How do glasses flow and fail in dimensions from two to infinity?

March 7, 2019

Simons Collaboration on Cracking the Glass Problem

Lisa Manning
Contributors to this work:
This is a large thrust of the collaboration

<table>
<thead>
<tr>
<th>PIs</th>
<th>Affiliates</th>
<th>Associates and Alumni</th>
</tr>
</thead>
<tbody>
<tr>
<td>L. Berthier</td>
<td>A. Ikeda</td>
<td>E. Agoritsas</td>
</tr>
<tr>
<td>G. Biroli</td>
<td>C. Brito</td>
<td>H.H. Boltz</td>
</tr>
<tr>
<td>E. Corwin</td>
<td>E. Lerner</td>
<td>E. DeGiuli</td>
</tr>
<tr>
<td>S. Franz</td>
<td>S. Sastry</td>
<td>T. DeGeus</td>
</tr>
<tr>
<td>L. Manning</td>
<td>G. Tarjus</td>
<td>D. Hexner</td>
</tr>
<tr>
<td>S. Nagel</td>
<td>H. Yoshino</td>
<td>Y. Jin</td>
</tr>
<tr>
<td>A. Liu</td>
<td></td>
<td>C. Lupo</td>
</tr>
<tr>
<td>G. Parisi</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M. Wyart</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F. Zamponi</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


SIMONS FOUNDATION
Motivation: some disordered materials fail catastrophically. some do not.

How can we predict which is which?
Motivation: applied strain/stress should be an axis on a glass phase diagram.

Can we put this phase diagram on stronger theoretical footing?

Does a material’s preparation (path through this phase space) change how it will fail?

Rigidity is a collective phenomena

“We are so accustomed to the rigidity of solid bodies – the idea for instance that when we move one end of ruler the other end moves the same distance ...

... that we don’t accept its almost miraculous nature, that it is an ‘emergent property’ not contained in the simple law of physics, although it is a consequence of them.”
To understand the breakdown of an emergent property like rigidity, physicists have learned to look at the collective excitations.
Part I. What are the linear low-energy excitations?

• Crystalline solid
  • Spatially extended phonons are the low-frequency excitations
    • Goldstone’s theorem: broken continuous symmetries generate low-energy long-wavelength excitations
  • Caveat: in crystals with defects, there are resonant modes at the defects

• Disordered solids
  • What are the low-energy excitations?
  • Are they extended or localized?
Part II: What types of excitations allow system to escape? Yielding

- under external driving, system must eventually find a new basin
- what are the collective nonlinear excitations and energy barriers that facilitate yielding?
- how do the properties of these excitations change with dimension?
Part III: Dynamics of traversing the potential energy landscape

- Under continued driving, the system must visit many minima after it crosses the initial energy barrier.
- What are the dynamics post-yielding, and how does that change as a function of dimension?
Part I

What are the linear excitations in disordered solids?
Defining linear excitations in all dimensions

• Start with a two-body interaction potential, e.g. soft spheres:

\[
V(r) = \begin{cases} 
\frac{\varepsilon}{\alpha} \left(1 - \frac{r}{\sigma}\right)^\alpha & r \leq \sigma \\
0 & r > \sigma 
\end{cases}
\]

• The dynamical matrix \( M \) defines how particle displacements \( u \) give rise to forces \( f \): \( f = Mu \)

• Unlike a crystalline solid, \( M \) is a \((Nd)\times(Nd)\) random matrix
• Study the spectrum of eigenvalues and eigenvectors

\[
M_{i\alpha j\beta}^{\neq} = \frac{\partial^2 V(|r_i - r_j|)}{\partial r_{i\alpha} \partial r_{j\beta}}
\]
Exact infinite-dimensional solution: $D(\omega) \sim \omega^2$

Soft perceptron model: Franz et al PNAS 2015

Eigenvalues: Density of states $D(\omega)$ is given by

$$D(\omega) \sim \begin{cases} \omega^{d-1} & \omega \ll \omega_0 \\ \omega^2/\omega_*^2 & \omega_0 \ll \omega \ll \omega_* \\ \text{constant} & \omega \gg \omega_* \end{cases}$$

