CFT Universality, Black Holes & Quantum Chaos

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Emergence of Geometry

**Question**: Does gravity emerge from *averaging* of quantum theories?

This could mean either:

- Ensemble average over many different theories, or
- Average over many configurations of the same theory.

We will consider the second:

- Geometry from *coarse-graining* over quantum states.

This is familiar in AdS, where

- A typical heavy state is a black hole microstate (complicated)
- Coarse grained state is a black hole geometry (simple).
AdS/CFT

The prototypical example is Cardy’s formula in AdS$_3$/CFT$_2$:

$$\rho(h, \bar{h}) \approx \left| \exp \left\{ 4\pi \sqrt{\frac{c}{24}} \right\} \right|^2 \quad h, \bar{h} \to \infty$$

which is universal, in that it depends only on

$$\Delta = h + \bar{h}, \quad J = h - \bar{h}, \quad c = \frac{3\ell_{AdS}}{2G}$$

and not on any details of the theory.

It has a gravitational interpretation as black hole entropy:

$$\log \rho(h, \bar{h}) = \frac{A}{4G}$$
Goal: Develop a universal theory of coarse-grained CFT$_2$ data.

We seek to:

- Understand which features of CFT data are universal.
- Give them a gravitational interpretation (when we can).

They will be *statistical* properties of the CFT data:

$$\{h_i, \bar{h}_i, C_{ijk}\}_{primaies}$$

which arise from coarse graining over BH microstates.

Chaotic and Integrable theories will realize them in different ways.
Plan for Today

- Universal Dynamics of Heavy Operators
- Interpretation (Chaotic vs. Integrable Theories)
- A Toy Model of Quantum Geometry
- Extensions & Speculations
Basic Strategy

Consider a correlation function as a sum over intermediate states:

\[
Z(q) = \sum_{\{O_i\}} C_{\{O_i\}} |F(\{h_i\}|q)|^2 \\
= \sum_{\{O_j\}} \tilde{C}_{\{O_j\}} |\tilde{F}(\{h_j\}|\tilde{q})|^2
\]

where \(i\) and \(j\) label operators propagating in different channels.

The \(C_{\{O_i\}}\) depend on the spectrum and OPE coefficients.

The conformal blocks are purely kinematic, and are related by

\[
F(\{h_i\}|q) = \int dh_j \mathcal{S}_{h_j,h_i} F(\{h_j\}|\tilde{q})
\]

where \(\mathcal{S}_{h_j,h_i}\) is a crossing kernel.
A CFT “No Hair” Theorem

If the the identity dominates in one channel

\[ Z(q) \approx | \mathcal{F}(1|q)|^2 = \left| \int dh_j S_{h_j,1} \mathcal{F}(\{h_j\}|\tilde{q}) \right|^2 \]

then \( S_{h_j,1} \) is the density of OPE coefficients in the dual channel:

\[ \overline{C_{\{O_j\}}} \approx |S_{h_j;1}|^2. \]

The result depends only on \( c \) and the dimensions/spins of the \( O_i \).

Result:

- Every observable dominated by \( 1 \) in one channel leads to a universal formula in the cross channel.
- It is “gravitational” in that it depends only on mass & spin.
Cardy’s Formula

Applying this to the partition function gives Cardy’s formula.

The crossing kernel $\mathcal{S}_{h,h'}$ of characters is a Fourier transform:

$$\chi_h \left( -\frac{1}{\tau} \right) = \int_0^\infty dP' \cos \left( 4\pi PP' \right) \chi_{h'}(\tau).$$

where

$$c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2, \quad h = \alpha(Q - \alpha), \quad \alpha = \frac{Q}{2} + iP.$$

The result is a version of Cardy’s formula

$$\rho(h, \bar{h}) \approx |\mathcal{S}_{h,1}|^2 = |\sinh(2\pi bP) \sinh(2\pi b^{-1} P)|^2$$

for the density of primaries.
Another example is the asymptotic density of OPE coefficients:

\[
\overline{C_{ijk}}^2 \approx |C_0(h_i, h_j, h_k)|^2
\]

\[
= \left| \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3} \prod_{\pm\pm\pm} \frac{\Gamma_b \left( \frac{Q}{2} \pm iP_i \pm iP_j \pm iP_k \right)}{\prod_{a \in \{i, j, k\}} \Gamma_b \left( \frac{Q}{2} + 2iP_a \right) \Gamma_b \left( \frac{Q}{2} - 2iP_a \right)} \right|^2
\]

which is valid in three different regimes:

- One operator heavy and the others fixed
- Two operators heavy and the other fixed
- All three operators heavy.

where we average only over the heavy primaries.

