M-theory and String Theory S-Matrix From CFT

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Based on:

- arXiv:1804.00949 with S. Chester and X. Yin
- arXiv:1808.10554 with D. Binder and S. Chester
- work in progress with D. Binder, S. Chester, and Y. Wang

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Motivation

- Two dreams of AdS/CFT:
  - learn about quantum gravity / string theory / M-theory from CFT
  - learn about strongly-interacting QFTs from the bulk

  This talk: a bit of both, although new results only for the 2nd
- Most well-established examples of AdS/CFT:
  - 4d $SU(N)$ $\mathcal{N} = 4$ SYM at large N and large 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ / type IIB strings on $AdS_5 \times S^5$
  - 3d $U(N)_k \times U(N)_{-k}$ ABJM theory at large N / M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. (I'll take $k = 1$.)
- Both have maximal SUSY.

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Goal: make a connection between three topics:

1. 4-pt scattering amplitudes in 10d and 11d (of gravitons and superpartners)
2. CFT 4-pt functions (crossing, Ward ids, analytical and numerical bootstrap)
3. exact results from SUSic localization

In particular:

- Use (2) and (3) to reconstruct string theory / M-theory S-matrix perturbatively at small momentum. Or, equivalently, reconstruct effective action beyond SUGRA

$$S = \int d^d x \sqrt{g} \left[R + \text{Riem}^4 + \cdots + (\text{SUSic completion}) \right] .$$

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S-matrix in M-theory

- In 11d, we can scatter: gravitons, gravitini, 3-form gauge particles.
- For 4 particles, momenta can be taken to lie within a 4d subspace.
- Momenta within 4d $\Longrightarrow$ can use 4d $\mathcal{N} = 8$ language. We scatter:
  - graviton (1);
  - gravitinos (8);
  - gravi-photons (28);
  - gravi-photinos (56);
  - scalars ($70 = 35 + 35$)
- At **leading** order in small momentum (i.e. p^2), scattering amplitudes are those in $\mathcal{N} = 8$ SUGRA at tree level. Example:

$$\mathcal{A}_{\text{SUGRA, tree}}(S_1 S_1 S_2 S_2) = \frac{tu}{s},$$

where $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_1 + p_3)^2$.
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Momentum expansion

- Momentum expansion takes a universal form (independent of the type of particle):

$$\mathcal{A} = \mathcal{A}_{\text{SUGRA, tree}} \Big(1 + \ell_p^6 f_{R^4}(s,t) + \ell_p^9 f_{\text{1-loop}}(s,t) + \ell_p^{12} f_{D^6 R^4}(s,t) + \ell_p^{14} f_{D^8 R^4}(s,t) + \cdots \Big) \,.$$

- $f_{D^{2n} R^4}$ = symmetric polyn in s, t, u of degree $n + 3$
- Known from type II string theory + SUSY [Green, Tseytlin, Gutperle, Vanhove, Russo, Pioline, ...] :

$$f_{R^4}(s,t) = \frac{stu}{3 \cdot 2^7} \,, \qquad f_{D^6 R^4}(s,t) = \frac{(stu)^2}{15 \cdot 2^{15}} \,.$$

- $\ell_p^{10} f_{D^4 R^4}$ allowed by SUSY, but known to vanish.

S-matrix in type IIB string theory

- In 10d (type IIB), we scatter: gravitons, gravitini, ...
- Take momenta to lie within a 5d subspace $\Longrightarrow$ can use 5d $\mathcal{N} = 8$ language.
  - Scatter: graviton (1); gravitino (8); ...; scalars ($42 = 20 + 20 + 2$)
- Double expansion in string length ℓ_s and string coupling g_s:

$$\mathcal{A} = \mathcal{A}^{\text{tree}}_{\text{SUGRA}} \left[\left(1 + \ell_s^6 f_{R^4}(s,t) + \ell_s^{10} f_{D^4R^4}(s,t) + \ell_s^{12} f_{D^6R^4}(s,t) + \cdots \right) + g_s^2(\cdots) + g_s^4(\cdots) + \cdots \right].$$

- At $O(g_s^0)$ this is completely known (ratio of Gamma functions). At each order in ℓ_s the coefficient is a polyn. in s, t, u.

$$f_{R^4} = \frac{\zeta(3)}{32} stu\,, \quad f_{D^4R^4} = \frac{\zeta(5)}{2^{10}} stu(s^2 + t^2 + u^2)\,, \quad f_{D^6R^4} = \frac{\zeta(3)^2}{2^{11}} (stu)^2\,, \ldots$$

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This talk:

Reproduce small momentum expansion from CFT:

- 11d: Reproduce f_{R^4} and $f_{D^4R^4} = 0$ from ABJM theory.
- 10d: Reproduce f_{R^4} from $\mathcal{N} = 4$ SYM.

