Black holes, random matrices, baby universes, and D-branes

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IfQ - Dec 6, 2019
Motivation

- What is the bulk explanation for the discreteness of the energy spectrum of quantum black holes in gauge/gravity duality?
- What bulk (as opposed to boundary) equations should one solve to find these energies?
- One of the deepest mysteries in quantum gravity.
- Discreteness affects simple observables, like the long time behavior of correlation functions [Maldacena].
Ensembles of quantum systems – the SFF

- A simpler question: what are **averaged** spectral properties? Consider an **ensemble** of unitary finite entropy quantum systems. A lesson from SYK. Aspects of discreteness still visible.

- Study the spectral form factor (SFF)

\[ \sum_{mn} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t} = Z(\beta + it)Z(\beta - it). \]

- Compute averaged SFF in SYK, \( \langle Z(\beta + it)Z(\beta - it) \rangle. \)

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[CGHPSSSST]

- Something that might be computable...
Averaged (microcanonical) SFF given by

$$\int dE dE' \langle \rho(E) \rho(E') \rangle e^{i(E-E')t}. $$

$$\langle \rho(E) \rho(E') \rangle$$ is the pair distribution function of eigenvalues.

A "fluctuating crystal" (the Dyson gas). Long range spectral rigidity, short ranged eigenvalue repulsion.

Eigenvalue spacing $\sim e^{-S} \sim e^{-N_{SYK}}$.

A smoking gun for discreteness in an averaged quantity.

[You-Ludwig-Xu; Garcia-Garcia–Verbaarschot; CGHPSSSSST]
The “Sine kernel formula” for the eigenvalue correlations in (GUE) random matrix theory [Dyson; Gaudin; Mehta].

Conjectured to be **universal** in quantum chaotic systems [ Wigner; Dyson; Berry; Bohigas-Giannoni-Schmit; . . . ].

What is the gravitational explanation for this pattern in SYK? (And then, in more general systems with a gauge/gravity dual?)
\[ \langle \rho(E) \rho(E') \rangle \sim e^{2S} - \frac{1}{2(\pi(E - E'))^2} (1 - \cos(2\pi e^S(E - E'))) \]

- \( e^{S(E)} e^{S(E')} \) gives the slope.
- \( 1/(E - E')^2 \) gives the ramp, (down by \( e^{-2S} \)).
- \( \cos(2\pi e^S(E - E')) \) gives the sharp break to the plateau, at \( t \sim e^S \).
- The ramp could be a nontrivial saddle point.
- What is the origin of the **doubly exponential** \( e^{ie^S} \sim e^{ie^N} \) behavior that signals the plateau?
JT gravity

- Try to compute the SFF in JT gravity, a model for SYK at low energies (not a uniform approximation, a model of the model).
- JT gravity:
  - 2D; $g_{\mu\nu}, \phi$. $G_N \sim 1/N_{SYK}$. Bulk constrained to $R = -2$. Fluctuating boundary with Schwarzian dynamics. Surfaces with Euler character $\chi$ weighted by $e^{S_0 \chi}$, Ground state entropy $S_0 \sim N_{SYK}$.
- Follow approach in [Saad, arXiv:1910.10311].
Write SFF as an overlap of time shifted TFD states:

\[ |Z(\beta + it)|^2 = |\langle TFD_\beta | e^{-i\frac{t}{2}(H_L+H_R)} | TFD_\beta \rangle|^2 \]

So as the SFF gets small the bra and ket are nearly, but not exactly, orthogonal. But should be no smaller than \(1/d \sim e^{-2S}\) (relative size).

The TFD state is dual to the Hartle-Hawking state \(|HH_\beta\rangle\). In JT gravity we can describe this in the length \(\ell\) basis, \(\psi_{HH}(\ell)\), where \(\ell\) is the geodesic length of the ER bridge [Harlow-Jafferis; Yang].
We can then write $Z(\beta + it)$ as the return amplitude of the Hartle-Hawking state:

$$Z(\beta + it) = \langle HH_\beta | e^{-i\frac{t}{2} H_{\text{Bulk}}} | HH_\beta \rangle$$

As $t$ gets larger the mean value of $\ell$ gets longer. The ERB grows in time. This means the overlap decreases, forever. This is the “slope.”

