$G_2$ instantons and the Seiberg-Witten monopoles

Andriy Haydys

New York

September 14, 2017
(Y^7, g) is a G_2 manifold, if Hol(g) \subset G_2 \implies 
there is \varphi \in \Omega^3(Y) s.t.

\diamond \text{Stab}(\varphi_y) = G_2 \subset \text{GL}(T_y Y);
\diamond d\varphi = 0 \text{ and } d(\ast \varphi) = 0.

Remark \varphi \text{ determines } g \text{ and is called the associative 3-form.}

Examples:

\diamond Bryant'87: local examples;
\diamond Bryant–Salamon'89: 3 complete examples;
\diamond Joyce'96: \sim 100 compact examples;
\diamond Kovalev'02, Corti–Haskins–Nordström–Pacini'15: \sim 500 000 000 compact examples.

This motivates the search for invariants of compact G_2 manifolds.
$G_2$ manifolds

$(Y^7, g)$ is a $G_2$ manifold, if $\text{Hol}(g) \subset G_2 \implies$ there is $\varphi \in \Omega^3(Y)$ s.t.

\begin{itemize}
  \item $\text{Stab}(\varphi_y) = G_2 \subset \text{GL}(T_y Y)$;
  \item $d\varphi = 0$ and $d(\ast \varphi) = 0$.
\end{itemize}

**Remark**

$\varphi$ determines $g$ and is called the associative 3-form.
\( \mathbb{G}_2 \) manifolds

\((Y^7, g)\) is a \( \mathbb{G}_2 \) manifold, if \( \text{Hol}(g) \subset \mathbb{G}_2 \implies \) there is \( \varphi \in \Omega^3(Y) \) s.t.

- \( \text{Stab}(\varphi_y) = \mathbb{G}_2 \subset \text{GL}(T_y Y) \);
- \( d\varphi = 0 \) and \( d(\ast \varphi) = 0 \).

Remark

\( \varphi \) determines \( g \) and is called the associative 3-form.

Examples:

- Bryant’87: local examples;
- Bryant–Salamon’89: 3 complete examples;
- Joyce’96: \( \sim 100 \) compact examples;
- Kovalev’02, Corti–Haskins–Nordström–Pacini’15: \( \sim 500\,000\,000 \) compact examples?

This motivates the search for invariants of compact \( \mathbb{G}_2 \) manifolds.
$(Y^7, g)$ is a $G_2$ manifold, if $\text{Hol}(g) \subset G_2 \quad \implies \quad$ there is $\varphi \in \Omega^3(Y)$ s.t.

- $\text{Stab}(\varphi_y) = G_2 \subset \text{GL}(T_y Y)$;
- $d\varphi = 0$ and $d(\ast \varphi) = 0$.

**Remark**

$\varphi$ determines $g$ and is called the associative 3-form.

**Examples:**
- Bryant’87: local examples;
- Bryant–Salamon’89: 3 complete examples;
- Joyce’96: $\sim 100$ compact examples;
- Kovalev’02, Corti–Haskins–Nordström–Pacini’15: $\sim 500\,000\,000$ compact examples?

This motivates the search for invariants of compact $G_2$ manifolds.
\( G_2 \) manifolds

\((Y^7, g)\) is a \( G_2 \) manifold, if \( \text{Hol}(g) \subset G_2 \implies \)
there is \( \varphi \in \Omega^3(Y) \) s.t.

\[ \diamond \text{Stab}(\varphi_y) = G_2 \subset \text{GL}(T_y Y); \]
\[ \diamond d\varphi = 0 \text{ and } d(*\varphi) = 0. \]

**Remark**

\( \varphi \) determines \( g \) and is called the associative 3-form.

