Genus One Fibrations, Meromorphic Jacobi Forms and 6d Theories

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Introduction

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2. The geometries

Jacobi forms

1. Definition of Jacobi forms
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Introduction

Purpose of the talk is to give an overview over new techniques to calculate all genus results for the topological string partition function on various types of elliptically fibred Calabi-Yau 3-folds $M$ in terms of modular forms.

A-model Definition of the Topological string partition

Perturbative string theory is defined by map

$$x : \Sigma_g \rightarrow M \times \mathbb{R}_{3,1}$$
from a 2d world-sheet $\Sigma_g$ of genus $g$ into a target space $M \times \mathbb{R}_{3,1}$. $\Sigma_g$ is equipped with a 2d super diffeomorphism invariant action $S$, of type IIA or IIB. The partition function is formally defined by

$$Z(G, B) = \int DxDhD\text{ferm} \ e^{\frac{i}{\hbar}S(x,h,\phi,\text{ferm},G,B)}.$$

Here the bosonic part of the action is the complexified area $\text{area}_C \sim \int_{\Sigma_g} d\sigma^2 \partial_\alpha x^i \partial_\beta x^j (h^{\alpha\beta} \sqrt{h}G_{ij} + i\epsilon^{\alpha\beta} B_{ij})$ and a genus counting term $-\frac{1}{2\pi} \phi \int_{\Sigma_g} R_2 \sim (2g - 2)g_s$.

- For $d_{\text{crit}} = 10$ the variational integral $\int Dh$ over
the world-sheet metric becomes a discrete integral
\[ \sum_g \int_{\mathcal{M}_g} d\mu_g \]
over the \(3g - 3\) dimensional moduli space of complex structure on \(\Sigma_g\).

- If \(M\) is a Calabi-Yau 3-fold \(\exists (\omega_{11}, \Omega_{30}) \& c_1(TM) = 0\)

  \[ S \] has \((2, 2)\) super conformal symmetry with four nilpotent operators \(Q^\pm, \bar{Q}^\pm\). Vector- and axial \(U(1)\) allow to define twisted nilpotent scalar operators \(Q_A\) and \(Q_B\), which define two inequivalent cohomological topological string theories, called \(A\) and \(B\) model.

  \[ S \] In the A-model, super symmetric localisation localizes
$x$ to holomorphic maps $x_{hol}$, so that

$$\int \mathcal{D}x \mathcal{D}h \to \sum_{g, \beta \in H_2(M, \mathbb{Z})} \int_{\mathcal{M}_{g, \beta}} \overline{c}^{vir}_{g, \beta} = \sum_{g, \beta \in H_2(M, \mathbb{Z})} r^\beta_g$$

localises to a sum over finite dimensional

$$\dim_{\mathbb{C}}(\mathcal{M}_{g, \beta}) = \int_{\mathcal{C}_\beta} c_1(T_M) + (\dim(M) - 3)(1 - g)$$

integrals over the moduli space of $x_{hol} : \Sigma_g \to [\mathcal{C}_\beta] \in M$, $\beta \in H_2(M, \mathbb{Z})$. The latter are symplectic invariants (Gromov-Witten invariants: $r^\beta_g \in \mathbb{Q}$), that lead to mathematical definition of the topological
string partition function in a large \( \text{vol} = \int_{[C_a]} \omega + iB \)
expansion

\[
Z = \exp \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(Q),
\]

with

\[
\mathcal{F}_g(Q) = \sum_{\beta \in H_2(M,\mathbb{Z})} r_g^\beta Q^\beta
\]

where \( Q^\beta = \exp(2\pi i \sum_a t_a \beta^a) \) and \( t_a = \int_{[C_a]} (B + i\omega) \).
**Integrality Properties:** \( F(g, t) = \log Z(g, t) \) has an alternative expansion in terms of \( D2 - D0 \) BPS indices \( I_\beta^g \in \mathbb{Z} \) (stable pair invariants)

\[
F(g_s, t) = \sum_{g \geq 0, m \geq 1} \frac{I_\beta^g}{m} \left( 2 \sin \frac{m g_s}{2} \right)^{2g-2} Q^\beta m .
\]

