$G_2$ manifolds and mirror symmetry

Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics – First Annual Meeting, New York, 9/14/2017

Andreas Braun
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based on

- [1602.03521]
- [1701.05202] + [1710.xxxxx] with Michele del Zotto (Stony Brook)
- [1708.07215] with Sakura Schäfer-Nameki (Oxford)
$G_2$ manifolds and string dualities

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Dualities

- $\text{Theory } \#1$ on background $F_1 \times \mathbb{R}_{1,n}$
- $\text{Theory } \#2$ on background $F_2 \times \mathbb{R}_{1,k}$

The 'backgrounds' $F_1/2$ often include more data than just geometry!

The geometric data is often that of a manifold of special holonomy.
theory \#1 on background $F_1 \times \mathbb{R}^{1,n}$

\[ \sim \]

theory \#2 on background $F_2 \times \mathbb{R}^{1,k}$

\[ \begin{array}{c}
\{ \text{the same physics} \}
\end{array} \]
Dualities

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$\sim$

theory #2 on background $F_2 \times \mathbb{R}^{1,k}$

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The ‘backgrounds’ $F_{1/2}$ often include more data than just geometry!
The geometric data is often that of a manifold of special holonomy.

consequences:

- moduli spaces agree: $\mathcal{M}_1 = \mathcal{M}_2$, i.e. massless states agree
- massive states agree
- dynamics and observables agree

However: \textbf{often need to work in specific limits/truncations to have tractable situation}
Dualities

theory \#1 on background \( F_1 \times \mathbb{R}^{1,n} \)
\[
\sim
\]
theory \#2 on background \( F_2 \times \mathbb{R}^{1,k} \)
\[
\begin{align*}
\text{the same physics} \\
\end{align*}
\]

The ‘backgrounds’ \( F_{1/2} \) often include more data than just geometry!
The geometric data is often that of a manifold of special holonomy.

Dualities can be related by fibrations (with singular fibres):

\[
F_{1/2} \rightarrow X_{1/2} \rightarrow B_m
\]

theory \#1 on background \( X_1 \times \mathbb{R}^{1,n-m} \)
\[
\sim
\]
theory \#2 on background \( X_2 \times \mathbb{R}^{1,k-m} \)
\[
\begin{align*}
\text{the same physics} \\
\end{align*}
\]
Example: Mirror Symmetry is T-duality

type IIA string on $S^1_r \times \mathbb{R}^{1,8}$

$\sim$

type IIB string on $S^1_{1/r} \times \mathbb{R}^{1,8}$

\{ ‘ T-duality ’ \}
Example: Mirror Symmetry is T-duality

type IIA string on $T^3 \times \mathbb{R}^{1,6}$

\[ \sim \]

type IIB string on $T'_3 \times \mathbb{R}^{1,6}$

\[
\{ \quad 3 \text{ T-dualities} \quad \}
\]

extra data: $B$-field!
Example: Mirror Symmetry is T-duality

\[
\begin{align*}
\text{type IIA string on } X \times \mathbb{R}^{1,3} & \sim \\
\text{type IIB string on } X^\vee \times \mathbb{R}^{1,3} & \end{align*}
\]

\[
\text{'mirror symmetry' }
\]

extra data: \(B\)-field!

from SYZ fibrations \cite{Strominger,Yau,Zaslow} of Calabi-Yau mirror manifolds \(X\) and \(X^\vee\):

\[
\begin{align*}
T^3 & \to X \to S^3 \\
T^3_\vee & \to X^\vee \to S^3.
\end{align*}
\]
Example: Mirror Symmetry is T-duality

\[
\begin{align*}
\text{type IIA string on } & \quad X \times \mathbb{R}^{1,3} \\
\sim & \quad \text{‘mirror symmetry’} \\
\text{type IIB string on } & \quad X^\vee \times \mathbb{R}^{1,3}
\end{align*}
\]

extra data: \( B \)-field!

\[
\mathcal{M}_{IIA} = \mathcal{M}_{IIB} = \mathcal{M}_V \times \mathcal{M}_H
\]

where

\[
\begin{align*}
\mathcal{M}_V &= \text{complexified Kähler moduli of } X = \text{complex structure of } X^\vee \\
\mathcal{M}_H &= \text{complex structure of } X = \text{complexified Kähler moduli of } X^\vee
\end{align*}
\]

comparing more refined properties leads e.g. to:

- Gromov-Witten invariants of \( X \) from periods of the mirror \( X^\vee \)
- Homological mirror symmetry
Story 1: Mirror Symmetry and $G_2$ manifolds

