Einstein equations from $p$-adic strings

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...and work in progress
1740: **Euler** introduces the Zeta function, which encapsulates properties of prime numbers.

1968: **Veneziano** amplitudes: certain 4-point scattering amplitudes can be described by Euler Gamma functions. These were later recognized as the amplitudes of open string tachyon scattering.
The Gamma function is a number-theoretic object: it is the proportionality coefficient between a multiplicative character and its Fourier transform.

Is there a connection between strings and number theory?
The Riemann Zeta function

\[ \zeta(\mu) := \sum_{n=1}^{\infty} \frac{1}{n^\mu} = \prod_{\text{prime}} \frac{1}{1 - p^{-\mu}} \]

encodes deep properties of prime numbers.

It obeys a functional equation between \( \mu \) and \( 1 - \mu \):

\[ \zeta(\mu) = 2^\mu \pi^{\mu - 1} \sin \left( \frac{\pi \mu}{2} \right) \Gamma(1 - \mu) \zeta(1 - \mu) \]
Riemann Zeta in the complex plane

- **Critical strip:** $0 \leq \Re(\mu) \leq 1$
- **Critical line:** $\Re(\mu) = 1/2$
- **Pole at $\mu = 1$**
- **Trivial zeros at negative even integers**
- **Nontrivial zeros in the critical strip**
Riemann hypothesis

Conjecture (Riemann hypothesis, RH, 1859)
Every zero of $\zeta(\mu)$ in the critical strip lies on the critical line $1/2 + it$.

The Riemann Zeta is the prototypical $L$-function.

Riemann hypotheses are expected to hold for all $L$-functions.
**Conjecture (Hilbert-Pólya, HP)**

The imaginary parts of the zeros of the Riemann Zeta function are the eigenvalues of an unbounded self-adjoint operator.

Was proposed by Pólya, 1912-1914.

HP implies RH.

Suggests connection to quantum mechanics, or to physics more broadly.
String theory in the universe of physical theories

A hierarchy of theories.
String theory is a quantum theory of strings and extended objects.

Contains gravity (... and many other things).

Different versions exist.

In this talk: **bosonic string theory**. Does not contain fermions (matter).
Imagine an open or closed string propagating through ambient spacetime $\mathbb{R}^d$. The one-dimensional string traces out a two-dimensional \textit{worldsheet} $W$.

$X^a(\sigma_i)$: ambient spacetime coordinates of a point $(\sigma_1, \sigma_2)$ on the worldsheet.

Coordinates $X$ are functions $X : W \rightarrow \mathbb{R}^d$. 
Dynamics of bosonic string theory

Polyakov action (Deser-Zumino, Brink-Di Vecchia-Howe 1976):

\[ S_{\text{Polyakov}} = \int_W d^2 \sigma \sqrt{h} h^{ij} g_{ab} \partial_i X^a \partial_j X^b \]

\( g_{ab} = \) target space metric, indices \( h_{ij} = \) worldsheet metric, indices

Equations of motion for the string:

\[ \Delta X = 0, \quad \Delta \cdot = \frac{1}{\sqrt{h}} \partial_i \left( \sqrt{h} h^{ij} \partial_j \cdot \right) \]

\( \Delta \) is the worldsheet Laplacian.
The scaling

\[ h_{ij} \rightarrow e^{\phi(\sigma_i)} h_{ij}, \quad X \rightarrow X, \]

is a symmetry of the Polyakov action. This is known as Weyl invariance.

Weyl invariance is a symmetry of critical string theory in \( d = 26 \) dimensions.

Consequence: worldsheet mass terms \( m^2 X^2 / 2 \) in the Polyakov action are forbidden.
Archimedean strings

The Veneziano amplitudes

\[ A(k_1, \ldots k_4) = \int_{\mathbb{R}} dx |x|_\infty^{k_1 k_2} |1 - x|_\infty^{k_1 k_3} = \frac{\Gamma_\infty (1 + k_1 k_2) \Gamma_\infty (1 + k_1 k_4)}{\Gamma_\infty (2 + k_1 k_2 + k_1 k_4)} \]

make sense when the Gamma functions are Euler or Gelfand-Graev.

The real case corresponds to tachyon scattering for open strings.