Note that \( F(g_s, t) \) is shift invariant under \( g_s \to g_s + 4\pi \) and has poles at \( g_s = 2\pi \mathbb{Q} \) from the genus zero sector.
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**Table 1:** $I_g^d$ on the Quintic CY
**Refinement:** If $M$ has an $U(1)$ action $F(g_s, t)$ has a canonical motivic refinement $g_s^2 = ? \epsilon_1 \epsilon_2$

$$F(\epsilon_1, \epsilon_2, t) = \sum_{m \geq 1, \beta \in H_2} \frac{N_{j_l j_r}^\beta}{m} \frac{[j_l]_s [j_r]_r Q^{\beta m}}{(q_1^2 - q_1^{-2})(q_2^2 - q_2^{-2})}$$

The $N_{j_l j_r}^\beta \in \mathbb{N}$ are the dimensions of vectors spaces that correspond to an $sl_l(2) \times sl_r(2)$ Lefshetz decompostion of the cohomology of the moduli space of stable pairs. Choi, Katz, AK : 1210.4403 and

$$[j]_x = \frac{x^{2j-1} - x^{-2j-1}}{x - x^{-1}}$$ and $q_k = e^{2\pi i \epsilon_k}$, $k = 1, 2$. 
$N_{j_l, j_r}^2$ for local CY 3-fold $\mathcal{O}(-K_S) \rightarrow B_8 \mathbb{P}^2$

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Generally the $N_{j_l, j_r}^\beta$ decompose into Weyl orbits of $E_8$. 
Modular and analytic properties:
Recall that $Q_B$ that defines a different cohomological topological string theory, called $B$ model.
– In the B-model, super symmetric localisation localizes $x$ to constant maps $x_c : \Sigma_g \rightarrow \text{pt.} \in W$.
– The integration over $Dh$ with the corresponding measure can be analysed and leads to the holomorphic anomaly equations.
– For $g = 0, 1$ the measure is special due to the killing symmetries on $\Sigma_{0/1}$
– In particular for $g = 0$ the $\mathcal{F}_0$ is determined by the
periods over a symplectic basis \((A^I, B^I)\) of \(H_3(W, \mathbb{Z})\)

\[
\Pi = \begin{pmatrix} F_0 \\ F_1 \\ X^0 \\ X^1 \end{pmatrix} = \begin{pmatrix} \int_{B_0} \Omega \\ \int_{B_1} \Omega \\ \int_{A^0} \Omega \\ \int_{A^1} \Omega \end{pmatrix} = X^0 \begin{pmatrix} 2F_0 - t \partial_t F_0 \\ \partial_t F \\ 1 \\ t \end{pmatrix}
\]

which are flat section w.r.t the Gauss-Manin connection given by solutions of the Picard Fuchs equations.

**Mondromy action**

The Picard-Fuchs equations are essentially
determined by the monodromy group $\Gamma \in \text{Sp}(h_3(W), \mathbb{Z})$ which is generated by analytic continuation of the periods around the singular divisors $D_i \in \mathcal{M}_{cs}$

$$\Pi \mapsto M_i \Pi$$

with $M_i^T \Sigma M_i = \Sigma$ and $\Sigma$ the symplectic paring on $H_3(W, \mathbb{Z})$

What are the expected invariances of $Z$? Within the $A$ model we have presented $Z$ for in a specific Kähler gauge in the Kähler line bundle $\mathcal{L}$, i.e. $\Omega \in \mathcal{L}^{-1}$ as $\Omega \rightarrow e^{f} \Omega$, expanded at a large radius point in a
specific real polarisation in $H_3(W, \mathbb{R})$.
The holomorphic anomaly for $F_1$ predicts that $Z$ is a
section of $\mathcal{L}^{\chi_{24}^{-1}}$ given by

$$Z = g_s^{\chi_{24}^{-1}} \exp \sum_{g=0}^{\infty} g_s^{2g-2} F_g(z(t)).$$

Note that $g_s \in \mathcal{L}$ and $F_g \in \mathcal{L}^{2-2g}$ to the right Kähler
transformation.
The monodromy action changes the polarisation
and the holomorphic anomaly equations have been interpreted as the infinitesimal change of polarisation
on $Z$ viewed as a wave function he wave function,

$$\langle X^I | Z \rangle = Z(X^I)$$

that arises when quantising $H_3(W, \mathbb{R})$ Witten 93 with a commutator (momenta $P_I = \partial_{X^I} Z(X^I)$)

$$[P_I, X^J] = g_s^2 \delta^J_I$$

The infinitessimal change of polarisation gives rise to
a heat like equation for $Z$

$$\left(\bar{\partial}_i - \left(\frac{\chi}{24} - 1\right) K_i - f_i - C_{ij}^i \left(\frac{g_s^2}{2} D_i D_j - D_i F_0 D_j\right)\right) Z = 0$$