Eigenvectors:
- delocalized
- asymptotically distributed according to the uniform Haar measure

Charbonneau et al 117 PRL 2016
But, in low dimensions (d=2,3) there are new features: eigenvectors are not all delocalized
Quasi-localized modes have different universal scaling $D(\omega) \sim \omega^4$

Wang... Flenner, Nat Comm (2019)
Mizuno, Shiba, Ikeda PNAS (2017)

Major discovery of our collaboration: universal across many different preparation protocols
How to predict universal $\omega^4$ scaling?

sparse random matrices (random graph Laplacian)

Structure of underlying graph is really important: only find $\omega^4$ for a random regular (isostatic) graph + fluctuations

Big open question: is there an analytic derivation?
Linear spectrum: Summary and future directions

For the first time ever, we have a firm understanding of the universal spectrum of low-frequency excitations in \( d = \infty \) and low dimensions \( (d = 2,3) \). Excitations are delocalized in high dimensions and quasi-localized in low dimensions.

Opens new avenues for future research:
What dynamics/processes generate the special “fine-tuning” in the spectrum of low-dimensional solids?
Part 2: Yielding

Under an applied force, what excitations allow the system to escape?
Applied strain allows the system to explore the potential energy landscape
Exact solutions in $d = \infty$ for jammed spheres under an applied strain

- Avoid solving dynamical equations and study Franz-Parisi potential, which can be computed exactly using the replica method in $d = \infty$

- Within replica method can also calculate $F$ under applied strain

---

**Diagram:**

- $F(\Delta_r, T)$ for liquid
- $F(\Delta_r, T)$ for glass

- Increasing strain

---
Motivation: applied strain/stress should be an axis on a glass phase diagram.


Can we put this phase diagram on stronger theoretical footing?

Does a material’s preparation (path through this phase space) change how it will fail?
Exact analytic jamming phase diagram

$d=\infty$ theory

$\frac{1}{\tilde{\varphi}}$ vs $\tilde{T}$

- Dynamical line
- Gardner line
- Yielding line

$d=3$ numerics

Density vs $\frac{1}{\tilde{\varphi}}$

- Stable
- Unstable
- Forbidden

Biroli, Urbani, Sci Post 2018

What about the nature of yielding?
Exact solutions in $d=\infty$:
Exhibit brittle failure at spinodal point
In experiments and simulations in low dimensions \((d=2,3)\) different materials fail differently

Brittle failure

in between

homogeneous flow

Yoo et al 2010

Amann et al 2013

Lauridsen et al 2002

What is the simplest model that can explain this?
Elasto-plastic models can describe this behavior.

Mean-field elastic interactions between quasi-localized yielding regions

Each yielding region is assumed to have an energy barrier that can be overcome by applied stress:

\[ x_i = \sum_i^y - \sum_i \]

stress to local yield instability:

distribution of stress gaps:

\[ P(x) \sim x^\theta \]
Elastoplastic models can be solved analytically and exhibit a brittle-to-ductile transition as a function of $P_0(x)$

Big open question: what features of $P_0(x)$ determine ductility?
New idea: Brittle-to-ductile transition is a 1\textsuperscript{st} order transition separated by a critical point.
What type of critical point? Random Field Ising Model (RFIM)

Model
\[ \mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - \sum_i (h_i + H(t)) S_i \]

Discontinuous transition
Discontinuous transition

\[ R < R_c \]

Critical point

Crossover

\[ R > R_c \]

External field

Magnetization

Gaussian random fields

\[ \overline{h_i} = 0 \]

\[ \overline{h_i^2} = R^2 : \text{Strength of disorder} \]

Analogy

RFIM | Yielding
--- | ---
Magnetization | Stress
External field | Strain

\[ R < R_c \]

Brittle yielding

\[ R > R_c \]

Ductile yielding

Rare droplet | Soft spot

Sethna, Dahmen, and Perkovic, The science of Hysterisis, 2005

Nandi, Biroli, and Tarjus, PRL 2016
With swap Monte Carlo, we can test this idea carefully, because we can finally generate very brittle glasses on a computer.
Numerics demonstrate this is a 1\textsuperscript{st} order transition

\begin{itemize}
\item For low 'enough' $T_{ini}$, a discontinuous transition becomes easily observable from macroscopic rheology.

