Energetic particle effects in magnetic confinement fusion devices are commonly studied by hybrid kinetic-fluid simulation codes whose underlying continuum evolution equations often lack the correct energy balance. While two different kinetic-fluid coupling options are available (current-coupling and pressure-coupling), in this work we apply the Euler-Poincaré variational approach to formulate a new conservative hybrid model in the pressure-coupling scheme.

1. Motivation

- MHD-like models are invalidated by energetic particles
- Small scales may control large-scale phenomenology
- Hybrid models: fluid interacts with a hot particle gas

2. State of the art

Two hybrid models are currently available in the ideal case

- **Current-coupling scheme** (CSS): same hypotheses as MHD, conserves energy exactly. Fluid equation:
  \[ \rho \left( \frac{DU}{Dt} + u \cdot \nabla U \right) = - \nabla P + \rho_0 (u \times \nabla) \times B, \]
  where \( \rho_0 \) and \( J_0 \) are the hot charge and current, resp.

- **Pressure-coupling scheme** (PCS): neglects inertia of the hot mean flow, energy is not conserved.
  \[ \rho \left( \frac{DU}{Dt} + u \cdot \nabla U \right) = - \nabla P - \rho_0 (u \times \nabla) \times B, \]
  where \( P_0 = \rho v f v \) is the hot pressure tensor.

- Fluid equations are coupled to Faraday’s law
  \[ \frac{\partial B}{\partial t} = - \nabla \times E, \quad E = - \nabla \times B \quad (\text{ideal Ohm’s law}) \]

- Hot particle dynamics is computed by the same kinetic equation in both models. For Vlasov dynamics one writes
  \[ \frac{\partial F}{\partial t} + v \cdot \nabla F = \frac{1}{\rho_0} \left( E - B \frac{v}{\rho} \right) \quad \frac{\partial \rho_0}{\partial t} = 0 \]

3. Remarks on the PCS

- PCS does not conserve total energy \( \mathcal{H} \) exactly (see (5))
  \[ \frac{\partial}{\partial t} \int w \left| \mathcal{K} \right| d^4x, \quad \text{with} \quad \mathcal{K} = v f d \nu v. \]

- Uncontrolled approximation: PCS gives \( \mathcal{H} \) under the hypothesis \( \delta K \equiv 0 \), which is not preserved in time.

4. Vlasov-MHD: Hamiltonian method & CCS

- Hamiltonian systems are characterized by
  \[ \pm \text{a conserved Hamiltonian functional} \ H = \text{a Poisson bracket} \ \{ \cdot , \cdot \} \text{ satisfying Jacobi and Leibniz rules.} \]
  - The Maxwell-Vlasov system possesses a well defined Hamiltonian structure and so does ideal MHD.
  - In [4] it was shown that (for Vlasov-MHD) CCS possesses a Hamiltonian structure \( \{ \cdot , \cdot \} \) with Hamiltonian (total energy, incompressible case)

  \[ \mathcal{H} = \frac{1}{2} \left( w | \mathcal{K} | d^4x \right) + \frac{2}{3} \int f \sqrt{v^2 + \mathcal{L}} d^3v + \frac{1}{2} \left( \frac{1}{m} B^2 \right) \]


- Seek a hybrid Hamiltonian system that conserves \( \mathcal{H} \).
- In [4], the CCS Poisson bracket was written in terms of the total momentum \( m_\nu v_\nu / \sqrt{m} \), thereby obtaining another Poisson bracket \( \{ \cdot , \cdot \}_{\mathcal{H}} \) to conserve \( \mathcal{H} \).
- However, the Hamiltonian structure produces extra terms in the accompanying kinetic equation [4], i.e.
  \[ \frac{\partial F}{\partial t} + v \cdot \nabla F = \frac{1}{\rho_0} \left( E + B f \right) \quad \frac{\partial \rho_0}{\partial t} = 0 \]

- Fluid and Faraday’s eqns are (2) and (3), respectively.
- Unlike previous models, cross helicity \( h \) is conserved [3].


- Linearize the incompressible PCS equations around static isochoric equilibrium with \( f_1 = f_0 (v^2) \).
- We consider longitudinal wave propagation.
- We considered a hydrogen bulk with density \( 10^{20} \text{ cm}^{-3} \), a magnetic field of 50 Gauss, and \( v \)-particles with 3.6 MeV and fractional density \( 5 \times 10^{-3} \).
- The non-Hamiltonian model exhibits an instability of static \( \mathcal{K} \)-equilibria above the cyclotron frequency.
- No physical source of energy can be identified, which can possibly drive this instability.
- This type of instability must be nonphysical and reflects the lack of energy conservation in the model.