Here $e^{-K} = i \int_W \Omega \wedge \bar{\Omega}$, $C_{ijk} = \int_W \Omega \wedge \partial_i \partial_j \partial_k \Omega$, $G_{i\bar{j}} = \partial_i \bar{\partial}_j K$, $C_{ij}^k = e^{2K} G^{\bar{i} \bar{m}} G^{\bar{j} \bar{l}} \bar{C}_{\bar{i} \bar{j} \bar{k}}$ and $D_i$ are covariant w.r.t. to the Weil-Petersson– $G_{i\bar{j}}$ and the Kähler gauge connection.
In order to understand the monodromy

\[
\begin{pmatrix}
\tilde{P}_I \\
\tilde{X}^I
\end{pmatrix}
= \begin{pmatrix}
A^J_I \\
C^{IJ}_I
\end{pmatrix}
\begin{pmatrix}
P_I \\
X^I
\end{pmatrix}
\]

we need the global version of the wave function transformation for the transformation \( X^I \mapsto \tilde{X}^I \) generated by \( S(X, \tilde{X}) \) with

\[
dS = P_I dX^I - \tilde{P}_I d\tilde{X}^I,
\]

i.e.

\[
S(X\tilde{X}) = -\frac{1}{2}(C^{-1}D)_{JK}X^JX^K + (C^{-1})_{JK}X^J\tilde{X}^K - \frac{1}{2}(AC^{-1})_{JK}\tilde{X}^J\tilde{X}^K.
\]
Then the wave function transformation can by
\[
\tilde{Z}(\tilde{X}) = \int dX \ e^{-S(X,\tilde{X})/g_s^2} Z(X).
\]
now be approximated by a saddle point expansion, which gives rise to a BCOV like graph expansion as explained in Aganagic, Bouchard and AK 06.

**Analytic properties:**
If follows further from the fact that \( \Pi \) is solution of the Picard Fuchs eqations \( F_g(a) \) are a convergent series in \( z \) for \( |z| < c \). However note that \( Z \) is an asymptotic expansion in \( g_s \) for all values of \( z \). This follows
already in the A-model for $Q(z) = 0$ the constant map contribution $g > 1$ is $F_g(0) = (-1)^g \frac{\chi G_{B_{2g}B_{2g-2}}}{2(2g(2g-2))!}$.

In the following we would like to understand a physically well established property of $Z$ for elliptically fibred Calabi Yau using this general theory of the modular action on the wave function $Z$. 
1 Geometries: Principle geometrical properties of $M$

- **Local elliptic CY 3-folds:** $M$ is the total space of $O(-K_S) \to S$ with $S$ a (rational) elliptic surface $E \to \mathbb{P}^1$. $M$ has an global $U(1)$ symmetry which makes a refined topological partition function canonical. Earliest example: $E$-string, $S = \frac{1}{2}K3$. First modular properties Lerche, Mayr, AK 96, Refined solved in JHKK 17

- **Global elliptic CY 3-folds:** $M$ is the total space $E \to S$. Refined topological partition function depends on the complex structure. Earliest example $S = \mathbb{P}^2$
Fano, degree 18 hypersurface in weighted projective space $X_{18}(11169)$. First observation of modular properties \cite{Candelas:1993}, all genus results up to high base degree \cite{Huang:2015}.

Properties of the elliptic fibration

– Global properties: elliptic fibration can have a single section as in the examples above, or multi-sections or multiple sections or no sections the latter three are called genus one fibrations \cite{Fierro:2019}.