**Examples:**

\[ \diamond \text{Bryant’87: local examples; } \]
\[ \diamond \text{Bryant–Salamon’89: 3 complete examples; } \]
\[ \diamond \text{Joyce’96: } \sim 100 \text{ compact examples; } \]
\( G_2 \) manifolds

\((Y^7, g)\) is a \( G_2 \) manifold, if \( \text{Hol}(g) \subset G_2 \implies \)

there is \( \varphi \in \Omega^3(Y) \) s.t.

\begin{itemize}
  \item \( \text{Stab}(\varphi_y) = G_2 \subset \text{GL}(T_y Y); \)
  \item \( d\varphi = 0 \) and \( d(\ast \varphi) = 0. \)
\end{itemize}

Remark

\( \varphi \) determines \( g \) and is called the associative 3-form.

Examples:

\begin{itemize}
  \item Bryant’87: local examples;
  \item Bryant–Salamon’89: 3 complete examples;
  \item Joyce’96: \( \sim 100 \) compact examples;
  \item Kovalev’02, Corti–Haskins–Nordström–Pacini’15: \( \sim 500 \ 000 \ 000 \) compact examples?
\end{itemize}
\( \mathbb{G}_2 \) manifolds

\((Y^7, g)\) is a \( \mathbb{G}_2 \) manifold, if \( \text{Hol}(g) \subset \mathbb{G}_2 \implies \)
there is \( \varphi \in \Omega^3(Y) \) s.t.

\( \diamond \text{Stab}(\varphi_y) = \mathbb{G}_2 \subset \text{GL}(T_y Y); \)
\( \diamond d\varphi = 0 \) and \( d(\ast\varphi) = 0. \)

Remark

\( \varphi \) determines \( g \) and is called the associative 3-form.

Examples:

\( \diamond \) Bryant’87: local examples;
\( \diamond \) Bryant–Salamon’89: 3 complete examples;
\( \diamond \) Joyce’96: \(~100 \) compact examples;
\( \diamond \) Kovalev’02, Corti–Haskins–Nordström–Pacini’15: \(~500\ 000\ 000 \) compact examples ?

This motivates the search for invariants of compact \( \mathbb{G}_2 \) manifolds.
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(\mathcal{S}_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(\mathcal{S}_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

$\diamond$ Represent $Y = \partial W$;
Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(S_Y)$ s.t. $
abla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend $s$ to a transverse section $\hat{s}$ of $\mathcal{S}_W$ (NB. rk $\mathcal{S}_W = 8$);
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(\mathcal{S}_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial \mathcal{W}$;
- Extend $s$ to a transverse section $\hat{s}$ of $\mathcal{S}_W^+$ (NB. $\text{rk } \mathcal{S}_W^+ = 8$);
- Define $\nu(Y, g) = -2 \#(\hat{s}^{-1}(0)) + \chi(W) - 3\sigma(W) \mod 48$. 

Crowley–Goette–Nordström: $\bar{\nu}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Important property: If $g_t$ is a 1-parameter family of $G_2$ metrics on $Y$,
then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

Theorem: There are closed 7-manifolds admitting at least two inequivalent metrics with holonomy $G_2$.

Theorem: For all $G_2$ manifolds constructed previously by CHNP we have $\bar{\nu} = 0$. 

Andriy Haydys (Freiburg University)
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(S_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend $s$ to a transverse section $\hat{s}$ of $\mathcal{S}_W^+$ (NB. $\text{rk } \mathcal{S}_W^+ = 8$);
- Define $\nu(Y, g) = -2 \#(\hat{s}^{-1}(0)) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$. 
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(S_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend $s$ to a transverse section $\hat{s}$ of $S^+_W$ (NB. $\text{rk } S^+_W = 8$);
- Define $\nu(Y, g) = -2 \#(\hat{s}^{-1}(0)) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\overline{\nu}$ s.t. $\nu = \overline{\nu} + 24 \mod 48$.

**Important property:** If $g_t$ is a 1-parameter family of $G_2$ metrics on $Y$, then $\overline{\nu}(Y, g_0) = \overline{\nu}(Y, g_1)$. 
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(S_Y)$ s.t. $\nabla s = 0 \implies s$ vanishes nowhere.