For (co)-associative fibrations

$$T^3 \to J \to M_4$$

or

$$T^4 \to J \to M_3$$

on a $G_2$ manifold $J$, can use the same logic leading to [Acharya]:

\[
\begin{align*}
\text{type IIA string on } J \times \mathbb{R}^{1,2} &\sim \text{type IIB string on } J^\vee \times \mathbb{R}^{1,2} \\
\text{or} &
\end{align*}
\]

exploiting $T^3$ fibration

or

\[
\begin{align*}
\text{type IIA/B string on } J \times \mathbb{R}^{1,2} &\sim \text{type IIA/B string on } J^\vee \times \mathbb{R}^{1,2} \\
\text{or} &
\end{align*}
\]

exploiting $T^4$ fibration

a coarse invariant is [Shatashvili, Vafa]

$$\dim \mathcal{M}_C = b_2 + b_3$$
Mirror Symmetry for TCS

$G_2$ twisted connected sums [Kovalev; Corti,Haskins,Nordström, Pacini] are defined by the data:

acyl Calabi-Yau threefolds $X_\pm = Z_\pm \setminus S_{0\pm}$

a matching $\phi : S_{0+} \rightarrow S_{0-}$

\[ \{ \text{a } G_2 \text{ manifold } J \} \]
Mirror Symmetry for TCS: $T^4$

acyl Calabi-Yau threefolds $X_\pm = Z_\pm \setminus S_{0\pm}$

a matching $\phi : S_{0+} \to S_{0-}$

\[
\begin{aligned}
\text{expectation:}
\text{ there exist coasscitive } T^4 \text{ fibrations restricting to the SYZ fibrations of } X_\pm \\
\text{ (see [Donaldson]). The corresponding mirror } J^\vee \text{ is constructed from }
\end{aligned}
\]

acyl Calabi-Yau threefolds $X^\vee_\pm = Z^\vee_\pm \setminus S_{0^\vee\pm}$

a matching $\phi^\vee : S_{0^\vee+} \to S_{0^\vee-}$

\[
\begin{aligned}
\text{the } G_2 \text{ manifold } J^\vee
\end{aligned}
\]

Found an explicit construction for $Z_\pm = \text{toric hypersurface } [\text{AB,del Zotto}]$. In general:

- $b_2(J) + b_3(J) = b_2(J^\vee) + b_3(J^\vee)$
- $H^\bullet(J, \mathbb{Z}) = H^\bullet(J^\vee, \mathbb{Z})$
Mirror Symmetry for TCS: $T^3$

acyl Calabi-Yau threefolds $X_\pm = Z_\pm \setminus S_{0\pm}$
\[ + \]
a matching $\phi : S_{0+} \to S_{0-}$ \( \left\{ \right. \)

a $G_2$ manifold $J$

**expectation:** if $X_+$ (and $S_+$) is elliptic, there exist an associative $T^3$ fibration restricting to the SYZ fibration of $X_-$ and the elliptic fibration on $X_+$. The corresponding mirror $J^\wedge$ is constructed from

acyl Calabi-Yau threefolds $X_{\vee}$ and $X_+$
\[ + \]
a matching $\phi^\wedge : S_{0+}^\vee \to S_{0-}$ \( \left\{ \right. \)

the $G_2$ manifold $J^\wedge$

Found an explicit construction for $Z_\pm = \text{toric hypersurface} \ [AB, \text{del Zotto}]$. In general:

- $b_2(J) + b_3(J) = b_2(J^\vee) + b_3(J^\vee)$
- $H^\bullet(J, \mathbb{Z}) = H^\bullet(J^\vee, \mathbb{Z})$
Mirror Symmetry for TCS: $T^3$

The acyl Calabi-Yau threefolds $X_\pm = Z_\pm \setminus S_{0\pm}$

$$a \ G_2 \text{ manifold } J$$

A matching $\phi : S_{0+} \rightarrow S_{0-}$

**expectation:** if $X_+$ (and $S_+$) is elliptic, there exist an associative $T^3$ fibration restricting to the SYZ fibration of $X_-$ and the elliptic fibration on $X_+$. The corresponding mirror $J^\wedge$ is constructed from

acyl Calabi-Yau threefolds $X_\wedge$ and $X_+$

$$\begin{align*}
\text{a matching } & \phi^\wedge : S_{0\wedge}^+ \rightarrow S_{0\wedge}^- \\
\text{the } G_2 \text{ manifold } & J^\wedge
\end{align*}$$

Found an explicit construction for $Z_\pm = \text{toric hypersurface} \ [AB, \text{del Zotto}].$ In general:

- $b_2(J) + b_3(J) = b_2(J^\wedge) + b_3(J^\wedge)$
- $H_\bullet(J, \mathbb{Z}) = H_\bullet(J^\wedge, \mathbb{Z})$

For [Joyce] orbifolds, our mirror map is consistent with CFT realization [Acharya; Gaberdiel,Kaste; AB, del Zotto].
Lessons from $G_2$ mirror symmetry

Starting with a smooth $J$, $J^\vee$ or $J^\wedge$ can be singular.
Lessons from $G_2$ mirror symmetry

Starting with a smooth $J$, $J^\vee$ or $J^\wedge$ can be singular.