– Semi-local properties: elliptic fibration can be smooth i.e. it has only $I_1$ fibres, or higher Kodaira fibres in various co-dimensions in the base.
In particular for elliptic fibration over $C^2 = -k$, $k = 2, 3, 4, 5, 6, 7, 8, 12$ in the base: Minimal 6d SCFTs $k = 4$ solved in [Haghighat, Lockhart, Vafa, AK 04] using elliptic genus of $D_4$ quivers. With the modular approach solutions for all $k$ low base winding in [JKKLZ 17]. The Nakajima blow up formula can be generalised to the 6d gauge theories [J.Haghighat,SW, JKSW '19].
The general form of the result:
Let $M$ be an elliptically fibred 3-fold over a 2d fano surface $B$

$$\mathcal{E} \longrightarrow M \longrightarrow B$$

To be concrete we consider here the simplest case $B = \mathbb{P}^2$ and call $\tau$ and $T$ be the Kähler parameters of the elliptic fiber $\mathcal{E}$ and a line $l \subset \mathbb{P}^2$, $q = \exp(2\pi i \tau)$ and $Q = \exp(2\pi i T)$. We expand $Z$ in terms of the base degrees $b$ as

$$Z(t, g_s) = Z_0(\tau, \lambda) \left(1 + \sum_{b=1}^{\infty} Z_b(\tau, g_s) Q^b\right).$$

Then $Z_{b>0}$ is a quotient of even weak Jacobi forms
of the following form HKK’15

\[ Z_b(\tau, z) = \frac{\varphi_b(\tau, z)}{\eta^{36b}(\tau) \prod_{k=1}^{b} \varphi_{-2,1}(\tau, kz)}. \]

Here \( \eta(\tau) \) is the Dedekind function and \( \varphi_b(\tau, z) \) is an even weak Jacobi form of index \( \frac{1}{3}b(b - 1)(b + 4) \) and weight \( 16b \). For more general base classes \( \beta \in H_2(B, Z) \) one has

\[
Z_\beta = \frac{1}{\eta^{12\beta \cdot K_B}} \frac{\varphi_{m_\beta, k_\beta}(\tau, z)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \varphi_{-2,1}(\tau, sz)}.
\]
where $\varphi_{m_\beta,k_\beta}(\tau, z)$ is a weak Jacobi form of weight

$$k_\beta = 6\beta \cdot K_B - 2 \sum_{l=1}^{b_2(B)} \beta_l$$

and index

$$m_\beta = \frac{1}{6} \sum_{l=1}^{b_2(B)} \beta_l(1 + \beta_l)(1 + 2\beta_l) - \frac{1}{2}\beta \cdot (\beta - K_B).$$
☆ Jacobi forms

Definition of Jacobi forms
Jacobi forms \( \varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \) depend on a modular parameter \( \tau \in \mathbb{H} \) and an elliptic parameter \( z \in \mathbb{C} \). They transform under the modular group (Eichler & Zagier)

\[
\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \quad z \mapsto z_\gamma = \frac{z}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}) =: \Gamma
\]

as

\[
\varphi (\tau_\gamma, z_\gamma) = (c\tau + d)^k e^{\frac{2\pi imcz^2}{c\tau + d}} \varphi (\tau, z)
\]

and under quasi periodicity in the elliptic parameter
as

\[ \varphi(\tau, z + \lambda \tau + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \varphi(\tau, z), \quad \forall \lambda, \mu \in \mathbb{Z}. \]

Here \( k \in \mathbb{Z} \) is called the \textit{weight} and \( Bm \in \mathbb{Z}_{>0} \) is called the \textit{index} of the Jacobi form.

The Jacobi forms have a Fourier expansion

\[ \phi(\tau, z) = \sum_{n,r} c(n, r) q^n y^r, \quad \text{where} \ q = e^{2\pi i \tau}, \ y = e^{2\pi i z}. \]

Because of the quasi peridicity one has \( c(n, r) =: C(4nm - r^2, r) \), which depends on \( r \) only modulo \( 2m \). For a \textbf{holomorphic} Jacobi form \( c(n, r) = 0 \)
unless $4mn \geq r^2$, for cusp forms $c(n, r) = 0$ unless $4mn > r^2$, while for weak Jacobi forms one has only the condition $c(n, r) = 0$ unless $n \geq 0$. 