$\diamond$ Represent $Y = \partial W$;
$\diamond$ Extend $s$ to a transverse section $\hat{s}$ of $S_W^+$ (NB. $\text{rk } S_W^+ = 8$);
$\diamond$ Define $\nu(Y, g) = -2 \#(\hat{s}^{-1}(0)) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

**Important property:** If $g_t$ is a 1-parameter family of $G_2$ metrics on $Y$, then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

**Theorem**

*There are closed 7-manifolds admitting at least two inequivalent metrics with holonomy $G_2$.***
Crowley–Nordström’s $\nu$-invariant and its refinement

Observation: Any $G_2$ manifold is equipped with a spinor $s \in \Gamma(\mathcal{S}_Y)$ s.t. $\nabla s = 0$ $\implies$ $s$ vanishes nowhere.

- Represent $Y = \partial W$;
- Extend $s$ to a transverse section $\hat{s}$ of $\mathcal{S}_W^+$ (NB. $\text{rk } \mathcal{S}_W^+ = 8$);
- Define $\nu(Y, g) = -2 \#(\hat{s}^{-1}(0)) + \chi(W) - 3\sigma(W) \mod 48$.

Crowley–Goette–Nordström: $\mathbb{Z}$-valued inv. $\bar{\nu}$ s.t. $\nu = \bar{\nu} + 24 \mod 48$.

Important property: If $g_t$ is a 1-parameter family of $G_2$ metrics on $Y$, then $\bar{\nu}(Y, g_0) = \bar{\nu}(Y, g_1)$.

Theorem

There are closed 7-manifolds admitting at least two inequivalent metrics with holonomy $G_2$.

Theorem

For all $G_2$ manifolds constructed previously by CHNP we have $\bar{\nu} = 0$. 
A gauge-theoretic approach

Pick a $G$-bundle $P \to Y$ (can assume $G = SU(2)$).

**Definition**

A connection $A$ on $P$ is called a $G_2$-instanton, if

$$F_A \wedge \ast \varphi = 0.$$  \hfill (\ast)

Remark

- $\Lambda^2 T^* M = \Lambda^2 14 T^* M \oplus \Lambda^2 7 T^* M$;
- $\pi_7 (F_A) = 0$;
- $(\ast)$ can be viewed as a first order non-linear elliptic PDE.

Fact

The moduli space of $G_2$-instantons $M = \{ A | F_A \wedge \ast \varphi = 0 \} / G(P)$ is finite dimensional; Moreover, $v$-dim $M = 0$. 

Andriy Haydys (Freiburg University)
A gauge-theoretic approach

Pick a $G$-bundle $P \to Y$ (can assume $G = SU(2))$.

**Definition**

A connection $A$ on $P$ is called a $G_2$-instanton, if

$$F_A \wedge \ast \varphi = 0.$$  

\((*)\)

**Remark**

- $\Lambda^2 T^* M = \Lambda^2_{14} T^* M \oplus \Lambda^2_7 T^* M$;  

\((*) \iff \pi_7(F_A) = 0;\)
A gauge-theoretic approach

Pick a $G$-bundle $P \to Y$ (can assume $G = SU(2)$).

**Definition**

A connection $A$ on $P$ is called a $G_2$-instanton, if

$$F_A \wedge \ast \varphi = 0.$$  \hspace{1cm} (\ast)

**Remark**

- $\Lambda^2 T^* M = \Lambda^2_{14} T^* M \oplus \Lambda^2_7 T^* M$; \hspace{1cm} ($\ast$) $\iff \pi_7(F_A) = 0$;
- ($\ast$) can be viewed as a first order non-linear elliptic PDE.
A gauge-theoretic approach

Pick a $G$-bundle $P \to Y$ (can assume $G = SU(2)$).

**Definition**

A connection $A$ on $P$ is called a $G_2$-instanton, if

$$F_A \wedge \star \varphi = 0. \quad (\star)$$

**Remark**

- $\Lambda^2 T^* M = \Lambda^2_{14} T^* M \oplus \Lambda^2_7 T^* M$; $\quad (\star) \iff \pi_7(F_A) = 0$;
- $(\star)$ can be viewed as a first order non-linear elliptic PDE.