What do the \( p \)-adic cases correspond to?
The Veneziano amplitudes

\[ A(k_1, \ldots, k_4) = \int_{\mathbb{Q}_p} dx \ |x|_p^{k_1 k_2} |1 - x|_p^{k_1 k_3} = \frac{\Gamma_p(1 + k_1 k_2) \Gamma_p(1 + k_1 k_4)}{\Gamma_p(2 + k_1 k_2 + k_1 k_4)} \]

make sense when the Gamma functions are Euler or Gelfand-Graev (Freund-Olson 1987).

The real case corresponds to tachyon scattering for open strings.

What do the \( p \)-adic cases correspond to?
$p$-adic string worldsheet

Zabrodin 1989: the $p$-adic Veneziano amplitudes follow if worldsheet is the Bruhat-Tits tree $T_p$, via $e^{ik^aX^a}$ insertions on the boundary.

Bruhat-Tits tree: an infinite tree of uniform degree $p + 1$. Boundary is $\mathbb{P}^1(\mathbb{Q}_p)$. 
Distance action for the $p$-adic string

Zabrodin’s $p$-adic Polyakov action:

$$S_{Zabrodin} = \frac{1}{V} \sum_{(ij) \in E(T_p)} \frac{1}{a^2_{(ij)}} \eta_{ab} \left( X^a_j - X^a_i \right) \left( X^b_j - X^b_i \right)$$

A more general action [Huang, S., Yau ’19]:

$$S = \frac{1}{V} \sum_{(ij) \in E(T_p)} \frac{d^2(X_i, X_j)}{a^2_{(ij)}}$$

Distance $d$ is on the target space. When the target manifold is flat this action reduces to Zabrodin’s action.

String equation of motion, in terms of the tree Laplacian:

$$\Delta X = 0, \quad \Delta X_i = \sum_{j \sim i} X_j - (p + 1)X_i$$
Eigenfunctions of the Laplacian

Bruhat-Tits tree with two marked points: \( \infty \) and \( O \). \( i \) is any vertex on the tree.

**Theorem (Zabrodin)**

The momentum operator eigenfunctions \( \phi_{\mu}(i) := p^{\mu}_{i,\infty} \) satisfy the Laplacian eigenequation \( \Delta \phi_{\mu} = \lambda_{\mu} \phi_{\mu} \), with eigenvalue

\[
\lambda_{\mu} = p^{\mu} + p^{1-\mu} - p - 1, \quad \mu \in \mathbb{C}.
\]
The eigenvalues $\lambda_\mu$ have $\mu \leftrightarrow 1 - \mu$ symmetry.

Up to phase, $\lambda_\mu$ is real if $\mu \in \mathbb{R}$, or if $\mu = 1/2 + it$ with $t \in \mathbb{R}$ (or if $\mu = i\pi / \log p$).
Define an inner product for functions $\phi, \psi : V(T_p) \rightarrow \mathbb{C}$ as

$$\langle \phi | \psi \rangle = \sum_{i \in V(T_p)} \phi^*(i)\psi(i).$$

**Theorem (Eigenfunction orthonormality [Huang, S., Yau ’19])**

The inner product of eigenfunctions $\phi_\mu(i) = p^\mu \langle i, \infty \rangle$ is

$$\langle \phi_\mu | \phi_\nu \rangle = \frac{1}{1 - p} \delta_{\mu^* + \nu} + \frac{p}{1 - p} \delta_{\mu^* + \nu - 1}.$$
Naively, the inner product of momentum eigenfunctions diverges.

Needs regularization:

$$\sum_{k=1}^{\infty} p^{-k\mu} = \frac{1}{p^\mu - 1}, \quad \mu \neq \frac{2\pi i k}{\log p}$$

This regularization can be understood as expanding a meromorphic function at a point and discarding the diverging term (if any).