The ring of weak Jacobi forms

A weak Jacobi form of given index \( m \) and even modular weight \( k \) is freely generated over the ring of modular forms of level one, i.e. polynomials in \( Q = E_4(\tau) \), \( R = E_6(\tau) \) and \( A = \varphi_{0,1}(\tau,z) \), \( B = \varphi_{-2,1}(\tau,z) \) as

\[
J_{k,m}^{\text{weak}} = \bigoplus_{j=0}^{m} M_{k+2j}(\Gamma_0) \varphi_{-2,1}^j \varphi_{0,1}^{m-j}.
\]

The generators are the Eisenstein series \( E_4, E_6 \)

\[
E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{\substack{m,n \in \mathbb{Z} \atop (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n
\]
as well as

\[ A = -\frac{\theta_1(\tau, z)^2}{\eta^6(\tau)}, \quad B = 4 \left( \frac{\theta_2(\tau, z)^2}{\theta_2(0, \tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(0, \tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(0, \tau)^2} \right). \]

To summarize generators and quantities defining the topological string partition function

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& Q & R & A & B & \varphi_b & Z_b(\tau, z) \\
\hline
\text{weight } k: & 4 & 6 & -2 & 0 & 16b & 0 \\
\text{index } m: & 0 & 0 & 1 & 1 & \frac{1}{3}b(b-1)(b+4) & \frac{b(b-3)}{2} \\
\hline
\end{array}
\]

Since the numerator in

\[ Z_b(\tau, z) = \frac{\varphi_b(\tau, z)}{\eta^{36b}(\tau) \prod_{k=1}^{b} \varphi_{-2,1}(\tau, k z)}. \]
is finitely generated, we can get for each $b$ the full genus answer based on a finite number of boundary data
– the conifold gap condition and regularity at the orbifold Huang, Quakenbush, A.K. hep-th/0612125
– the involution symmetry on $\mathcal{M} I : \Omega \mapsto i\Omega \leftrightarrow$ fibre modularity
– the parametrization of $Z$ in terms of weak Jacobi-Forms
we can solve the compact elliptic fibration over $\mathbb{P}^2$ to $b = 20$ for all $d_E$ and $\forall g$ or to genus $189 \ \forall b$ and $\forall d_E$. 
Witten’s wave function and weak Jacobi-forms

Witten gave a wave function interpretation the topological string partition function, which implies

\[
\left( \frac{\partial}{\partial (t')^{\bar{a}}} + \frac{i}{2} g_s^2 e^{2K} C_{\bar{a} \bar{b} \bar{c}} G^{b \bar{b}} G^{c \bar{c}} \frac{D}{Dt^{b}} \frac{D}{Dt^{c}} \right) Z(g_s, \tau, b) = 0 ,
\]

and summarizes all holomorphic anomaly equations. We want to study this in limit of large base \( B \). Let A.K, Manschot, Wotschke 1205.1795

\[ K_i \cdot K_j = c_{ij} \]
the intersection in the base $B$ and

$$K_B = -c_1(b) = - \sum_i a^i K_i$$

then with $s$ the number of rational sections the classical intersections $\kappa_{klm}$ of the elliptic CY $M$ are

$$\kappa_{eee} = s \int_B c_1^2(T_B)$$

$$\kappa_{eii} = s a_i$$

$$\kappa_{eij} = s c_{ij}$$

$$\kappa_{ijk} = 0$$

Now the Kähler $K$ potential the and Weil Petersson
metric $G^{i\bar{j}}$ follows from the prepotential

$$F^{(0)} \sim -\frac{\kappa_{abc}}{3} t^a t^b t^c + \chi(M) \frac{\zeta(3)}{2(2\pi i)^3} + \sum_{\beta \in H_2(M,Z)} n_0^\beta \text{Li}_3(q^\beta).$$