This geometry has the topology of the disk, $\chi = 1$, so is of order $e^{S_0}$.

The SFF in this region is just the modulus squared of this. The geometry is two disks, of order $e^{2S_0}$.
The full theory involves arbitrary spacetime topologies, consistent with the boundary condition. In particular one copy of the HH state can “emit” a closed circular baby universe. This changes the support of \( \ell \) in the state. The emission process “shortens” the ERB, allowing significant overlap. The baby universe is then “absorbed” by the other copy.

This spacetime has the topology of the cylinder, a spacetime wormhole. It has \( \chi = 0 \) and is of order \( e^0 \), exponentially small. But it is nondecreasing. The baby universe can rotate before being reabsorbed, giving a factor of \( t \) from this collective coordinate.

This is the ramp. This spacetime wormhole limits how close the states can be to orthogonal.
Arbitrary topologies

- The plateau is much more subtle.
- To begin with we need to study spacetimes with arbitrary topology. Let's focus for now on Euclidean manifolds with one boundary, computing $Z(\beta)$.

Surfaces with any number ($g$) of handles with $\chi = 1 - 2g$. 
Sum over topologies

$Z(\beta) = \sum_g Z^{(g)}(\beta) \times e^{(1-2g)S_0} = e^{S_0} \sum_g Z^{(g)}(\beta) \times (e^{-S_0})^{2g}$

- Looks like a perturbative string genus expansion (the "JT string"), but here $g_s = e^{-S_0} \sim e^{-1/G_N} \sim e^{-N}$.
- These are nonperturbative effects in $G_N$. Joining and splitting of baby JT universes, a "third quantized" description.
- We need to sum them up.
There are powerful techniques for summing over all topologies of certain types of 2D gravity coupled to "minimal" matter.

These techniques (which originated in the 1980s) use the ‘t Hooft double-line diagram expansion of large rank ($L$) matrix integrals to describe triangulations of 2D surfaces.

Initial application: YM perturbation theory with $N \times N$ matrices. String world sheets.

Surfaces weighted by $N^\chi$. 
The JT gravity partition function summed over all genera, $Z_{JT}(\beta)$, can be calculated using a certain random matrix ensemble for Hermitian $L \times L$ matrices $H$ [Saad-SS-Stanford].

$$Z_{JT}(\beta) = \langle e^{-\beta H} \rangle_{\text{matrix}}$$

The matrix ensemble is defined by a partition function

$$Z = \int dH e^{-L \text{Tr} V(H)}.$$ 

Rather than specify the potential $V$ we specify the large $L$ density of eigenvalues $\rho_{0}^{\text{total}}(E)$ (the analog of the Wigner semicircle).
Choose a $\rho_0^{\text{total}}(E)$ like

$$\rho_0^{\text{total}}(E) = e^{S_0} \sinh(2\pi\sqrt{E}) \frac{1}{4\pi^2}.$$ 

Zoom in on the lower edge, taking $L \to \infty$ to keep the local density of eigenvalues fixed, $\sim e^{S_0}$, the genus counting parameter. Adjust things carefully so that the edge density equals that calculated from JT gravity on the disk

Then we have $Z_{JT}(\beta) = \langle e^{-\beta H} \rangle_{\text{matrix}}$, to all orders in $e^{-S_0}$. 

JT gravity as a matrix integral, contd.

- \( Z_{JT}(\beta) = \langle e^{-\beta H} \rangle_{\text{matrix}} \).

- The boundary dual of JT gravity is an ensemble of quantum systems specified by the random matrix ensemble for \( H \), the boundary Hamiltonian.

- The effective dimension of the matrices is \( e^{S_0} \sim e^N \), not \( N \). A consequence of the third quantized nature of the description.

- This relationship gives a (non-unique) nonperturbative definition of JT gravity.

- This is a random matrix ensemble, so (local) random matrix statistics is manifest.
D-brane effects

- The JT string perturbative series of $e^{-S_0}$ nonperturbative effects, $Z(\beta) = \sum_g Z^{(g)}(\beta) \times e^{(1-2g)S_0}$, is described by the small fluctuations in the smoothed eigenvalue density $\rho(E)$. These describe the ramp.

- But the eigenvalues are actually discrete. This discreteness is described in the JT string by the third quantized analog of D-brane effects (cf. work in the early 2000s).