Use known data on scattering amplitudes + other tools:

- 11d: Use known $f_{D^6R^4}$ + other info to determine the first few terms in the $1/N$ expansion of the stress tensor multiplet 4-pt function.
- 10d: Use known $f_{D^6R^4}$ + other info to determine:
  - Full $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle_{\text{conn}}$ correlator (where $\mathcal{O}_2 = \mathcal{O}_{\mathbf{20}'}$ is the $1/2$-BPS superconformal primary of the stress tensor multiplet) in planar limit up (order $1/N^2$) to order $1/\lambda^{5/2}$ (3 orders in the $1/\lambda$ expansion: 1, $1/\lambda^{3/2}$, $1/\lambda^{5/2}$)
  - Full planar $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_p\mathcal{O}_p \rangle_{\text{conn}}$ correlator to same order.
  - Extract leading non-planar corrections (order $1/N^4$) to anomalous dimensions of unprotected operators to order $1/\lambda^{5/2}$.

Flat space limit of CFT correlators

- Idea: Flat space scattering amplitudes can be obtained as limit of CFT correlators [Polchinski '99; Susskind '99; Giddings '99; Penedones '10; Fitzpatrick, Kaplan '11] .
- Example: For a CFT operator $\phi(x)$,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_{\text{conn}} = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} g(U, V)$$

with $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$, go to Mellin space

$$g(U, V) = \int \frac{ds\, dt}{(4\pi i)^2} U^{s/2} V^{(t-2\Delta_\phi)/2} \Gamma^2\left(\Delta_\phi - \frac{s}{2}\right) \Gamma^2\left(\Delta_\phi - \frac{t}{2}\right) \Gamma^2\left(\Delta_\phi - \frac{u}{2}\right)$$

where $s + t + u = 4\Delta_\phi$.

- From the large s, t limit of $M(s, t)$ one can extract scattering amplitude $\mathcal{A}(s, t)$ [Penedones '10; Fitzpatrick, Kaplan '11] .

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I'll first talk about 3d ABJM / 11d, then about 4d $\mathcal{N} = 4$ SYM / 10d.

Flat space limit of CFT correlators: M-theory case

If L is the radius of AdS (and $L/2$ is the radius of S^7), then

$$\mathcal{A}(s,t) \propto \lim_{L\to\infty} \mathcal{N}(L) \int_{c-i\infty}^{c+i\infty} d\alpha\, e^{\alpha} \alpha^{-1/2} M\left(-\frac{L^2}{4\alpha}s, -\frac{L^2}{4\alpha}t\right)$$

- Expect $M(s,t)$ to have a series expansion in $\ell_p/L \propto N^{-1/6}$.
- At leach order in ℓ_p/L, it is only the large s, t behavior of $M(s,t)$ that contributes to $\mathcal{A}(s,t)$.
- In order for $\mathcal{A}(s,t)$ to have an expansion in ℓ_p times momentum, we need

$$M = \frac{1}{\mathcal{N}(L)} \sum_{k=1}^{\infty} \left(\frac{\ell_p}{L}\right)^{2k} \text{(function that grows as } k\text{th power of } s,t,u)$$

- Instead of $1/N$ or ℓ_p/L I will use $1/c_T$, where $\langle T_{\mu\nu} T_{\rho\sigma}\rangle \propto c_T$.