Now analyze $C_{\bar{a}bc} := e^{2K} c_{\bar{a}bc} G^{\bar{b}\bar{c}}$ in the limit $\text{Im}(T^k) = h \to \infty$

$$C_{\bar{a}}^{\alpha\beta} = \begin{pmatrix} -\frac{2\tau_2^2 h^4}{V} + O(h^5) & A^1 h^3 + O(h^5) & \cdots & A^{r-1} h^3 + O(h^5) \\ A^1 h^3 + O(h^5) & \vdots & \ddots & \vdots \\ A^{r-1} h^3 + O(h^5) & \vdots & \ddots & -\frac{1}{4\tau_2^2} c^{kl} + O(h) \\ \end{pmatrix}$$

where $V = (c_{kl} T_2^k T_2^l)^2$ and $A^i = \frac{(3E + 4\tau_2^3 T_2^i)}{6\tau_2^2 V}$, with
\[ E = \frac{\zeta(3)}{2(2\pi)^3} \chi(M) \]. The \( c_{kl} = \tilde{D}_k \cdot \tilde{D}_l \) and \( c^{kl} = C^k \cdot C^l \) are the intersections on the base.

Applied to the wave function equation of \( Z \) with \( (t')^{\bar{a}} = \bar{\tau} \) and \( Q^\beta = e^{2\pi i b T} \), we get in the large base limit, because of the special form of the intersection matrix of elliptically fibered Calabi-Yau 3 folds only derivatives in the base direction \( T^i \) for \( t^b \) and \( t^c \).

Identifying \( g_s \) with \( 2\pi i z \) and using the fact that the
only $\bar{\tau}$ dependence is in $\hat{E}_2$ this becomes

$$
\left( \partial_{\hat{E}_2} + \frac{\beta \cdot (\beta - K_B)}{24} z^2 \right) Z_\beta(\tau, z) = 0
$$

which is solved by a weak Jacobi form of index $m = \frac{b(b-3)}{2}$ as we argue below.

Because of modularity and quasiperiodicity given a weak Jacobi form $\varphi_{k,m}(\tau, z)$ one can always define modular form of weight $k$ as follows

$$
\tilde{\varphi}_k(\tau, z) = e^{\frac{\pi^2}{3} m z^2 E_2(\tau)} \varphi_{k,m}(\tau, z)
$$
It follows that the weak Jacobi forms $\varphi_{k,m}(\tau, z)$ have a Taylor expansion in $z$ with coefficients that are quasi-modular forms as in Eichler and Zagier $^1$.

$$\varphi_{k,m} = \xi_0(\tau) + \left( \frac{\xi_0(\tau)}{2} + \frac{m\xi'_0(\tau)}{k} \right) z^2 + \left( \frac{\xi_2(\tau)}{24} + \frac{m\xi'_1(\tau)}{2(k+2)} + \frac{m^2\xi''_0(\tau)}{2k(k+1)} \right) z^4 + O(z^6).$$

Moreover one has

$$\left( \partial_{E_2} + \frac{mg_s^2}{12} \right) \varphi_{k,m}(\tau, z) = 0.$$

In particular $A$ and $B$ are quasi-modular forms that

$^1$E.g. $\phi_{-2,1}(\tau, z) = -z^2 + \frac{E_2 z^4}{12} + \frac{-5E_2^2 + E_4}{1440} z^6 + \frac{35E_2^3 - 21E_2E_4 + 4E_6}{362880} z^8 + O(z^{10}).$
satisfy the modular anomaly equation

\[ \partial_{E_2} A = -\frac{g_s^2}{12} A, \quad \partial_{E_2} B = -\frac{g_s^2}{12} B. \] (1)

We can write this as the holomorphic anomaly equation

\[ \left(2\pi i \text{Im}^2(\tau) \bar{\partial}_\tau - \frac{mg_s^2}{4}\right) \hat{\varphi}_{k,m}(\tau, z) = 0. \] (2)
Elliptically fibred CY- manifolds

Global fibration over $\mathbb{P}^2$

The formalism leads to a series of all genus predictions of BPS invariants for low base degree HKK’15. E.g. for $b = 1$ and $b = 2$ the numerator is

$$\varphi_1 = -\frac{Q(31Q^3 + 113P^2)}{48},$$

which leads to the following prediction of BPS invariants
Table 2: Some BPS invariants $n^g_{(d_E, 1)}$ for base degree $b = 1$ and $g, d_E \leq 6$. 