- Such effects are of order $e^{-c/g_s}$, which here is of order $e^{-ce^{S_0}}$. Doubly exponential.

- For the (FZZT brane) effects that describe the Sine kernel formula, $c$ is imaginary. These effects are not small, but oscillate very quickly. Like the “allowed” region of WKB.

- These D-brane effects describe the plateau.
Spacetimes that end

- D-brane effects are described by arbitrary numbers of disconnected world sheets, ending on the D-brane.

\[ e^{-c/g_s} = -\frac{c}{g_s} + \frac{1}{2} \left( \frac{c}{g_s} \right)^2 + \ldots \]

- The third quantized D-brane effects are described by arbitrary numbers of disconnected spacetimes. What they “end on” is not so clear...
This analysis raises **lots** of questions. (See [Saad-SS-Stanford, arxiv:1903.11115] Section 6.)

Let’s focus on one of them.
Non-averaged systems

- What happens when one does **not** average, like standard SYM? For a non-averaged system the spectral form factor is very erratic [Prange].
- Universal (a consequence of random matrix statistics).
- What is the bulk explanation for this erratic behavior?  ⋆

\[
\langle Z(\beta + it)Z(\beta - it) \rangle
\]
Moments from wormholes

- Can compute the size of fluctuations gravitationally (schematic).
- Compute the second moment, \( \langle (Z(\beta + it)Z(\beta - it))^2 \rangle \).

\[
\langle (Z(\beta + it)Z(\beta - it))^2 \rangle = 2(\langle Z(\beta + it)Z(\beta - it) \rangle)^2,
\]
so fluctuations are the same size as the signal, \( \langle (ZZ^*)^2 \rangle - (\langle ZZ^* \rangle)^2 = (\langle ZZ^* \rangle)^2 \).

- \( \langle (Z(\beta + it)Z(\beta - it))^k \rangle = k!(\langle Z(\beta + it)Z(\beta - it) \rangle)^k \).
- An exponential distribution ("Porter-Thomas").
- \( Z = x + iy \), where \( x, y \) are each independently gaussian distributed.
- All simple smooth statistics should probably be accessible this way, but not the actual erratic signal. ∗
Semiclassical quantum chaos

- An analogy: semiclassical chaos in an ordinary few body quantum mechanical systems, like a quantum billiard.
- Use the path integral (Gutzwiller trace formula), summing over periodic orbits (schematically)

\[
\text{Tre}^{-iHt/\hbar} \sim \sum_a e^{i\frac{\hbar}{\hbar} S_a}
\]

- In the analogy \( H \) is the boundary quantum system, the orbit sum is the microscopic bulk description.
- The spectral form factor becomes:

\[
\text{Tre}^{-iHt/\hbar} \ Tr^{iHt/\hbar} \sim \sum_{ab} e^{i\frac{\hbar}{\hbar}(S_a - S_b)}
\]
Semiclassical quantum chaos, contd.

\[ \sum_{ab} e^{i\frac{\hbar}{(S_a - S_b)}} \]

- Long times \( t \to \) long orbits \( \to \) large phases \( \to \) large fluctuations.
- But on averaging (over time, say) in the ramp region the only terms that survive are the ones where \( a = b \), up to a time translation [Berry].
- There are \( e^t \) such paths, multiplied by an \( e^{-t} \) one loop determinant, giving a bulk microscopic derivation of the order one (times \( t \)) value of the ramp. The pattern of pairing –not the microscopics – seems analogous to the spacetime wormhole geometry. An effective description.
Wormholes conflict with factorization [Maldacena-Maoz].

The non-averaged spectral form factor obviously factorizes, because of the double sum.

$$\sum_{ab} e^{i \frac{\hbar}{\hbar} (S_a - S_b)}$$

But because averaging picks out the diagonal terms $a = b$, it destroys factorization. It makes the wormhole connection [Coleman].

How to restore factorization? Don’t get rid of the wormhole. Add back in the off-diagonal terms [Maldacena-Maoz]. These are responsible for the erratic behavior.

What is the bulk realization of these contributions?
Breaking the wormhole

- What breaks the wormhole?
- Microscopic phase space semiclassically determines all the microstates of the system.
- The Fuzzball program [Mathur...]