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- To obtain scattering amplitude of graviton + superpartners in M-theory, look at **stress tensor multiplet** in $k = 1$ ABJM theory:
- Stress tensor multiplet of any $\mathcal{N} = 8$ SCFT (R-symm is $\mathfrak{so}(8)$):

	dimension	spin	$\mathfrak{so}(8)_R$	couples to
focus on this $\longrightarrow$	1	0	$\mathbf{35}_c$	scalars
	3/2	1/2	$\mathbf{56}_v$	gravi-photinos
focus on this $\longrightarrow$	2	0	$\mathbf{35}_s$	pseudo-scalars
	2	1	$\mathbf{28}$	gravi-photons
	5/2	3/2	$\mathbf{8}_v$	gravitinos
	3	2	$\mathbf{1}$	graviton

- Easier to look at scalars than at operators with spin.
- There are 35 $\Delta = 1$ scalars S_{IJ} (traceless symmetric) and 35 $\Delta = 2$ pseudo-scalars P_{AB} (traceless symmetric).
- Task: find the Mellin amplitudes M_{SSSS} (6 fns), M_{SSPP} (3 fns), M_{PPPP} (6 fns) in the $1/c_T$ expansion, and then take flat space limit.

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Ward identity

- $\langle SSSS \rangle = \frac{1}{x_{12}^2 x_{34}^2} \times$ 6 functions $\mathcal{S}_i(U, V)$, $i = 1, \ldots 6$
- $\langle SSPP \rangle = \frac{1}{x_{12}^2 x_{34}^4} \times$ 3 functions $\mathcal{R}_i(U, V)$, $i = 1, \ldots 3$
- $\langle PPPP \rangle = \frac{1}{x_{12}^4 x_{34}^4} \times$ 6 functions $\mathcal{P}_i(U, V)$, $i = 1, \ldots 6$

- The $\mathcal{S}_i(U, V)$ obey differential relations [Dolan, Gallot, Sokatchev '04] . Example:

$$\partial_U \mathcal{S}_4(U, V) = \frac{1}{U}\mathcal{S}_4(U, V) + \left(\frac{1}{U} - \partial_U - \partial_V\right)\mathcal{S}_2(U, V) + \left(\frac{1}{U} + (U-1)\partial_U + V\partial_V\right)\mathcal{S}_3(U, V)\,,$$

$$\partial_V \mathcal{S}_4(U, V) = -\frac{1}{2V}\mathcal{S}_4(U, V) - \frac{1}{V}\left(1 - U\partial_U + (U-1)\partial_V\right)\mathcal{S}_2(U, V) - (\partial_U + \partial_V)\,\mathcal{S}_3(U, V)\,.$$

- One can derive differential relations relating $\mathcal{R}_i$ and $\mathcal{P}_i$ to $\mathcal{S}_i$ [Binder, Chester, SSP '18] . (Pretty hard, but we wrote Mathematica code!)

$$\mathcal{R}_1(U, V) = \frac{1}{4}\Big[4 + \left(U^2 - 4U\right)\partial_U + \left(4 + U - 2U^2 + 7UV - 4V^2\right)\partial_V$$
$$+ 2U(2V - U + 2)(U\partial_U^2 + (U + V - 1)\partial_U\partial_V + V\partial_V^2)\Big]\mathcal{S}_1(U, V)\,,\ \text{etc.}$$

CFT 4-pt correlators at large N

- The requirements
  - The Mellin amplitudes obey SUSY Ward identities;
  - The Mellin amplitudes are consistent with crossing symmetry;
  - At order $1/c_T^{\frac{7}{9}+\frac{2}{9}n}$, the Mellin amplitude grows at most as the n^{th} power of s, t, u;
  - The Mellin amplitudes have the analytic properties appropriate for tree-level Witten diagrams

determine M_{SSSS}, M_{SSPP}, M_{PPPP} up to a few constants whose number depends on n.

- Number of solutions (only deg. 1 one has poles):

degree in s, t, u	1	2	3	4	5	6	7	...
11D vertex	R			R^4		$D^4 R^4$	$D^6 R^4$	...
scaling	c_T^{-1}			$c_T^{-\frac{5}{3}}$		$(0\times)c_T^{-\frac{19}{9}}$	$c_T^{-\frac{7}{3}}$	
# of params	1			2		3	4	...

(degree 1 in [Zhou '18]); degree $\geq$ 2 in [Chester, SSP, Yin '18] .)