\item $\langle \sigma \rangle$ becomes sharper, susceptibilities $\chi_{dis} = N(\langle \sigma^2 \rangle - \langle \sigma \rangle^2)$ and $\chi_{con} = -d\langle \sigma \rangle / d\gamma$ diverge as $N \to \infty$.

\item Ab o n afi de discontinuous phase transition.
\end{itemize}
Motivation: applied strain/stress should be an axis on a glass phase diagram.


Can we put this phase diagram on stronger theoretical footing?

Does a material’s preparation (path through this phase space) change how it will fail?

e.g. how does $P_0(x)$ depend on $T_{\text{init}}$ for swap Monte Carlo simulations?

and do yielding regions always interact via simple elasticity kernel?

Does a material’s preparation (path through this phase space) change how it will fail?
Local yield stress: literally $P_0(x)$
Try to do something more clever:

- Linear vibrational modes and extensions
- Local specific heat (e.g. weighted nonlinear modes)
- Machine learning

Are these structural indicators good predictors of where the material will fail?
Work in progress: comparison of structural indicators

Logistics: project involving 20 scientists, 5 PIs or affiliates within the Simons Collaboration common data formats, metadata, correlation of structure with deformation

rearrangement event
correlation between structural indicator and localized plastic event

Ultra-stable glass

Poorly annealed glass

next 1 > 2 > 5 > 10 plastic events
Yielding transition: Summary and future directions

• exact jamming phase diagram in $d=\infty$ in good agreement with low-dimension simulations

• brittle-to-ductile transition is a 1$^{\text{st}}$ order transition in the RFIM universality class

• ongoing large collaborative project to develop and test numerical methods to identify localized excitations

Opens new avenues for future research:

• How do properties of localized excitations change with material preparation?

• How do localized excitations interact?
Part III: dynamics under shear

What is the behavior of the material post-yielding?

How does the dynamics of the driving force (finite strain rate, quantum vs. classical) change the behavior?

Connections to *dynamics* talk by Jorge Kurchan later in the workshop
Work in progress:
What happens after you go over the energy barrier?

In low dimensions, quasi-localized excitations clearly play an important role.

Big open question: How to best describe their dynamics and interactions?
Work in progress:
Study evolution of structural indicators during an avalanche

particle displacements
weighted average of eigenvectors
What happens when the system visits lots of minima? Empirical observation:

- There is no first-principles description of the out-of-equilibrium steady state regime.
  - Currently rely on phenomenological models: Elasto-plastic, Mode Coupling Theories (MCT)
New! Exact formulation in $d=\infty$

- Builds upon work in equilibrium (cf. talk by Jorge Kurchan)
  - For the first time ever, we have the correct dynamical mean field theory for glasses

- We have derived Dynamical Mean Field Theory (DMFT) for out-of-equilibrium pairwise interacting particles in $d=\infty$
  - Converts a many body physics problem into an effective scalar stochastic process (self-consistently defined)

- Work in progress: numerical/analytical study of these equations and the corresponding effective dynamics
  - A corollary: In $d=\infty$, many out-of-equilibrium driving mechanisms (shear, active forces) are similar

Agoritsas et al, ArXiv:1808.00236 2019
Work in progress:
Is it also true that random active forces are similar to shear in low dimensions? (Connections to active matter)

Preliminary evidence suggests they are very similar
At low temperatures, quantum effects allow new mechanisms for traversing the landscape