A “long string” at the bottom of an AdS$_2$ throat. Quantized semiclassically, it describes the set of (extremal) microstates.
Orbits and fuzzballs?

Speculation:

Instead of a particle orbit, perhaps the long string executes a periodic orbit.

But we need chaos; we need to go away from semiclassics; and we probably need more (brane?) degrees of freedom.

Perhaps we can get something like the erratic red curve.

And if we average perhaps we recover the smooth double trumpet and the ramp.....
A simple model: “Eigenbranes”

[Blommaert-Mertens-Verschelde, 1911.11603]

Freeze a subset of the matrix integral eigenvalues to mock up some non-averaged microstates.

Trumpets can end on eigenvalues (FZZT branes)

Get an erratic contribution to the red curve...
What causes the plateau in the periodic orbit expansion, after averaging?

At long times orbits traverse all of phase space. Orbits inevitably overlap so their actions are correlated. Allows lots of additional \( a \neq b \) pairings. A massive matching failure. The diagonal pairing (analogous to the wormhole) stays, but cancelled by many other contributions.

How to represent? Maybe geometry is a bad set of variables. Mechanism for plateau in \( G, \Sigma? \)

Running out of Hilbert space is often represented by branes. Giant graviton: \( \text{Tr} \phi^\ell \rightarrow \det \phi, \ell \rightarrow N. \) D-brane wrapping \( S^5. \)

FZZT branes, bose-fermi duality...
Why don’t wormhole pinwheels like these that compute $\text{Tr} \rho^n$ suffer from “erratic red curve disease” in a non-averaged theory?

Compute the variance in an ensemble:

$$\langle \text{Tr} \rho^n \text{Tr} \rho^n \rangle - \langle \text{Tr} \rho^n \rangle \langle \text{Tr} \rho^n \rangle$$

A handle connecting two pinwheels – of relative magnitude $e^{-2S_0}$. The variance is small!

We say $\text{Tr} \rho^n$ is a self-averaging quantity. A single element of the ensemble, a non-averaged system, gives a result close to the pinwheel value.
The purities are “single trace” expectation values. Study the simple model.

$H_{\alpha\beta}$ is a matrix in “color” space. Here $\alpha, \beta = 1 \ldots e^S$. EOW branes should be described by fields that are color fundamentals. We need $k$ “flavors” of them. So let $C_{i\alpha}$ be a bifundamental describing the $k$ EOW branes, $i = 1 \ldots k$. (i.i.d complex gaussian variables)

In the simple model (at large $\mu$)

$$\rho_{ij} \sim C_{i\alpha} (e^{-\beta H})_{\alpha\beta} C^*_{\beta j}.$$ 

$$Tr \rho^n \sim Tr[(Ce^{-\beta H}C^\dagger)^n],$$ a single trace object.

So the purity fluctuations are small by large $N$ factorization. They are “self-averaging.”
The ramp is not self averaging

- In contrast, the ramp is a fluctuation correction to single trace quantities $\langle T e^{-i\hat{H}t} T e^{i\hat{H}t} \rangle - \langle T e^{-i\hat{H}t} \rangle \langle T e^{i\hat{H}t} \rangle$, itself with order one fluctuations.
- Like the variance of the variance in gaussian integrals.
- The analog in the simple model is the microstate overlap $|\langle \psi_i | \psi_j \rangle|^2$, which has order one fluctuations. The purity is the sum of a large number of such terms, where the fluctuations average out.
The disk is a single trace quantity

- The large $N$ counting analog of the purity in the spectral form factor story is one partition function, $\langle Z(\beta + it) \rangle = \langle \text{Tre}^{-(\beta + it)H} \rangle$ at relatively short time. The analog of the pinwheel is just the Euclidean black hole, with disk topology.

- At small times the disk geometry gives an accurate answer. The correction is measured by the variance, computed by the double trumpet. It is of relative magnitude $e^{-2S_0}$.
Microstate holes?

- In a non-averaged system there will be small errors, described by (one half of) the broken trumpet.

- Here, and in the purity case, we expect “microstate holes” to contribute a small amount. We don’t know how to do an arbitrarily accurate calculation.
We are learning how to calculate the Page curve.

The erratic red curve is going to take some new insights...
Thank you

Thank You