- The number of solutions matches number of solutions to the Ward identity for the flat space scattering amplitudes.
  So:

- To determine $M(s,t)$ to order $1/c_T$ we should compute **one** CFT quantity.
- To determine $M(s,t)$ to order $1/c_T^{5/3}$ we should compute **two** CFT quantities. Etc.
- Luckily, we can compute **three** CFT quantities (one being c_T) and relate them to $M(s,t)$.

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What to compute

- What we can compute: mass deformed S^3 partition function $Z(m_1, m_2) = e^{-F(m_1, m_2)}$ as a function of two masses m_1, m_2.
- How to compute: use SUSic localization to write $Z(m_1, m_2)$ as a finite dim'l integral [Kapustin, Willett, Yaakov '09] ; then use statistical physics techniques to extract $F(m_1, m_2)$ to all orders in the $1/N$ expansion! [Marino, Putrov '11; Nosaka '15]
- The following quantities

$$\left.\frac{\partial^2 F}{\partial m_1^2}\right|_{m_1=m_2=0}, \qquad \left.\frac{\partial^4 F}{\partial m_1^4}\right|_{m_1=m_2=0}, \qquad \left.\frac{\partial^4 F}{\partial m_1^2 \partial m_2^2}\right|_{m_1=m_2=0}$$

can be related to $\langle SSSS \rangle$, $\langle SSPP \rangle$, and $\langle PPPP \rangle$ and thus can be used to determine $M(s, t)$ up to order $1/c_T^{19/9}$.

Localization result

- Using supersymmetric localization [Kapustin, Willett, Yaakov '09] :

$$Z_{S^3}(m_1, m_2) = \int d^N\lambda\, d^N\mu\, \frac{e^{ik\sum_i(\lambda_i^2-\mu_i^2)} \prod_{i<j}\sinh^2(\lambda_i-\lambda_j)\sinh^2(\mu_i-\mu_j)}{\prod_{i,j}\cosh(\lambda_i-\mu_j+\frac{m_1}{2})\cosh(\lambda_i-\mu_j+\frac{m_2}{2})}$$

- Small N: can evaluate integral exactly.
- Large N: rewrite $Z_{S^3}(m)$ as the partition function of non-interacting Fermi gas of N particles with [Marino, Putrov '11; Nosaka '15]

$$U(x) = \log(2\cosh x) - m_1 x\,, \qquad T(p) = \log(2\cosh p) - m_2 p\,.$$

Resummed perturbative expansion [Nosaka '15] :

$$Z_{S^3}(m) \sim \mathrm{Ai}\left(f_1(m_1, m_2)N - f_2(m_1, m_2)\right)$$

for some known functions f_1 and f_2. ($\log Z \propto N^{3/2}$ as expected from 11d SUGRA)

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Integrated correlators

How to relate mass derivatives to $\langle SSSS \rangle$, $\langle SSPP \rangle$, $\langle PPPP \rangle$?

- In Lagrangian, mass deformations are

$$\sum_{i=1}^{2} m_i \int d^3x \left(\frac{i}{r} J_i + K_i \right) + O(m^2)$$

- The $\Delta = 1$ ops J_i are linear combinations of the $\mathbf{35}_c$ scalars S_{IJ}.
- The $\Delta = 2$ ops K_i are linear combinations of the $\mathbf{35}_s$ pseudoscalars P_{IJ}.
- $\frac{\partial^2 F}{\partial m_i^2} = -\left\langle \left(\int \frac{i}{r} J_i + \int K_i \right)^2 \right\rangle$ = integrated 2-pt fn.
- $\frac{\partial^4 F}{\partial m_i^2 \partial m_j^2} = -\left\langle \left(\int \frac{i}{r} J_i + \int K_i \right)^2 \left(\int \frac{i}{r} J_j + \int K_j \right)^2 \right\rangle$ = integrated 4-pt fn.

Integrated 2-pt function

- Because S and P belong to the same multiplet as the stress tensor, $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \propto c_T$ implies $\langle SS \rangle \propto c_T$ and $\langle PP \rangle \propto c_T$.
- Then $\langle JJ \rangle \propto c_T$ and $\langle KK \rangle \propto c_T$, and

$$\left.\frac{\partial^2 F}{\partial m_i^2}\right|_{m=0} = \frac{\pi^2}{32} c_T \,.$$

But what does this have to do with the 4-pt function?