- Theory of two-level systems (Anderson, Halperin, Varma) was invented to explain universal anomalous low-temperature scaling of specific heat ($C_p \sim T$) in glasses
- Can be explained by quantum tunneling between two states with a very small energy barrier
- Big open question: what are these two-level systems in glasses?
Quasi-localized excitations and two-level systems are closely connected

\[ u_i(s) = \frac{1}{2!}\lambda_is^2 + \frac{1}{3!}\kappa_is^3 + \frac{1}{4!}\chi_is^4 + o(s^4) \]

- elasto-plastic models generate a flow in \( \lambda \) and \( \kappa \)
- together, \( \lambda \) and \( \kappa \) govern statistics of \( D(\omega) \sim \omega^\alpha \) and energy barriers \( P(x) \sim x^\theta \)

Big open question: Does this elasto-plastic result apply to real disordered solids?
Steady state dynamics: Summary and future directions

• We have derived exact Dynamical Mean Field Theory (DMFT) for out-of-equilibrium in \( d=\infty \).

• We have developed numerical tools to study the dynamics of avalanches in low dimensions after the yielding point.

Opens new avenues for future research:
• What are the transient dynamics of the DMFT in \( d=\infty \)？
• How do quasi-localized excitations in low dimensions affect the dynamics?
Major accomplishments . . .

• discovery of the universal spectrum of low-frequency excitations in $d=\infty$ and low dimensions
• an exact jamming phase diagram in $d=\infty$
• demonstration that the brittle-to-ductile transition is a 1$^{\text{st}}$ order transition in the RFIM universality class
• exact Dynamical Mean Field Theory (DMFT) for out-of-equilibrium in $d=\infty$
• new numerical tools to study dynamics of quasi-localized excitations in poorly annealed and ultra-stable glasses

. . . that pave the way for new questions

• what dynamics/processes generate the special “fine-tuning” in the spectrum of low-dimensional solids?
• What are the transient dynamics of the exact DMFT in $d=\infty$?
• In low dimensions, how do quasi-localized excitations affect the dynamics?

Thank you for your attention!
Extra slides
Thanks so much for your attention!

<table>
<thead>
<tr>
<th>PIs</th>
<th>Affiliates</th>
<th>Associates and Alumni</th>
</tr>
</thead>
<tbody>
<tr>
<td>L. Berthier</td>
<td>A. Ikeda</td>
<td>E. Agoritsas</td>
</tr>
<tr>
<td>G. Biroli</td>
<td>C. Brito</td>
<td>H.H. Boltz</td>
</tr>
<tr>
<td>E. Corwin</td>
<td>E. Lerner</td>
<td>E. DeGiuli</td>
</tr>
<tr>
<td>S. Franz</td>
<td>S. Sastry</td>
<td>T. DeGeus</td>
</tr>
<tr>
<td>L. Manning</td>
<td>G. Tarjus</td>
<td>D. Hexner</td>
</tr>
<tr>
<td>S. Nagel</td>
<td>H. Yoshino</td>
<td>Y. Jin</td>
</tr>
<tr>
<td>A. Liu</td>
<td></td>
<td>C. Lupo</td>
</tr>
<tr>
<td>G. Parisi</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M. Wyart</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F. Zamponi</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**SIMONS FOUNDATION**
Preliminary comparison of structural indicators

Logistics: project involving 20 scientists, 5 PIs or affiliates within the Simons Collaboration
common data formats, metadata, correlation of structure with deformation
Summary and future directions:

• For the first time ever, we have a firm understanding of the universal spectrum of low-frequency excitations in $d=\infty$ and low dimensions ($d=2,3$). Excitations are delocalized in high dimensions and quasi-localized in low dimensions.

• What are the connections between quasi-localized modes and two-level systems? (more at the end of this talk)

• Can we develop analytic results for sparse random graph Laplacians?
  • from dense limit?
  • from 1d limit?

• Is the pseudo-gap important in low dimensions?
  • Special preparation protocols (e.g. breathing particles) can open up a true gap in the density of states, but dynamics are similar to pseudo-gapped systems
  • many types of dynamics seem to “fill in” the gap.

