Spherical spin glass models

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Spin glasses

**Glass** *(positional disorder)*
Amorphous solid lacking periodic structure. Formed by rapid cooling.

**Spin Glass** *(magnetic disorder)*
Disordered magnet. Spins are not aligned in a regular structure.
Spin glass models on the lattice (Edwards-Anderson)

- **Configuration space:**

Assignments of $+1$ and $-1$ to the sites of the lattice.

$x = (x_1, x_2, \ldots, x_N), \ x_i \in \{+1, -1\}, \ N = n^2$

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- **(Random) Energy function:**
  
  Associate to each configuration $x$ an energy:
  
  $$H_N(x) = \sum_{i \sim j} J_{ij} x_i x_j.$$

  $J_{ij} \sim N(0, 1)$, independent, $i \sim j = “$neighboring sites”$. 
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Configurations with low energy?
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Configurations with low energy? **Too difficult!!!**
Spherical spin glass models

We modify the model:

1. "Mean-field" model: all spins interact with each other,
   \[ H_N(x) = \frac{1}{\sqrt{N}} \sum_{1 \leq i, j \leq N} J_{ij} x_i x_j. \]

2. Change the configuration space: the spins \( x_i \) are allowed to take real values instead of \( \pm 1 \), however, still with \( \|x\| = \sqrt{\sum x_i^2} = \sqrt{N} \).

3. Generalize to \( p \)-body interactions with general \( p \geq 2 \),
   \[ H_N(x) = \frac{1}{N^{p-1}/2} \sum_{1 \leq i_1, \ldots, i_p \leq N} J_{i_1, \ldots, i_p} x_{i_1} \cdots x_{i_p}. \]
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is a random energy landscape on \( S^{N-1}(\sqrt{N}) \) the sphere of radius \( \sqrt{N} \) in \( \mathbb{R}^N \). We are interested in very large dimension \( N \).
- At equilibrium the spin system should have low energy.
Low energies

- At equilibrium the spin system should have low energy.
- How many “valleys” are there at any depth?
Local minima

Theorem (Auffinger-Ben Arous-Černý, ‘13; S. ‘16)

The number of local minima below $-NE \approx e^{N\Theta_p(-E)}$. 
Equilibrium (Gibbs/Boltzmann) distribution

- The equilibrium (Gibbs/Boltzmann) density at temperature $T$ is

$$f(x) = e^{-\frac{1}{T}H_N(x)/Z}.$$ 

- $Z$ is chosen so that the total mass is 1.

- At equilibrium

$$\text{Prob}\left\{ \text{spins configuration } \in A \right\} = \int_A f(x) \, dx.$$
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What are the most probable configurations?
Equilibrium (Gibbs/Boltzmann) distribution

- Which valleys carry most of the distribution?

$$f(x) = e^{-\frac{1}{T}H_N(x) / Z}$$

- exp. more valleys
- exp. larger density
Equilibrium (Gibbs/Boltzmann) distribution

**Theorem**

For low temperature $T$, 99% of the distribution concentrates on a few of the deepest valleys.
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Thank You!