- (Super)Conformal block decomposition

$$\langle SSSS \rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_{\text{multiplets } \mathcal{M}} \lambda_{\mathcal{M}}^2 g_{\mathcal{M}}(U, V)$$

- Then, in a normalization where $\lambda_{\text{id}}^2 = 1$,

$$\lambda_{\text{stress}}^2 = \frac{\langle SST_{\mu\nu} \rangle \langle SST_{\rho\sigma} \rangle}{\langle T_{\mu\nu} T_{\rho\sigma} \rangle} = \frac{256}{c_T} \,.$$

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Four mass derivatives

Expressing the four mass derivatives in terms of c_T:

$$-\frac{1}{c_T^2}\frac{\partial^4 F}{\partial m_1^4} = \frac{3\pi^2}{64}\frac{1}{c_T} + \frac{(3\pi)^{4/3}}{2^{10/3}}\frac{1}{c_T^{5/3}} + \frac{(\cdots)}{c_T^2} - (18\pi^2)^{1/3}\frac{1}{c_T^{7/3}} + \cdots$$

$$-\frac{1}{c_T^2}\frac{\partial^4 F}{\partial m_1^2 \partial m_2^2} = -\frac{\pi^2}{64}\frac{1}{c_T} + \frac{5\pi^{4/3}}{2^{10/3}3^{2/3}}\frac{1}{c_T^{5/3}} + \frac{(\cdots)}{c_T^2} - \frac{4(2\pi^2)^{1/3}}{3^{10/3}}\frac{1}{c_T^{7/3}} + \cdots$$

Impose these constraints on the 4-pt functions $\langle SSSS\rangle$, $\langle SSPP\rangle$, $\langle PPPP\rangle$ using

$$\frac{\partial^4 F}{\partial m_i^2 \partial m_j^2} = -\left\langle \left(\int \frac{i}{r} J_i + \int K_i\right)^2 \left(\int \frac{i}{r} J_j + \int K_j\right)^2 \right\rangle$$

Integrated 4-pt function

- If $\langle AABB \rangle = \frac{1}{x_{12}^{2\Delta_A} x_{34}^{2\Delta_B}} G(U, V)$ in flat space, then the integrated correlator on S^3 (with metric $ds^2 = \Omega^{-2}(\vec{x}) d\vec{x}^2$) is

$$\left\langle \left(\int A \right)^2 \left(\int B \right)^2 \right\rangle = \int \prod_{i=1}^{4} d^3 \vec{x}_i \frac{(\Omega(\vec{x}_1)\Omega(\vec{x}_2))^{\Delta_A - 3} (\Omega(\vec{x}_3)\Omega(\vec{x}_4))^{\Delta_B - 3}}{x_{12}^{2\Delta_A} x_{34}^{2\Delta_B}} G$$

  with $\Omega(\vec{x}) = 1 + \frac{\vec{x}^2}{4r^2}$.

- This non-conformal integral can be written as

$$\left\langle \left(\int A \right)^2 \left(\int B \right)^2 \right\rangle \propto \int dU\, dV\, \bar{D}_{3-\Delta_A, 3-\Delta_A, 3-\Delta_B, 3-\Delta_B}(U, V) \frac{G(U, V)}{U^{\Delta_A}} .$$

  where $\bar{D}$ function is the (Euclidean) AdS contact Witten diagram for the 4-pt function of ops of dim $3 - \Delta_A$, $3 - \Delta_A$, $3 - \Delta_B$, $3 - \Delta_B$.

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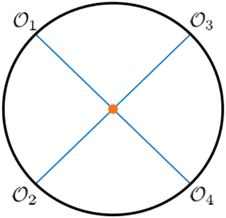
$\bar{D}$ function



$$= D_{r_1,r_2,r_3,r_4}(\vec{x}_i) = \int \frac{dz_0 d^d\vec{z}}{z_0^{d+1}} \prod_{i=1}^4 G_{B\partial}^{r_i}(z_0,\vec{z};\vec{x}_i)$$

$$G_{B\partial}^{r_i}(z_0,\vec{z};\vec{x}_i) = \left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x}_i)^2}\right)^{r_i}$$

The $\bar{D}$ function is defined as

$$\bar{D}_{r_1,r_2,r_3,r_4}(U,V) = \frac{x_{13}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_4} x_{24}^{r_2}}{x_{14}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_1 - r_4} x_{34}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_3 - r_4}} \frac{2\prod_{i=1}^4 \Gamma(r_i)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{-d+\sum_{i=1}^4 r_i}{2}\right)} D_{r_1,r_2,r_3,r_4}(x_i)$$

- The reason why $\bar{D}$ functions appear in the integrated 4-pt functions on S^3 is that $SO(4,1)/SO(4) = \mathbb{H}^4$.