What dynamics/processes generate the special “fine-tuning” in the spectrum of low-dimensional solids?
Thanks and contributors

• Thanks to the Simons Foundation for supporting this work
Quasi-localized excitations and two-level systems are closely connected:

\[ u_i(s) = \frac{1}{2!} \lambda_i s^2 + \frac{1}{3!} \kappa_i s^3 + \frac{1}{4!} \chi_i s^4 + o(s^4) \]

**Idea:** elastoplastic models generate a flow in space of linear response -- \( D(\omega) \sim \omega^\alpha \) -- and energy barriers \( P(x) \sim x^\theta \)
Part III: Dynamics of traversing the potential energy landscape

• Under continued driving, the system must visit many minima after it crosses the initial energy barrier
• What are the dynamics post-yielding, and how does that change as a function of dimension?
  • Avalanches: what are the dynamics when the system is “going down hill”?
  • What happens when the system visits a very large number of minima at a finite rate?
  • What happens when we consider quantum effects for traversing the potential energy landscape?
Part III: Summary and future directions

• We have derived exact Dynamical Mean Field Theory (DMFT) for out-of-equilibrium in $d=\infty$.

• We have developed numerical tools to study the dynamics of avalanches in low dimensions after the yielding point.

Open questions:
• What are the transient dynamics under shear in $d=\infty$?
• How do quasi-localized excitations in low dimensions affect the dynamics?
What happens to the structure? What types of landscapes are visited?

- Weighted average of eigenvectors
- Particle displacements
Previous work under quasistatic shear

<table>
<thead>
<tr>
<th>Finite Shear Rate</th>
<th>Quasistatic Shear</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{\gamma} &gt; 0$</td>
<td>$\dot{\gamma} = 0$</td>
</tr>
</tbody>
</table>

- Configurations at each $t$
- Minimized snapshots at each $\gamma$
Define stress $\sigma = \frac{V}{2} \frac{dU}{d\gamma}$, see similar stress drops
Derivation of dynamical mean-field equations (DMFE) for out-of-equilibrium system of pairwise interacting particles, in the thermodynamic & infinite-dimensional limits ($N \to \infty$, $d \to \infty$).

Models of dense assemblies of particles with pairwise interactions $v(|\vec{x}_i(t) - \vec{x}_j(t)|)$.

Spatial fluctuations are suppressed $\Rightarrow$ mean-field description exact in this limit.

Effective scalar stochastic process (self-consistently defined).

Related publications / previous works


Equilibrium dynamics:


Continuous random perceptron (similar derivations: dynamical cavity & supersymmetric path integral)

Derivation of dynamical mean-field equations (DMFE) for out-of-equilibrium system of pairwise interacting particles, in the thermodynamic & infinite-dimensional limits $(N \to \infty, d \to \infty)$

Models of dense assemblies of particles with pairwise interactions $v(|\vec{x}_i(t) - \vec{x}_j(t)|)$

- Spatial fluctuations are suppressed $\Rightarrow$ mean-field description exact in this limit

- Example: 'flowing' phase at constant shear rate

- Out-of-equilibrium steady-state regime
  - Exact first-principle description still missing, instead:
    - Elasto-plastic models
    - Mode-coupling theories (MCTs)
  - Here first step towards an exact benchmark
A few key physical assumptions specific to high dimensions, but in the co-shearing frame:

- Particles stay ‘close’ to their initial position:
  \[ \mathbf{u}_i(t) = \mathbf{x}_i(t) - \hat{S}_\gamma(t) \mathbf{x}_i(0) \sim \mathcal{O}(1/d) \]
  \[ \mathbf{w}_{ij}(t) = \mathbf{u}_i(t) - \mathbf{u}_j(t) \sim \mathcal{O}(1/d) \]
- Each particle has numerous uncorrelated neighbours.
- Statistical isotropy dominant for \((d - 2)\) directions.

