Computation of the $\nu$-invariant—Modular functions

The logarithm of the Dedekind $\eta$-function is given by

$$L(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \sum_{d|n} \frac{1}{d} e^{2\pi in\tau}$$

**Theorem (G-Nordström-Zagier)**

There exists a constant $c_{k,\pm,\pm} \in \mathbb{Q}$ such that

$$F_{k,\pm,\pm}(s_{\pm}) = \frac{144}{\pi} \left( iL \left( \frac{s_{\pm}i + \varepsilon_{\pm}}{k_{\pm}} \right) - iL \left( \frac{s_{\pm}i - \varepsilon_{\pm}}{k_{\pm}} \right) + c_{k,\pm,\pm} \right)$$

Compute the variational term using the functional equations

$$L(\tau + 1) = \frac{\pi i}{12} + L(\tau) \quad \text{and} \quad L \left( \frac{-1}{\tau} \right) = \frac{1}{2} \log \frac{\tau}{i} + L(\tau)$$
Example. The example with $\cos \vartheta = \frac{1}{\sqrt{6}}$ has $\bar{\nu}(M, g) = -65$

In particular, $3 \nmid \bar{\nu}(M, g)$, so it is not $G_2$-nullbordant
Example. The example with \( \cos \vartheta = \frac{1}{\sqrt{6}} \) has \( \bar{\nu}(M, g) = -65 \). In particular, \( 3 \nmid \bar{\nu}(M, g) \), so it is not \( G_2 \)-nullbordant.

Conjecture

All values in \( \mathbb{Z}/48 \) occur as \( \nu \)-invariants of \( G_2 \)-holonomy metrics.
Example. The example with $\cos \vartheta = \frac{1}{\sqrt{6}}$ has $\bar{\nu}(M, g) = -65$

In particular, $3 \nmid \bar{\nu}(M, g)$, so it is not $G_2$-nullbordant

Conjecture

All values in $\mathbb{Z}/48$ occur as $\nu$-invariants of $G_2$-holonomy metrics

Questions

- How many different $G_2$-metrics exist on one 7-manifold?
- Are different $G_2$-metrics on a fixed 7-manifold $G_2$-bordant?
Example. The example with $\cos \vartheta = \frac{1}{\sqrt{6}}$ has $\bar{\nu}(M, g) = -65$
In particular, $3 \nmid \bar{\nu}(M, g)$, so it is not $G_2$-nullbordant

Conjecture

All values in $\mathbb{Z}/48$ occur as $\nu$-invariants of $G_2$-holonomy metrics

Questions

- How many different $G_2$-metrics exist on one 7-manifold?
- Are different $G_2$-metrics on a fixed 7-manifold $G_2$-bordant?

Construct more examples

- Find more asymptotically cylindrical Calabi-Yau manifolds
- Understand their moduli space, make the K3 surfaces match
- Consider other constructions
Thanks for your attention!