This is similar to mirror symmetry for $K3$ surfaces; every $K3$ fibre of $X_\pm$ and hence of $J$

$$S \to J \to S^3$$

has ADE singularities.
Lessons from $G_2$ mirror symmetry

Starting with a smooth $J$, $J^\vee$ or $J^\wedge$ can be singular.

This is similar to mirror symmetry for $K3$ surfaces; every $K3$ fibre of $X_\pm$ and hence of $J$

$$S \to J \to S^3$$

has ADE singularities. These are perfectly well-behaved due to the B-field, so we

**conjecture:** There exist metrics of $G_2$ holonomy on twisted connected sum $G_2$ manifolds $J$ in which every K3 fibre of $X_\pm$ (and $J$) has ADE singularities.
Starting point [Witten] (see also talks by [Acharya; Morrison]):

\[
\text{M-Theory on } K3 \times \mathbb{R}^{1,6} \\
\sim \\
\text{Heterotic String Theory on } T^3 \times \mathbb{R}^{1,6}
\]

with a flat $E_8 \times E_8$ bundle $W$ on $T^3$

\{ the same physics \}

\[ T^3 \rightarrow X \rightarrow S^3 \]

with $X$ a Calabi-Yau manifold

$K3 \rightarrow J \rightarrow S^3$ with $J$ a $G_2$ manifold

expectation: $\exists$ coassociative $K3$ fibration for $J$ a TCS $G_2$ manifold
Story 2: M-Theory/Heterotic Duality

Starting point [Witten] (see also talks by [Acharya; Morrison]):

\[
\begin{align*}
\text{M-Theory on } & K3 \times \mathbb{R}^{1,6} \\
\sim & \\
\text{Heterotic String Theory on } & T^3 \times \mathbb{R}^{1,6} \\
& \text{the same physics} \\
& \text{with a flat } E_8 \times E_8 \text{ bundle } W \text{ on } T^3
\end{align*}
\]

Let’s fiber this data over \( S^3 \) (see [Donaldson]):

\[
\begin{align*}
\text{M-Theory on } & J \times \mathbb{R}^{1,3} \\
\sim & \\
\text{Heterotic String Theory on } & X \times \mathbb{R}^{1,3} \\
& \text{the same physics} \\
& \text{with a holomorphic } E_8 \times E_8 \text{ bundle } V \text{ on } X \\
& T^3 \rightarrow X \rightarrow S^3 \text{ with } X \text{ a Calabi-Yau manifold} \\
& K3 \rightarrow J \rightarrow S^3 \text{ with } J \text{ a } G_2 \text{ manifold}
\end{align*}
\]

**expectation:** \( \exists \) coassocitive \( K3 \) fibration for \( J \) a TCS \( G_2 \) manifold
The moduli spaces are

\[ \mathcal{M}_M = \left\{ \int_L C_3 + i\Phi_3 \mid L \in H^3(J) \right\} \rightarrow \dim_{\mathbb{C}} \mathcal{M}_M = b_3(J) \]

\[ \mathcal{M}_{het} = \{ \text{geometry of } X + \text{bundle } V + \text{NS5-branes} \} \]
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One can work out \( X \) and \( V \) explicitly for TCS \( G_2 \) manifolds, assuming \( S_\pm \) are elliptic:

\[ J = \{ X_+ \times S^1 \} \cup \{ X_- \times S^1 \} \]
\[ X = \{ (dP_9 \setminus T^2) \times T^2 \} \cup \{ (dP_9 \setminus T^2) \times T^2 \} \]
\[ V = \{ V_+ \} \cup \{ V_- \} \]

\( X \) is the ‘Schoen’ Calabi-Yau threefold with \( h^{1,1} = h^{2,1} = 19 \), [Kovalev,AB, Schäfer-Nameki].
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We were able to prove:

\[ \dim \mathcal{M}_{het} = \dim \mathcal{M}_M \]
Lessons from heterotic/M-Theory duality

- If the structure group $G$ of $V$ has non-abelian commutant $G^\perp$ in $E_8 \times E_8$, there are $ADE$ singularities in every K3 fibre of $J$.

**conjecture** (again) : There exist metrics of $G_2$ holonomy on twisted connected sum $G_2$ manifolds $J$ in which every K3 fibre of $X_\pm$ has $ADE$ singularities.
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- There are cases in which $G^\perp \neq 0$ for arbitrary (small) deformations, i.e.

| conjecture: $\exists$ codimension$_\mathbb{R} = 4$ non-smoothable singularities on $G_2$. |

Question: There are quantum corrections $\sim$ counting curves on $X$ on heterotic side which sum to an $E_8 \Theta$-function [Donagi,Grassi,Witten] ... what do these become on $J$? Do they count something interesting (such as associatives) there? What is the relation to recent work of [Joyce]? 

→ Thank you! ←
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• The (complicated) structure of the bundle moduli space corresponds to (a subspace of) an extended $G_2$ moduli space.

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