Infinite-dimensional limit: exact mean-field dynamics

- Scalar stoch. process: \( y(t) = \frac{d}{\ell} \dot{r}_{0,ij}(t) \cdot \mathbf{w}_{ij}(t) \)
- Isotropic kernels:
  \[ \{ \hat{k}(t), \hat{M}_C(t,s), \hat{M}_R(t,s) \} \]
- Vectorial stoch. processes: \( \{ \mathbf{u}(t), \mathbf{w}(t) \} \)
- Matricial correlation/response
  \[ \{ \hat{C}(t,t'), \hat{R}(t,t') \} \]

Check that we recover dynamically the thermodynamic/static results, for equilibrium & state-following protocols (quench in temperature, random forces, finite strain)

ONGOING: numerical/analytical study of these equations & the corresponding effective dynamics
Structural indicators

• Approaches to identify structural indicators that can predicts where the material will become unstable:
  • Linear or nonlinear vibrational modes
    • soft spots, local specific heat, etc.
  • Machine Learning
  • Local yield stress (brute force)
  • Saddle point sampling
  • Local geometric calculations
    • free volume, Voronoi anisotropy, generalized bond–orientational order
Soft spots + filtering plane waves
Softness field

• Machine learning identifies local structural features that correlate strongly with excitations (Liu PRL 2015)

• 90-95% prediction accuracy for excitations in glass/ glassy liquid (Liu PRL 2015, Nat Phys in press)

• Can extract energy barriers for excitations
Coupling of energy landscape to elasto-plastic models makes predictions for flow in energy barrier space:

\[ u_i(s) = \frac{1}{2!} \lambda_i s^2 + \frac{1}{3!} \kappa_i s^3 + \frac{1}{4!} \chi_i s^4 + o(s^4) \]

Wencheng Ji, Marco Popovic, Tom De Geus, Edan Ilerer, MW
Density of quasi-localized modes: testing the theory
Exact solutions in $d = \infty$: Gardner transition before failure predicts existence and scaling of two different shear moduli.
Eigenvector statistics are rich for glasses

• density of vibrational states
  • recall: analytic prediction of specific heat $c \sim T^3$ for phonons
  • data for glasses shows universal linear scaling at the lowest frequencies:

\[ C_p = C_{\text{TLS}} T + C_D T^3 \]
but, can open up a gap at the very lowest frequencies with special zero-temperature dynamics

Geert Kapteijns, Wencheng Ji, Carolina Brito, MW, Edan Lerner PRE 2019
Numerics demonstrate this is a 1st order transition

\[ \langle \sigma \rangle \text{ becomes sharper, } \chi_{\text{dis}} = N \left( \langle \sigma^2 \rangle - \langle \sigma \rangle^2 \right) \text{ and } \chi_{\text{con}} = -\frac{d\langle \sigma \rangle}{d\gamma} \text{ diverge as } N \to \infty. \]
• swap with continuous polydispersity: breathing particles
• breathing particles must have a gap in QLM in any inherent structures
Hint: low frequency excitations different in low dimensions

Nope!

Doesn’t seem to be the mean field solution in low dimensions:

\[ D(\omega) \sim \begin{cases} 
\omega^{d-1} & \omega \ll \omega_0 \\
\omega^2/\omega_*^2 & \omega_0 \ll \omega \ll \omega_* \\
\text{constant} & \omega \gg \omega_* 
\end{cases} \]

So what is the origin of the low frequency behavior \( \sim \omega^4 \)?
Athermal Quasi-static Persistent Motion: “Strain”-like measurement

- During athermal quasi-static shear, we constrain the shear direction during minimization.
- To mimic this, translate along Nd-vector $|c\rangle$ and project out force and velocity along $|c\rangle$ during minimization.
- Force becomes $|F\rangle \rightarrow |F\rangle - \langle c | F | c \rangle$
- Velocity becomes $|v\rangle \rightarrow |v\rangle - \langle c | v | c \rangle$
Open(!!) questions

• How does a material system move around in this complex potential energy landscape under an applied force?
• What are the important elementary excitations?
• How do those elementary excitations interact?
• How do those answers depend on
  • “static” control parameters (pressure, material preparation, interaction potential)?
  • dynamic control parameters (temperature, finite strain rate)?
• How does this lead to emergent pattern formation such as shear bands and avalanches?
low-D eigenvectors: mix of phonon-like modes and quasi-localized excitations

