Computations on K3 surfaces
Past, Present and Future

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Motivation: torsion on elliptic curves

$E / \mathbb{Q}$ an elliptic curve:

\[ y^2 = x^3 + Ax + B \quad 4A^3 + 27B^2 \neq 0. \]
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Theorem (Mordell, 1922)
\[ E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r. \]
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Theorem (Mordell, 1922)

$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$.

Theorem (Mazur, 1977)

$E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of the following 15 groups:

$$\mathbb{Z}/n\mathbb{Z} \quad 1 \leq n \leq 10, \text{ or } n = 12, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \quad 1 \leq n \leq 4.$$
Motivation: torsion on elliptic curves

Theorem (Merel, 1996)

Fix \( d \in \mathbb{Z}_{>0} \). There is an integer \( c = c(d) \) such that:

For all number fields \( k \) with \( [k : \mathbb{Q}] = d \) and all elliptic curves \( E/k \),

\[ \#E(k)_{\text{tors}} < c. \]
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Question

Is there a Merel theorem for K3 surfaces?
K3 surfaces

**K3 surface:** smooth, projective, geometrically integral;
\[ \omega_X \cong \mathcal{O}_X \text{ and } h^1(X, \mathcal{O}_X) = 0. \]
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**Examples**

- \[ w^2 = x^6 + y^6 + z^6 \text{ in } \mathbb{P}(1,1,1,3) \text{ (degree 2).} \]
- \[ x^4 + y^4 = z^4 + w^4 \text{ in } \mathbb{P}^3 \text{ (degree 4).} \]
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**Problem:** K3 surfaces have no group structure.
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Problem: K3 surfaces have no group structure.

Replacement for \( E(k)_{\text{tors}} \)?
Reinterpreting $E(k)_{\text{tors}}$

$$E(k)_{\text{tors}} \simeq (\text{Pic}^0 E)_{\text{tors}}$$
Reinterpreting $E(k)_{\text{tors}}$

\[ E(k)_{\text{tors}} \cong (\text{Pic}^0 E)_{\text{tors}} \]
\[ = (\text{Pic} E)_{\text{tors}} \]
Reinterpreting $E(k)_{tors}$

$$E(k)_{tors} \cong (\text{Pic}^0 E)_{tors}$$
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E(k)_{\text{tors}} \cong (\text{Pic}^0 E)_{\text{tors}}
= (\text{Pic} E)_{\text{tors}}
\cong H^1(E, \mathcal{O}_E^\times)_{\text{tors}}
\cong H^1_{\text{et}}(E, \mathbb{G}_m)_{\text{tors}}
\]

Note: Hilbert 90 implies $\ker(H^1_{\text{et}}(E, \mathbb{G}_m)_{\text{tors}} \to H^1_{\text{et}}(E, \mathbb{G}_m)) \cong H^1_{\text{et}}(\text{Spec } k, \mathbb{G}_m) = 0$. 

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Transcendental Brauer groups

For a K3 surface over a number field $k$, use

$$\text{im}(H^2_{\text{et}}(X, \mathbb{G}_m)_{\text{tors}} \rightarrow H^2_{\text{et}}(\overline{X}, \mathbb{G}_m)_{\text{tors}})$$

$\text{Br}(X)$ $\rightarrow$ $\text{Br}(\overline{X})$
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First isomorphism theorem:

$$\text{im}(\text{Br}(X) \rightarrow \text{Br}(\overline{X})) \cong \frac{\text{Br}(X)}{\ker(\text{Br}(X) \rightarrow \text{Br}(\overline{X}))}$$

$Br_1(X)$ is the transcendental Brauer group of $X$. Note: There is a natural injection $\text{Br}(X)/Br_1(X) \rightarrow \text{Br}(X)_{\text{Gal}(k/k)}$. 
Transcendental Brauer groups

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Note: There is a natural injection $\text{Br}(X)/\text{Br}_1(X) \hookrightarrow \text{Br}(\bar{X})^{\text{Gal}(\bar{k}/k)}$. 
Is $\text{Br}(X)/\text{Br}_1(X)$ important?

There is a filtration of the Brauer group $\text{Br}_0(X) \subseteq \text{im}(\text{Br}(\text{Spec} k) \to \text{Br}(X)) \subseteq \text{Br}_1(X) \subseteq \ker(\text{Br}(X) \to \text{Br}(X)) \subseteq \text{Br}(X)$.

Conjecture (Skorogobatov 2009) The group $\text{Br}(X)/\text{Br}_0(X)$ controls all failures of the Hasse principle on K3 surfaces over number fields. We will come back to the quotient $\text{Br}_1(X)/\text{Br}_0(X)$ shortly.
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We will come back to the quotient $\text{Br}_1(X)/\text{Br}_0(X)$ shortly.
Is $\text{Br}(X)/\text{Br}_1(X)$ finite?