Summary of computation

- Superconformal Ward id + asymptotic growth in Mellin space + crossing symmetry + analytic structure of Mellin tree amplitudes $\Longrightarrow$ determine Mellin amplitudes M_{SSSS}, M_{SSPP}, M_{PPPP} in $1/c_T$ expansion up to a few undetermined constants at each order
- SUSic localization $\Longrightarrow Z_{S^3}(m_1, m_2) \Longrightarrow F(m_1, m_2)$ in $1/c_T$ expansion $\Longrightarrow$ 2nd and 4th derivatives of $F(m_1, m_2)$ in $1/c_T$ expansion $\Longrightarrow$ conditions on integrated correlators $\langle SSSS\rangle$, $\langle SSPP\rangle$, $\langle PPPP\rangle$.
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Precision test of AdS/CFT

- The flat space limit implies $f_{R^4}(s,t) = \frac{stu}{3 \cdot 2^7}$ and $f_{D^4 R^4} = 0$, as expected.

- **This is a precision test of AdS/CFT beyond supergravity!!**

- To go beyond $f_{D^4 R^4}$ one needs to compute more CFT quantities. But: if we'e interested in the CFT correlator at order $1/c_T^{7/3}$ (corresponding to $D^6 R^4$) one can use $f_{D^6 R^4} = (stu)^2/(3 \cdot 2^7)$ as **input**.
  - Extract CFT data at this order. Dim of lowest unprotected singlet

$$\Delta = 2 - \frac{1120}{\pi^2 c_T} - \frac{71680 \cdot 6^{\frac{1}{3}}}{\pi^{\frac{8}{3}} c_T^{\frac{5}{3}}} + \frac{a}{c_T^2} - \frac{75286.7}{c_T^{\frac{7}{3}}} + \cdots .$$

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4d $\mathcal{N} = 4$ SYM

- To obtain scattering amplitudes of graviton + superpartners in type IIB string theory, look at **stress tensor multiplet** of $\mathcal{N} = 4$ SYM:
- Stress tensor multiplet of any $\mathcal{N} = 8$ SCFT (R-symm is $\mathfrak{so}(8)$):

dimension	spin	$\mathfrak{so}(6)_R$	couples to
2	0	$\mathbf{20}'$ (real)	scalars
3	0	$\mathbf{10}$ (complex)	pseudo-scalars
4	0	$\mathbf{1}$ (complex)	dilaton, axion
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	2	$\mathbf{1}$ (real)	graviton

- There are 20 $\Delta = 2$ scalars S_{IJ} ($I, J = 1, \ldots, 6$); 10 $\Delta = 3$ pseudo-scalars P_{AB} and $\bar{P}^{AB}$ (traceless symmetric); two marginal operators Φ and $\bar{\Phi}$.

CFT 4-pt correlators at large N

- For $\langle SSSS\rangle$, $\langle SSP\bar{P}\rangle$, ... consider a double expansion in $1/N$ and $1/\lambda$, or equivalently in $1/c_T$ and $1/\lambda$. ($c_T = (N^2 - 1)/4$)
- To reproduce tree level amplitude, want to find $\mathcal{M}_{SSSS}$, $\mathcal{M}_{SSP\bar{P}}$, ... at order $1/c_T$ (leading order), and take flat space limit.
- The requirements
  - The Mellin amplitudes obey SUSY Ward identities;
  - The Mellin amplitudes are consistent with crossing symmetry;
  - At order $1/c_T \times 1/\lambda^{\frac{n-1}{2}}$, the Mellin amplitude grows at most as the n^{th} power of s, t, u;
  - The Mellin amplitudes have the analytic properties appropriate for tree-level Witten diagrams

  determine M_{SSSS}, $M_{SSP\bar{P}}$, ... up to a few constants whose number depends on n.