It is similar to the analytic continuation of the Euler Gamma function. It is the same as the analytic continuation of the Gelfand-Graev Gamma function.
When $\Re(\mu) = 1/2$, the momentum eigenfunctions $p^{\mu(i,\infty)}$ are the spherical vectors of the unitary principal series representation of $\text{PGL}(2, \mathbb{Q}_p)$. The principal series representations of $\text{PGL}(2, \mathbb{Q}_p)$ thus are plane waves on the Bruhat-Tits tree.
Laplacian eigenvalue product

The 4-point Veneziano amplitudes satisfy the product formula

\[ A_{(\infty)}^{-1}(k_i) = \prod_p A_p(k_i). \]

**Definition (Eigenvalue product)**

Define an Archimedean quantity as

\[ \lambda_{(\infty)} := \prod_p \lambda_{(p)}^{-1}. \]

This Archimedean quantity is just

\[ \lambda_{(\infty)} = \prod_p \frac{1}{p(1 - p^{-\mu})(1 - p^{\mu-1})} = \frac{1}{4\pi^2} \zeta(\mu)\zeta(1 - \mu). \]
Suppose Archimedean Weyl invariance (or another physical requirement) demands $\lambda(\infty) = 0$.

Self-consistency of string theory should imply that the tree Laplacian $\Delta_p$ is self-adjoint.

**HP (strengthened [Huang, S., Yau '19])**

Weyl invariance and self-adjointness of $\Delta_p$ imply RH, or that the zeros of $\zeta(\mu)$ are on the $\pi ik / \log p$ lines in the critical strip.

The converse also holds: RH implies the eigenvalues of $\Delta_p$ are real.

Intuition: the zeroes of Riemann Zeta are related to the spectrum of bosonic strings.
An $L$-function is

$$L(\mu, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\mu} = \prod_p \frac{1}{1 - \chi(p)p^{-\mu}}.$$

$\chi : \mathbb{Z} \rightarrow \mathbb{C}$ is a Dirichlet character. It is periodic (with conductor $q$), and completely multiplicative.

$L$-functions satisfy functional equations, are conjectured to obey RH.
Aside: \(L\)-functions from string theory

Consider the action [in progress]:

\[
S = \sum_{(ij) \in E(T_p)} \frac{d^2(X_i, X_j)}{a^2} + \sum_{i \in V(T_p)} m^2 d^2(X_i, \mathcal{O})
\]

\(\mathcal{O}\) is a special point on the target manifold.

The flat space limit:

\[
S = \sum_{(ij) \in E(T_p)} \frac{(X_i - X_j)^2}{a^2} + \sum_{i \in V(T_p)} m^2 X_i^2
\]

The equation of motion:

\[
\left(\Delta - m^2 a^2\right) X = 0
\]

String theory around a nontrivial background?
Aside: \(L\)-functions from string theory

Up to a log derivative of the \(L\)-function, the eigenfunctions of the equation of motion operator product into

\[
\prod_p \left[ 1 - \chi(p)p^{-\mu} \right]^{-1} \left[ 1 - \chi(p)p^{\mu-1} \right]^{-1} = L(\mu, \chi)L(1 - \mu, \chi).
\]

The mass parameter is related to the Dirichlet character:

\[
m^2 a^2 = p \chi^{-1}(p) [1 - \chi(p)] \left[ 1 - \chi(p)p^{-1} \right]
\]

HP:

\[
\lambda_{\mu} \in \mathbb{R} \iff m^2 a^2 \in \mathbb{R} \iff \chi(p) = \pm 1
\]

HP implies RH (up to parallel lines) for quadratic Dirichlet \(L\)-functions.
Back to string theory: path integrals on the Bruhat-Tits tree

An $n$-point function:

$$\langle X_{i_1}^{a_1} \ldots X_{i_n}^{a_n} \rangle = \frac{1}{Z} \int DX e^{iS[X]} X_{i_1}^{a_1} \ldots X_{i_n}^{a_n}$$

Evaluate such expressions by expanding in momentum eigenbasis

$$X_i^a = \int_\mu c_\mu^a \phi_\mu(i)$$

and using the orthonormality relations

$$\langle \phi_\mu | \phi_\nu \rangle = \frac{1}{1-p} \delta_{\mu^*+\nu} + \frac{p}{1-p} \delta_{\mu^*+\nu-1}.$$ 

**Theorem [in progress]**

Using the orthogonality relations, regularization, and boundary conditions, only the critical line and $\mu = 0$ contribute to $n$-point functions.
Hilbert-Pólya, path integral formulation

The points with $\lambda_{(\infty)}(\mu) = 0$ contribute to the path integral.
Contributions to the path integral + Archimedean Weyl invariance.