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**Theorem (Skorobogatov–Zarhin 2008)**

Let $k/\mathbb{Q}$ be a finitely generated field; let $X/k$ be a K3 surface. Then $\text{Br}(X)/\text{Br}_1(X)$ and $\text{Br}(\overline{X})^{\text{Gal}(\overline{k}/k)}$ are finite.
Uniform boundedness conjectures

For $X/C$ a K3 surface, always have $H^2(X,\mathbb{Z}) \cong U \oplus 3 \oplus E_8(-1) \oplus 2$.

We call $\Lambda_{K3} = U \oplus 3 \oplus E_8(-1) \oplus 2$ the K3 lattice.

$\text{NS}(X)$ embeds primitively in $H^2(X,\mathbb{Z})$.

Conjecture (Weak uniform boundedness (V.-A. 2015))

Fix a number field $k$ and a primitive sublattice $\Lambda \subset \Lambda_{K3}$.

There is an integer $B = B(k,\Lambda)$ such that:

For all K3 surfaces $X/k$ with $\Lambda \to \text{NS}(X)$,

$\# \text{Br}(X)/\text{Br}_1(X) < B$. 

Uniform boundedness conjectures

For $X/\mathbb{C}$ a K3 surface, always have $H^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1)^2$. 
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**Conjecture (Weak uniform boundedness (V.-A. 2015))**

Fix a number field $k$ and a primitive sublattice $\Lambda \subset \Lambda_{K3}$.

There is an integer $B = B(k, \Lambda)$ such that:

For all K3 surfaces $X/k$ with $\Lambda \hookrightarrow NS(X)$,

$$\#Br(X)/Br_1(X) < B.$$
Remarks

Could ask for $B(k, \Lambda)$ instead of $B(k, \Lambda)$ (strong uniform boundedness).

Weak Shafarevich conjecture (1994): for fixed number field $k$, there are only finitely many possibilities for $\text{NS}(X)$.

$\implies$ can dispense with $\Lambda$ in the conjecture.

Strong Shafarevich conjecture: for fixed $[k: \mathbb{Q}]$, there are only finitely many possibilities for $\text{NS}(X)$.

$\implies$ can dispense with $\Lambda$ in the strong version of the conjecture.
Remarks

- Could ask for $B([k : \mathbb{Q}], \Lambda)$ instead of $B(k, \Lambda)$ (strong uniform boundedness).
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- Strong Shafarevich conjecture: for fixed $[k : \mathbb{Q}]$, there are only finitely many possibilities for $\text{NS}(\mathcal{X})$.
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Conjecture (Strong unif. boundedness + strong Shafarevich)

Fix $d \in \mathbb{Z}_{>0}$. There is an integer $B = B(d)$ such that:
For all number fields $k$ with $[k : \mathbb{Q}] = d$ and all K3 surfaces $X/k$,

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This should be the K3 analogue of Merel’s theorem.
While we are dreaming...

Assume the K3 analogue of Merel's Theorem.
Assume Skorobogatov's conjecture.
Then work of Kresch–Tschinkel and Charles implies:

Conjecture
There is an effective algorithm that takes as input the equations of a K3 surface $X$ over a number field $k$ and determines whether $X(k)$ is empty.
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**Conjecture**

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What about $\text{Br}_1(X)/\text{Br}_0(X)$?
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Let $X$ be a smooth projective variety over a field $k$ with $\text{char } k = 0$. Assume that $\text{Pic}(\overline{X}) \cong \mathbb{Z}^r$. Then there is an integer $M = M(r)$, independent of $X$, such that $\# \text{Br}_1(X)/\text{Br}_0(X) < M$. 

Idea of the proof.

1. Pass to a finite Galois extension $K/k$ such that $\text{Pic}(\overline{X}_K) \cong \mathbb{Z}^r$.
2. Hochschild–Serre $\Rightarrow \text{Br}_1(X)/\text{Br}_0(X) \cong H^1(\text{Gal}(K/k), \mathbb{Z}^r)$.
3. $H^1(G, \mathbb{Z}^r) \cong (\mathbb{Z}^r/|G|) \otimes \mathbb{Z}/|G|$ where $G = \text{Gal}(K/k)$.
4. $G$ acts through a finite subgroup of $\text{GL}_r(\mathbb{Z})$ (only finitely many possibilities).
What about $\text{Br}_1(X)/\text{Br}_0(X)$?


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**Lemma (V.-A., Viray 2017)**

Let $X$ be a smooth projective variety over a field $k$ with $\text{char } k = 0$. Assume that $\text{Pic}(\overline{X}) \cong \mathbb{Z}^r$. Then there is an integer $M = M(r)$, independent of $X$, such that $\# \text{Br}_1(X)/\text{Br}_0(X) < M$.