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- Number of solutions:

degree in s, t, u	1	2	3	4	5	6	7	...
10D vertex	R			R^4		D^4R^4	D^6R^4	...
scaling	c_T^{-1}			$c_T^{-1}\lambda^{-\frac{3}{2}}$		$c_T^{-1}\lambda^{-\frac{5}{2}}$	$c_T^{-1}\lambda^{-3}$	
# of params	1			2		3	4	...

- The number of solutions matches number of solutions to the Ward identity for the flat space scattering amplitudes.
- In this case, we compute **one** quantity (besides $c_T = (N^2 - 1)/4$) and thus determine $M(s, t)$ to order $1/c_T \times 1/\lambda^{3/2}$.

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S^4 partition function

- For mass deformed $\mathcal{N} = 4$ SYM (a.k.a. $\mathcal{N} = 2^*$ theory) on S^4, compute $Z_{S^4}(m, \lambda) = e^{-F(m,\lambda)}$ with

$$Z(m, \lambda) = \int d^{N-1}a \prod_{i<j} \frac{(a_i - a_j)^2 H^2(a_i - a_j)}{H(a_i - a_j - m)H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2} |Z_{\text{inst}}|^2 \, .$$

- In the strong coupling limit the instantons don't contribute.
- Can compute at leading order in $1/N$:

$$-64\pi^2 \frac{\partial_m^2 \partial_\tau \partial_{\bar{\tau}} F}{\lambda^2}\bigg|_{m=0} = 1 - \frac{12\zeta(3)}{\lambda^{\frac{3}{2}}} + \frac{45\zeta(5)}{\lambda^{\frac{5}{2}}} + \cdots \, ,$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}$ and $\lambda = g_{\text{YM}}^2 N$.

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where $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g_{\text{YM}}^2}$ and $\lambda=g_{\text{YM}}^2N$.

- In $\mathcal{N}=4$ SYM, m couples to an integrated operator $\int\left(\frac{i}{r}J+K\right)$ where, if (ϕ_i,χ_i), $i=1,2,3$ are the 3 adjoint chirals of $\mathcal{N}=4$ SYM, we have

$$J=\operatorname{tr}\phi_1^2+\operatorname{tr}\phi_2^2+\text{h.c.}=\text{lin. combo of } \mathbf{20}' \text{ ops } S$$
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- SUSic Ward identities relating $\langle SSSS \rangle$ to $\langle SSP\bar{P} \rangle$ implies further

$$-\partial_m^2 \partial_\tau \partial_{\bar{\tau}} F = \int dr\, d\theta\, r^3 \sin^2\theta\, \frac{r^2 - 1 - 2r^2 \log r}{(r^2-1)^2} \frac{\mathcal{G}(1 + r^2 - 2r\cos\theta, r^2)}{(1 + r^2 - 2r\cos\theta)^2},$$

  where $\mathcal{G}$ is one of the six functions appearing in $\langle SSSS \rangle$.

- Using this, the general solution for the tree $\langle SSSS \rangle$ at order $1/c_T$ expanded in $1/\lambda$, and the asymptotic expansion in $1/\lambda$ of the LHS, we can fix $\langle SSSS \rangle$ to order $1/c_T \times 1/\lambda^{3/2}$. Then, from flat space limit, find

$$\mathcal{A} = \mathcal{A}_{\text{SUGRA}}^{\text{tree}} \left[1 + \frac{\zeta(3) stu}{32} + \cdots \right].$$

Precision test of $\text{AdS}_5/\text{CFT}_4$ beyond SUGRA!