**HP (Path integral version [in progress])**

The points with $\lambda_{(\infty)}(\mu) = 0$ contribute to the path integral.
HP suggests the nontrivial Riemann Zeta zeros are related to the spectrum of bosonic string theory.

$\mu = 0$ is not a principal series representation. Presumably corresponds to the vacuum energy.

Tree-level two-point function:

$$\langle X^a_i X^b_j \rangle_{\text{tree}} \sim \eta^{ab} \int \frac{1}{\lambda_{\frac{1}{2}+it}^{\frac{1}{2}+it}} \phi_{\frac{1}{2}+it}^{\frac{1}{2}+it}(i)\phi_{\frac{1}{2}-it}^{\frac{1}{2}-it}(j)$$

$\eta^{ab} =$ target space metric $\quad \lambda_{\frac{1}{2}+it} =$ Laplacian eigenvalue on the critical line

$\phi_{\frac{1}{2}+it} =$ momentum eigenvalue on the critical line
Curved target space

When the target space has nontrivial curvature, the action

$$S = \sum_{(ij) \in E(T_p)} \frac{d^2 (X_i, X_j)}{V a^2_{ij}}$$

reduces to [Huang, S., Yau '19]

$$S = \sum_{(ij) \in E(T_p)} \frac{1}{V a^2_{ij}} \left[ \eta_{ab} (X_j^a - X_i^a) (X_j^b - X_i^b) - \frac{1}{3} R_{abcd} X_i^c X_i^d X_j^a X_j^b \right].$$

$R_{abcd} = \text{target space Riemann tensor}$ \hspace{1cm} $V = \text{number of neighbors of each vertex}$

This is a field theory with \textit{quartic} interactions.
Add multiplicity to the edges:

\[ V \rightarrow \Lambda V, \quad \Delta \rightarrow \Lambda \Delta \]

The target manifold metric should not run with \( \Lambda \).

The one-loop correction to the two-point function \( \langle X_i^a X_j^b \rangle \) must vanish:

\[
\langle X_i^a X_j^b \rangle_{1\text{-loop}} = \frac{1}{Z} \int D X e^{i S_{\text{free}}[X]} \sum_{k_1 \sim k_2} \left( R_{mpnq} X_{k_1}^m X_{k_1}^n X_{k_2}^p X_{k_2}^q \right) X_i^a X_j^b
\]
Two-point function at one-loop

![Diagram of two-point function](image)

One-loop correction to the two-point function.

There are three types of **Wick contractions**:

1. $\{\mu_3, \mu_4\} = \{0, 0\}, \{\mu, \nu, \rho, \lambda\} = \{0, 0, 0, 0\}$.
2. $\{\mu_3, \mu_4\} = \{0, 0\}, \{\mu, \nu, \rho, \lambda\} = \{0, 0, \frac{1}{2} + it, \frac{1}{2} - it\}$.
3. $\{\mu_3, \mu_4\} = \{\frac{1}{2} + it, \frac{1}{2} - it\}, \{\mu, \nu, \rho, \lambda\} = \{0, 0, \frac{1}{2} + it, \frac{1}{2} - it\}$. 
Two-point function at one-loop

The one-loop correction takes the form of a sum of diverging power laws in the regulator $M$. The divergence always comes from the $\mu = 0$ eigenvalue.

All terms are proportional to Ricci tensor $R_{ab}$. Thus the target-space Einstein equations are obtained,

$$R_{ab} = 0.$$
Conclusion: Summary

What we have shown

**Riemann Zeta** is related to the eigenvalues of the $p$-adic string worldsheet Laplacian.

The worldsheet momentum eigenfunctions obey powerful orthogonality conditions.

Only the critical line and $\mu = 0$ contribute to $p$-adic string path integral.

The one-loop non-running of the target space metric implies the target space (Archimedean) **Einstein equations**.
Conclusion: Some conjectures

What is likely true

The *spectrum* of bosonic string theory is related to the zeroes of the Riemann Zeta. The *zeros* of the Riemann Zeta contribute to the string path integral (true if RH). *L*-functions naturally relate to string theory.
Why should $L$-functions and string theory be connected?

They are both hard problems on counting: $L$-functions count primes, string theory is about counting degrees of freedom.

The precise connection is made by the worldsheets Laplacian playing a central role on both sides.
Thank you!