- quasi-localized
- plane-wave-like
- hybrids
- extended + disordered
Outline

• Part I: Excitations in the glass state
• Part II: Yielding (escaping the glass state via applied strain)
• Part III: Steady state
Complex potential energy landscape

- The linear response of these systems is already very interesting and informative.
How to explain $\omega^4$ scaling?
Sparse Random Matrices (Random graph Laplacian)

Special type of localized mode that likely only exists in 1D
d = 1

Stanifer et al PRE 2018

Structure of underlying graph is really important: only find $\omega^4$ for a random regular (isostatic) graph + fluctuations

Benetti et al PRE 2018
A Gardner transition separates an “elastic” and a “plastic” regime
Partial irreversibility in the Gardner phase

Reversibility to HOME (the reference liquid state)

\[ \phi_g = 0.655 \quad \phi = 0.66 \]

\[ \gamma_G = 0.1 \]

MSD to the initial state

Exact calculations for strained systems in $d=\infty$

- Clever way to bypass dynamic calculations
- One equilibrated master replica $\{R_i\}$, One slaved replica $\{X_i\}$.
- Fix the distance $\Delta_r \sim \sum_i (X_i - R_i)^2$
- Franz-Parisi potential $F(\Delta_r, T)$ can be computed exactly in $d=\infty$

Out-of-equilibrium glassy states $T<T_g$ can be studied:
Slave at temperature $T$, Master at $T_g$
Exact calculations in $d = \infty$ for yielding transition

$F(\Delta_r, T_g, T)$

- A glass at $T < T_g$ can be sheared
- Increasing $\gamma$
- Before spinodal, response $\Sigma(\gamma)$ can be computed for any $T, T_g$

Hard spheres prepared in equilibrium at $2\phi/d = 7$
Exact calculations for strained systems in $d=\infty$

- Clever way to bypass dynamic calculations
- One equilibrated master replica $\{R_i\}$, One slaved replica $\{X_i\}$.
- Fix the distance $\Delta_r \sim \sum_i (X_i - R_i)^2$
- Franz-Parisi potential $F(\Delta_r, T)$ can be computed exactly in $d=\infty$

- Out-of-equilibrium glassy states $T<T_g$ can be studied:
  Slave at temperature $T$, Master at $T_g$
Exact calculations in \( d = \infty \) for yielding transition

\[ F(\Delta_r, T_g, T) \]

- A glass at \( T < T_g \) can be sheared

- Before spinodal, response \( \sum(\gamma) \) can be computed for any \( T, T_g \)

![Graph showing scaled stress and strain](image)
Explanation for reversibility under small applied strains:

3 scenarios for metastable state:

- Never plastic before yielding
- Always plastic
- Plastic after finite strain

• Gardner = buckling instability

Hard sphere simulations in $d=3$
Prediction for exponents before irreversible yielding (crackling noise)

- Power-law distribution of stress drop
  \[ \tau = 1 \quad \text{Soft potential} \]
  \[ \tau = 1.41 \ldots \quad \text{Hard potential} \]
  \[ \rho(\Delta E) \sim \Delta E^{-\tau} \]

- Two elastic moduli can be defined

Protocol-dependent shear modulus of amorphous solids
Exploring the complex free-energy landscape of the simplest glass by rheology
Examples of Amorphous solids

- window (soda-lime) glass
- dense polymers
- colloids
Part 1. What are the linear low-energy excitations?

• Crystalline solid
  • Spatially extended phonons are the low-frequency excitations
    • Goldstone’s theorem: broken continuous symmetries generate low-energy long-wavelength excitations
  • Caveat: in crystals with defects, there are resonant modes at the defects

• Disordered solids
  • What are the low-energy excitations?
  • Are the extended or localized?

Barker and Sievers, Rev. Mod. Phys 1975