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3. 
   \[
   H^1(G, \mathbb{Z}^r) \cong \frac{(\mathbb{Z}^r/|G|)^G}{(\mathbb{Z}^r)^G/(|G|)} \quad \text{where} \quad G = \text{Gal}(K/k).
   \]
   $\implies \#H^1(G, \mathbb{Z}^r)$ divides $|G|^r$, regardless of action.
What about $\text{Br}_1(X)/\text{Br}_0(X)$?


Let $X$ be a smooth projective variety over a field $k$ with $\text{char } k = 0$. Assume that $\text{Pic} (\overline{X}) \cong \mathbb{Z}^r$. Then there is an integer $M = M(r)$, independent of $X$, such that $\# \text{Br}_1(X)/\text{Br}_0(X) < M$.

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Corollary

There is an absolute constant $M$ such that, for all K3 surfaces $X$ over a field, we have

$$\#\text{Br}_1(X)/\text{Br}_0(X) < M.$$
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**Corollary**

*There is an absolute constant $M$ such that, for all K3 surfaces $X$ over a field, we have*

$$\#\operatorname{Br}_1(X)/\operatorname{Br}_0(X) < M.$$ 

**Proof.**

$\operatorname{Pic}(\overline{X}) = \operatorname{Pic}(X_{k_{\text{sep}}})$ is free of rank $r \leq 20$.

Apply (proof of!) the lemma for $1 \leq r \leq 20$; add up bounds to get $M$. \qed
What about $\text{Br}_1(X)/\text{Br}_0(X)$?

**Question**

*What can $\text{Br}_1(X)/\text{Br}_0(X)$ be for a K3 surface over a field?*
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For example, if $\text{Pic}(\overline{X}) \cong \mathbb{Z}$ then $\text{Br}_1(X)/\text{Br}_0(X) = 0$. 
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**Theorem (Wolff, 2018)**

*Let $X$ be a K3 surface over a field such that $\operatorname{Pic}(\bar{X}) \cong \mathbb{Z}^2$. Then $\operatorname{Br}_1(X)/\operatorname{Br}_0(X) = 0$.*
$\ell$-primary boundedness: an easier conjecture?

Conjecture ($\ell$-primary boundedness)
Fix a number field $k$, a prime $\ell$, and a primitive sublattice $\Lambda \subset \Lambda_{K3}$.

There is an integer $B = B(k, \Lambda, \ell)$ such that for all $K3$ surfaces $X/k$ with $\Lambda \hookrightarrow \text{NS}(X)$,

$$\#(\text{Br}(X)/\text{Br}_1(X))[\ell^\infty] < B.$$
ℓ-primary boundedness: an easier conjecture?

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Strong version: replace $B(k, \Lambda, \ell)$ with $B([k: \mathbb{Q}], \Lambda, \ell)$. 

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Strong version: replace $B(k, \Lambda, \ell)$ with $B([k:Q], \Lambda, \ell)$.

After all, before Merel, there was Manin...

Theorem (Manin 1969)

Fix a number field $k$ and a prime $\ell$. There is an integer $c = c(k, \ell)$ such that for all elliptic curves $E/k$,

$$\#E(k)[\ell^\infty] < c.$$
Evidence

I Kodaira dimension estimates for relevant moduli problem.
   Joint work with Tanimoto; Mckinnie, Sawon, and Tanimoto.

II Conditional analogues in the case of full-level structures for abelian varieties.
   Joint work with Abramovich.

III Special cases:
   i. Verification for some lattices $\Lambda$ of rank 19.
      Joint work with Viray.
   ii. The CM case. (Gives Merel-type result for $K3$s with $\rho = 20$.)
      Skorobogatov/Orr. Further work by Valloni.
   iii. $\ell$-primary boundedness for 1-dimensional families.
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III. Special cases: 1-parameter families of Kummer surfaces

Fix a number field $k$, as well as non-CM elliptic curves $E$, $E'$ with a cyclic isogeny of minimal degree $d$ between them.
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Fix a number field $k$, as well as non-CM elliptic curves $E$, $E'$ with a cyclic isogeny of minimal degree $d$ between them.

Let $X = \text{Kum}(E \times E') = \overline{E \times E'}/\iota$, where $\iota: x \mapsto -x$. 

$\Lambda_d = \text{NS}(X)$ has rank 19, discriminant $2^d$, independent of $E$, $E'$ and isogeny.

Theorem (V.-A. Viray 2017)

Fix a positive integer $r$, and a prime $\ell$.

There is a positive integer $B = B(r, d, \ell)$ such that for all K3 surfaces $X/k$ with $[k: \mathbb{Q}] = r$ and $\text{NS}(X) \simeq \Lambda_d$, $\#(\text{Br}(X)/\text{Br}_1(X)) \ll B$. 