- The previous method is applied to guiding-center theory upon exploiting the Littlejohn’s Hamiltonian approach.
- In this context, the momentum shift involves the first moment of the drift-kinetic distribution thereby leading to the total momentum \( \rho_0 v_\nu = m_\nu f_0 (v^2) \).
- Upon denoting
  \[ X = (U_\nu, u_\nu), \]
  this procedure leads to
  \[ \rho \frac{\partial X}{\partial t} + v \cdot \nabla X = - \nabla P_0 + \frac{\rho_0}{m_\nu} (u \times \nabla) \times B, \]
  \[ \frac{\partial E}{\partial t} + v \cdot \nabla E = \frac{\rho_0}{m_\nu} (u \times \nabla) \times B, \]

- Stability of sound waves (preliminary results): Upon linearizing around static isochoric equilibrium and by considering longitudinal propagation, the non-Hamiltonian variant is seen not to capture kinetic damping, which instead appears in the Hamiltonian model.

8. Variational approach to fluid dynamics – a tutorial

- Incompressible fluid flows as Euler-Lagrange equations:
  \[ \frac{\partial E}{\partial t} + v \cdot \nabla E = \frac{\rho_0}{m_\nu} (u \times \nabla) \times B, \]
  \[ \eta (x) \text{ is an invertible map (diffeomorphism) s.t.} \mu = 1. \]
- By the change of variable formula, one relates as follows
  \[ L (\eta, \eta) = L (\eta, \eta^{-1}) = L (U) \]
  where \( \eta \) and \( \eta^{-1} \) are so that \( U \) is the Eulerian velocity.
- Hamiltonian’s principle \( \delta (f (U) - \int v \cdot \nabla U \) yields Euler’s equation
  \[ \frac{\partial U}{\partial t} + (U \cdot \nabla) U = - \nabla p, \quad \nabla U = 0 \]
- Arbitrary variations \( \eta \) yield Euler-Lagrange eqns:
  \[ \frac{\partial f_\eta}{\partial \eta} \]
- Upon defining \( \zeta \sim \eta \), we obtain the variations:
  \[ \delta U = \delta (\eta, \eta^{-1}) \sim \delta (U, U^{-1}) \]
- These variations yield the Euler-Poincaré equations:
  \[ \frac{\partial f_\eta}{\partial \eta} \]
- This approach generalizes to Vlasov kinetic theory


- We combine Newcomb’s MHD Lagrangian with the phase-space Lagrangian (yielding the Hamiltonian PCS):
  \[ \delta L = \mathcal{F} (\eta, \eta^{-1}) \delta \eta + \int [\nabla \times A] \delta A \]
- For the hydrodynamic gauge \( A = 0 \).
- However, particles move in the Lagrangian bulk frame:
  \[ f_\eta = \int [\nabla \times (\eta^{-1})] \delta (U^{-1}) (U \cdot A) \cdot U + \int [\nabla \times (\eta^{-1})] \delta A \cdot U \]
- Problem: the tangent map does not act on the guiding-center phase-space – how about drift-kinetic particles?
  - We lift guiding-center motion to the standard phase-space and consider the total Lagrangian
  \[ \delta L = \int [\nabla \times (\eta^{-1})] \delta (U^{-1}) - \int [\nabla \times A] \delta A \]
  \[ + \int [\nabla \times (\eta^{-1})] \delta A \cdot U + \int [\nabla \times (\eta^{-1})] \delta A \cdot U \]
  - The Hamiltonian PCS is then found by integrating out \( \nu \), that is \( f (x, \eta, \eta^{-1}) = f (x, \nu, \eta^{-1}) \delta (\nu^2) \),
  - The energy-conservation requires new force terms
  - The new models preserve energy and cross helicity
  - Energy non-conservation leads to atypical instabilities or unphysical lack of kinetic (Landau) damping

10. Conclusions

- The energy-conservation requires new force terms
- The new models preserve energy and cross helicity
- Energy non-conservation leads to atypical instabilities or unphysical lack of kinetic (Landau) damping

References


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