These hold at finite $c$ and come from crossing kernels.
OPE Asymptotics 2

It follows from crossing of:

- Four point functions

- Torus two point functions:

- Genus two partition functions:
Which lead to asymptotic formulas:

- When $\Delta_i$ is large:
  \[
  (C_{\mathcal{O}i})^2 \approx 16^{-\Delta_i} e^{-2\pi \sqrt{\frac{c-1}{12} \Delta_i}} \Delta_i^2 (\Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'}) - \frac{c+1}{4}
  \]

- When $\Delta_i \approx \Delta_j$ are large:
  \[
  (C_{\mathcal{O}ij})^2 \approx e^{-4\pi \sqrt{\frac{c-1}{12} \Delta_i \Delta_{\mathcal{O}}}}
  \]

- When $\Delta_i \approx \Delta_j \approx \Delta_k$ are large:
  \[
  (C_{ijk})^2 \approx \left(\frac{27}{16}\right)^3 \Delta_i^3 e^{-6\pi \sqrt{\frac{c-1}{12} \Delta_i \Delta_{\mathcal{O}}}} \Delta_i^5 \frac{5c-11}{36}
  \]

These make predictions for black hole dynamics in AdS.

Chang & Lin
Das, Datta, Pal
Brehm Das, Datta
Hikida, Tusuki, Takayanagi, Romero-Bermdez
P. Sabella-Garnier, and K. Schalm
Cardy, A.M. & Maxfield
Many others!
Non-universal Quantities

An observable where the contribution of 1 vanishes leads to non-universal results in the dual channel.

For example, crossing of torus 1pt functions:

leads to

$$\bar{C_{\mathcal{O}ii}} \approx C_{\mathcal{O}\chi\chi} \Delta_i \frac{\Delta_{\mathcal{O}}}{2} e^{-2\pi \Delta_\chi} \sqrt{\frac{12\Delta_i}{c-1}}$$

when $\Delta_i$ is large. Here $\chi$ is the lightest non-$\mathbf{1}$ primary.

This matches dynamics in a black hole background.

Kraus, A.M.
Many others!
Chaos & Eigenstate Thermalization Hypothesis

Different theories should realize these averages in different ways:

- **Chaotic**: Average over window of size $\sim e^{-S}$
- **Integrable**: Average over window of size $\sim O(1)$

We can even organize matrix elements into

$$
\langle i | O | j \rangle \approx f^{O}(\Delta_i) \delta_{ij} + g^{O}(\Delta_i, \Delta_j) R_{ij}
$$

when $i$ and $j$ are heavy, with

$$
f^{O}(\Delta_i) = \overline{C_{Oii}}, \quad g^{O}(\Delta_i, \Delta_j) = \sqrt{\overline{C_{Oij}^2}} \sim e^{-S/2}
$$

**ETH**: In a chaotic theory $R_{ij}$ looks like a random matrix.
Holographic ETH

In a typical theory there is an exponential hierarchy

$$\langle i|O|j \rangle \approx f^O(\Delta_i)\delta_{ij} + g^O(\Delta_i, \Delta_j)R_{ij}$$

between diagonal and off diagonal terms, since $g^O \sim e^{-S/2}$.

But in 2D CFT $f^O$ is non-universal and exponentially small:

$$f^O \sim e^{-\Delta_{\text{gap}}\sqrt{\Delta_i/c}}$$

So the hierarchy disappears when

$$\Delta_{\text{gap}} = \frac{c - 1}{16}$$

This would be a theory of pure gravity at large $c$. 
Pure Gravity

In a theory of pure gravity there is no hierarchy between diagonal and off-diagonal terms.