III. Special cases: 1-parameter families of Kummer surfaces

Fix a number field $k$, as well as non-CM elliptic curves $E$, $E'$ with a cyclic isogeny of minimal degree $d$ between them.

Let $X = \text{Kum}(E \times E') = (\overline{E \times E'})/\iota$, where $\iota : x \mapsto -x$.

Let $\Lambda_d = \text{NS}(\overline{X})$.

$\Lambda_d$ has rank 19, discriminant $2d$, + indep. of $E$, $E'$ and isogeny.
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**Theorem (V.-A. Viray 2017)**

Fix a positive integer $r$, and a prime $\ell$.

There is a positive integer $B = B(r, d, \ell)$ such that for all K3 surfaces $X/k$ with $[k : \mathbb{Q}] = r$ and $\text{NS}(\overline{X}) \simeq \Lambda_d$,

$$\#(\text{Br}(X)/\text{Br}_1(X))[\ell^{\infty}] < B.$$
Key idea:

Given $X/k$ with $\text{NS}(\overline{X}) \cong \Lambda_d$, an element of order $n$ in $\text{Br}(X)/\text{Br}_1(X)$ can be used to construct a finite extension $L/k$ (degree independent of $X$) such that $\text{Gal}(L(E'_{n/c})/L)$ is an abelian group. Here $X_L \cong \text{Kum}(E \times E')$. 
III. Special cases: 1-parameter families of Kummer surfaces

Proposition (V.-A.–Viray 2017)

Let $L$ be a number field, $\ell$ a prime number. There is a constant $B := B(L, \ell)$ such that for all non-CM elliptic curves $E/L$, the extension $L(E_{\ell^s})/L$ is not abelian for $s > B$.
Proposition (V.-A.–Viray 2017)

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Question

Can we prove Serre-uniformity type results for non-abelianess of the mod $\ell$ Galois representation of elliptic curves over some concrete number fields?
III. Special cases: 1-parameter families

Theorem (Cadoret–Charles, 2018)

Fix $d \in \mathbb{Z}_{>0}$, a prime number $\ell$, a number field $k$. Let $X \to S$ be a K3 scheme over a $k$-curve $S$. There is a constant $C = C(d)$ such that for every $s \in S(K)$ with $[K : k] < d$,

$$\# \text{Br}(\overline{X}_s)^\text{Gal}(\overline{k}/K) < C.$$
III. Special cases: 1-parameter families

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Question

Can we prove a result along the lines of V.-A.–Viray or Cadoret–Charles for 2-parameter families?
E.g., For K3 surfaces $X/k$ such that $\bar{X} \simeq \text{Kum}(E_1 \times E_2)$ for non-isogenous, non-CM elliptic curves $E_1$, $E_2$?
III. Special cases: CM K3 surfaces

A K3 surface $X/\mathbb{C}$ has CM if

$$E(X) := \text{End}_{\text{Hdg}}(T(X)_\mathbb{Q})$$

is a CM field and $\dim_{E(X)} T(X)_\mathbb{Q} = 1$. 

Examples:

- K3's with $\rho(X) = 20$; $E(X)$ is an imaginary quadratic field.
- Kummer surfaces $X = \text{Kum}(A)$ where $A$ has CM.

Theorem (Orr–Skorobogatov, 2018)

Fix $d \in \mathbb{Z}_{>0}$. There is an integer $C = C(d)$ such that:

For all number fields $k$ with $[k:\mathbb{Q}] = d$ and all CM K3 surfaces $X/k$, $\#\text{Br}(X) \text{Gal}(\mathbb{Q}/k) < C$. 
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**Theorem (Orr–Skorobogatov, 2018)**

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$$\# \text{Br}(\overline{X})^{\text{Gal}(\overline{\mathbb{Q}}/k)} < C.$$
III. Special cases: CM K3 surfaces

Valloni has developed an explicit theory of CM for K3 surfaces.

**Theorem (Valloni, 2018)**

Let $X/\mathbb{Q}(i)$ be a K3 surface with CM by $\mathbb{Q}(i)$. Then $\text{Br}(\overline{X})^{\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(i))}$ is isomorphic to one of:

\[
0, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]
Questions

Question: Are all the above groups realizable for some $X/Q(i)$?

What is the co-kernel of the map $Br(X) \rightarrow Br(X)_{\text{Gal}(Q/Q(i))}$?

Question: Given a K3 surface $X/Q(i)$ with CM by $Q(i)$, can we devise a practical procedure to determine if $X(A)$ is empty? E.g., diagonal quartic surfaces.

Question: Given a quartic surface $X/k$ over a number field, can we determine algorithmically if $X$ has CM?

Question: What are the possibilities for $Br(X)_{\text{Gal}(Q/k)}$ when $X/k$ is a CM Kummer surface?

Database of CM K3 surfaces and their Brauer groups?
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