Learn about CFT

- To obtain planar $\langle SSSS \rangle_{\text{conn}}$ to order $1/\lambda^{5/2}$ we would need to know **two** quantities in addition to c_T.
- One is from $\partial_m^2 \partial_\tau \partial_{\bar\tau} F$. The other can be fixed from the flat space limit using the string theory amplitude $\mathcal{A}_{D^4R^4} = \mathcal{A}_R \frac{(s^2+t^2+u^2)stu\zeta(5)}{2^{10}}$.
- With $u = 4 - s - t$, we have

$$M_{SSSS} = (\text{free part}) + (\text{R-symm})\left[\frac{1}{2(s-2)(t-2)(u-2)} + \frac{15}{2}\zeta(3)\frac{1}{\lambda^{\frac{3}{2}}} + \frac{315}{8}\zeta(5)\left(s^2+t^2+u^2-3\right)\frac{1}{\lambda^{\frac{5}{2}}} + \cdots\right]\frac{1}{N^2} + O(N^{-4})$$

- Dimension of the lowest double trace operator is then

$$\Delta = 4 - \frac{1}{N^2}\left[16 + \frac{17280\zeta(3)}{7\lambda^{\frac{3}{2}}} + \frac{122400\zeta(5)}{\lambda^{\frac{5}{2}}} + O(\lambda^{-3})\right] + O(N^{-4}).$$

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Higher 1/2-BPS operators

- Can generalize the above analysis to higher BPS operators $\langle SSS_pS_p \rangle$ where "S_p" is the 1/2-BPS operator in $[0p0]$ of $SO(6)$.

Comments:

- S^4 partition function is a modification of Pestun's. At $m = 0$, Hermitian matrix model with polynomial potential [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, SSP]

$$Z(m,\lambda) = \int d^{N-1}a \prod_{i<j} \frac{(a_i - a_j)^2 H^2(a_i - a_j)}{H(a_i - a_j - m)H(a_i - a_j + m)} e^{-\frac{8\pi^2 N}{\lambda}\sum_i a_i^2} |e^{i\sum_p \tilde{\tau}_p \sum_i a_i^p}|^2 |Z_{\text{inst}}|^2 .$$

- Need to solve Hermitian matrix model with polynomial potential at $m = 0$ and do Gram-Schimidt on $\partial_{\tau_p}\partial_{\bar{\tau}_p} F\big|_{m=\tau_p=0}$. After this, perturb w.r.t. m. Surprisingly simple formula:

$$\partial_m^2 \partial_{\tau_p}\partial_{\bar{\tau}_p} F\Big|_{m=\tau_p=0} \propto \int d\omega \, \frac{J_1(\sqrt{\lambda}\omega/\pi)^2 - J_p(\sqrt{\lambda}\omega/\pi)^2}{\sinh^2 \omega} .$$

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Higher 1/2-BPS operators

- In the end, after using [Alday, Bissi, Perlmutter '18] , get

$$M_{SSS_pS_p} = (\text{free}) + (\text{R-symm})\left[\frac{1}{(s-2)(t-p)(u-p)} + \frac{(p+1)_3}{4}\zeta(3)\frac{1}{\lambda^{\frac{3}{2}}} + \frac{(p+1)_5}{32}\zeta(5)\left(s^2+t^2+u^2+A(p)s+B(p)\right)\frac{1}{\lambda^{\frac{5}{2}}} + \cdots\right]\frac{1}{N^2} + O(N^{-4})$$

  where $u = 2p - s - t$, and

$$A(p) = \frac{2p(p-2)}{5+p}\,, \qquad B(p) = -2p^2 + \frac{50+20p(p+2)}{(p+4)(p+5)}\,.$$

- From this expression, one can use [Alday Bissi; Aprile, Drummond, Heslop, Paul '17, '18; Alday, Bissi, Perlmutter '18] to extract anomalous dimensions at $O(1/N^4)$.

Conclusion

- A combination of techniques (supersymmetric localization, SUSY Ward identities, Mellin space) can be used to recover graviton scattering amplitudes (at small momentum) in 11d and 10d from ABJM theory in 3d and $\mathcal{N} = 4$ SYM in 4d.
- In 11d, we can reproduce the $f_{R^4}(s,t) = \frac{stu}{3 \cdot 2^7}$ and show that $f_{D^4 R^4} = 0$. In 10d, we can reproduce $f_{R^4}(s,t) = \frac{\zeta(3) stu}{32}$.
- We can use SUSic localization + flat space limit to compute $\langle 22pp \rangle$ in 4d $\mathcal{N} = 4$ SYM to three non-trivial orders in $1/\lambda$.

For the future:

- Generalize to other dimensions, other 4-point function, higher-point functions, less SUSY, etc. (See also [Chester, Perlmutter '18] in 6d.)
- More about loops in AdS.

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