\[
\varphi_2 = \frac{B^4Q^2 \left(31Q^3 + 113R^2\right)^2}{23887872} + \frac{1}{1146617856} \left[2507892B^3AQ^7R + 9070872B^3AQ^4R^3 \\
+ 2355828B^3AQR^5 + 36469B^2A^2Q^9 + 764613B^2A^2Q^6R^2 - 823017B^2A^2Q^3R^4 \\
+ 21935B^2A^2R^6 - 9004644BA^3Q^8R - 30250296BA^3Q^5R^3 - 6530148BA^3Q^2R^5 \\
+ 31A^4Q^{10} + 5986623A^4Q^7R^2 + 19960101A^4Q^4R^4 + 4908413A^4QR^6\right],
\]
Table 3: Some BPS invariants for $n^g_{(d_E,2)}$

② Checks form algebraic geometry:
Using the definition of BPS states as Hodge numbers of the BPS moduli space, we get vanishing conditions, from the Castelnouvo bounds, as well as explicite
results for non singular moduli spaces:

Figure 1: The figure shows the boundary of non-vanishing curves for the values of $b = 1, 2, 3, 4, 5$.

Computing the Euler characteristic of the BPS moduli space, we obtain for these values on the edges of the
figure

\[ n_{d_E,b}^{d_E b - (3b^2 - b - 2)/2} = (-1)^{d_E b - (1/2)(3b^2 + b - 4)} \]

\[ 3 \left( d_E b - \frac{3b^2 + b - 6}{2} \right) \cdot \]

which perfectly matches the prediction of the weak Jacobi forms.
Other basis:
Take $B = F_1$ the Hirzebruch surface $F_1$. This a rational fibration with a $(-1)$ curve as the section. Together with the elliptic fibre $M$ contains now the elliptic surface $\frac{1}{2}K3$ with $12$ $I_1$ fibres, which gives rise to the $E$-string.

$\beta = (b_1, b_2) \in H_2(F_1, \mathbb{Z})$, $b_1$ the degree w.r.t. to the $(-1)$ section $b_2$ the degree w.r.t. the fibre.

The Castnovo bound is sufficient to fix the mero–morphic Jacobiforms $Z_\beta(\tau, z)$ if $\beta \cdot (\beta - K_B) \leq 0$. 
Figure 2: $\beta(\beta - K_B)$ in the Kähler moduli space of $F_1$. 
Conclusions:

\[ Z_{d_B}(\tau, z) = \frac{\varphi_{d_B}(\tau, z)}{\eta^{36d_B(\tau)} \prod_{k=1}^{d_B} \varphi_{-2,1}(\tau, kz)} . \] (4)

- Since the elliptic argument \( z \) of the Jacobi forms is identified with the string coupling

\[ g_s = 2\pi i z \]

this expression captures all genus contributions for a given base class.

- From the transformation properties of weak Jacobi forms it follows that the dependence of \( Z \) on string the
coupling is coupling is quasi periodic.

- We can make infinitely many checks from algebraic geometry for those curves which have smooth moduli spaces, as seen above. But e.g. for $d_B = 1$ one can confirm the formulas for all classes.

- Generalizations

\[
Z(t_b, \tau, t_m, \epsilon_1, \epsilon_2) = Z_{\beta=0} \left( 1 + \sum_{\beta \neq 0} Z_\beta(\tau, t_m, \epsilon_1, \epsilon_2) Q^\beta \right)
\]

Now the $Z_\beta(\tau, t_m, \epsilon_1, \epsilon_2)$ are also meromorphic Jacobi forms and $t_m, \epsilon_1, \epsilon_2$ are all elliptic parameters and the $Z_\beta(\tau, t_m, \epsilon_1, \epsilon_2)$ are rings of Weyl invariant
Jacobi forms.

The index is related to the anomaly polynomials of the $(1,0)$ 6d theory

\[
M(\epsilon_+, \epsilon_-, m)_\beta = -\frac{\epsilon_+^2}{2} ((2 - A_{JJ}) \beta^J + A_{IJ} \beta^I \beta^J)
- \frac{\epsilon_+^2}{2} \left( ((2 - A_{JJ}) + 2h^\vee_{G,J}) \beta^J - A_{IJ} \beta^I \beta^J \right)
- \beta^J A_{Ja} \frac{1}{2} (m_a, m_a)_{g_a}.
\]