There is other evidence that

$\Delta_{gap} = \frac{c - 1}{16}$

may be the size of the gap in pure gravity in 2+1 dimensions:

- Positivity of the density of states
- Mass of the lightest ($\mathbb{Z}_2$) orbifold
- Gravity path integral seems well defined at classical and one-loop level

Benjamin, Ooguri, Shao & Wang
N. Benjamin, S. Coller, & A.M. (in progress)
Is there a quantum model of black holes where we can see chaotic behaviour explicitly?

- What features can be reproduced by quantizing geometry?

Reproducing BH entropy is difficult:

- Naive computations give $\infty$ unless we cut off.
- Must remove states which appear semi-classically sensible.
- Complicated non-perturbative physics.

Instead, let’s try to understand other features of the spectrum.

We will work in $2 + 1$ dimensions.
Geometric Model for Black Hole Microstates

Replace the ER bridge by a compact geometry behind the horizon:

\[ \text{The geometry outside the horizon matches the BTZ black hole.} \]
Phase Space of 3D Gravity

The phase space of 3D gravity at fixed $g$ is finite dimensional.

It is $T^*\mathcal{M}_{g,1}$, where $\mathcal{M}_{g,1}$ is the moduli space of genus $g$ Riemann surfaces with one boundary.

We can use length and twist coordinates $x^\alpha = (L_i, \theta_i)$ to see that

$$\dim(\mathcal{M}_{g,1}) = 6g - 4$$

Moncrief
Scarinci & Krasnov
Study dynamics of a free particle on $\mathcal{M}_{g,1}$:

$$H = \nabla^2$$

with respect to the Weil-Petersson metric $g_{\alpha\beta}$ on $\mathcal{M}_{g,1}$.

This is a toy model for a theory of gravity.

There are an infinite number of states, unless we cut off at high energy.

- Can we understand these states?

**Problem:** The metric $g_{\alpha\beta}$ is not known analytically.

But we can approximate it accurately using CFT techniques.
Example: $\mathcal{M}_{0,4}$

The Liouville 4pt function of operators with dimension $\Delta_i = \frac{c-1}{12}$

$$\langle V_1(0) V_2(x) V_3(1) V_4(\infty) \rangle = \int d\Delta C_{12\Delta} C_{34\Delta} |\mathcal{F}_\Delta(\Delta_i; x)|^2$$

$$\sim e^{-c S_L(x, \bar{x})}$$

is dominated by a saddle point at large $c$, with

$$\Delta(x, \bar{x}) = \frac{c - 1}{12} \left( 1 + (2\pi L_s(x, \bar{x}))^2 \right)$$

where $L_s(x, \bar{x})$ is a length coordinate on $\mathcal{M}_{0,4}$.

Recursion relations allow us to compute $\mathcal{F}_\Delta(\Delta_i; x)$ accurately. For example, $O(q^6)$ correctly gives

$$\cosh L_s \left( \frac{1}{2}, \frac{1}{2} \right) = 3 + O(10^{-9})$$

Hadasz, Jaskolski, Piatek
Weil-Petersson Metric

The resulting approximation for the Weil-Petersson metric

\[ ds^2 = \frac{8\pi^2}{\left(\log \frac{2^8}{qq}\right)^3} \left(1 + \frac{40\pi^2\zeta(3)}{\left(\log \frac{2^8}{qq}\right)^3} + \ldots\right) dqd\bar{q} \]

converges rapidly inside the fundamental domain.

We can accurately compute wave functions:
Spectrum

To probe the spectrum, compute:

\[ ZZ^* \]

\[ \left\langle \frac{E_{n+1} - E_n}{E_n - E_{n-1}} \right\rangle \]

Looks like a random matrix in Gaussian Orthogonal Ensemble?

Results are for entertainment purposes only.
Open Questions

Universal statistical properties of CFT data:

- Is there an analog of the HKS sparseness constraint?
- Can they be made rigorous with Tauberian theorems?
- What additional assumptions are needed to ensure ETH?
- Semi-classical understanding of, e.g. BH $\rightarrow$ light+light?
- Is averaging over states related to averaging over theories?
- Is pure gravity a single theory or an ensemble of theories?

Quantization of BTZ microstates:

- Can we sensibly quantize Einstein gravity on $\mathcal{M}_{g,1}$?
- Can we project out null states to get a finite answer?
- Can we reproduce $S_{BH}$ by quantizing geometry alone?