**Fact**

The moduli space of $G_2$ instantons

$$\mathcal{M} = \{ A \mid F_A \wedge \psi = 0 \} / G(P)$$

is finite dimensional; Moreover, $v \cdot \dim \mathcal{M} = 0$. 
Question (Donaldson–Thomas’98)

Can we define a $G_2$ Casson invariant $\lambda(Y, g)$ by “counting” $G_2$-instantons on $Y$ such that

$$\lambda(Y, g_0) = \lambda(Y, g_1),$$

provided there is a smooth path $g_t$ of $G_2$-metrics connecting $g_0$ and $g_1$. 

Problem: $M$ may be non-compact, i.e., $G_2$-instantons can degenerate.

Theorem (Tian’00; (Nakajima, Price, Uhlenbeck))

Let $A_i$ be any sequence of $G_2$-instantons. Then there is a closed subset $S \subset Y$, $\dim S \leq 3$, s.t.

$A_i$ converges in $C^\infty_{\text{loc}}(M \setminus S)$ to a $G_2$-instanton.

Moreover, $S = S_0 \cup M$ and

- (Regularity): $M$ is an integer multiplicity rectifiable current;
- (PDE): $M$ is associative, i.e. $\int_M \phi = \text{vol} M$;
- ($S_0$ is small): $H^3(S_0) = 0$. 

Andriy Haydys (Freiburg University)
Question (Donaldson–Thomas’98)

Can we define a $G_2$ Casson invariant $\lambda(Y, g)$ by “counting” $G_2$-instantons on $Y$ such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \quad (**)$$

provided there is a smooth path $g_t$ of $G_2$-metrics connecting $g_0$ and $g_1$.

Problem: $\mathcal{M}$ may be non-compact, i.e., $G_2$ instantons can degenerate.
Question (Donaldson–Thomas’98)

Can we define a $G_2$ Casson invariant $\lambda(Y, g)$ by “counting” $G_2$-instantons on $Y$ such that

$$\lambda(Y, g_0) = \lambda(Y, g_1), \quad (**)$$

provided there is a smooth path $g_t$ of $G_2$-metrics connecting $g_0$ and $g_1$.

Problem: $\mathcal{M}$ may be non-compact, i.e., $G_2$ instantons can degenerate.

Theorem (Tian’00; (Nakajima, Price, Uhlenbeck))

Let $A_i$ be any sequence of $G_2$ instantons. Then there is a closed subset $S \subset Y$, $\dim S \leq 3$, s.t. $A_i$ converges in $C^\infty_{loc}(M \setminus S)$ to a $G_2$ instanton.
Question (Donaldson–Thomas’98)

Can we define a $G_2$ Casson invariant $\lambda(Y, g)$ by “counting” $G_2$-instantons on $Y$ such that

$$\lambda(Y, g_0) = \lambda(Y, g_1),$$

provided there is a smooth path $g_t$ of $G_2$-metrics connecting $g_0$ and $g_1$.

Problem: $\mathcal{M}$ may be non-compact, i.e., $G_2$ instantons can degenerate.

Theorem (Tian’00; (Nakajima, Price, Uhlenbeck))

Let $A_i$ be any sequence of $G_2$ instantons. Then there is a closed subset $S \subset Y$, $\dim S \leq 3$, s.t. $A_i$ converges in $C_{loc}^\infty(M \setminus S)$ to a $G_2$ instanton. Moreover, $S = S_0 \cup M$ and

- (Regularity): $M$ is an integer multiplicity rectifiable current;
- (PDE): $M$ is associative, i.e. $\iota_M^* \varphi = \text{vol}_M$;
- ($S_0$ is small): $\mathcal{H}^3(S_0) = 0$. 
Degenerations of $G_2$ instantons

Conjecture (Tian)

$\dim S_0 \leq 1$. 
Degenerations of $G_2$ instantons

Conjecture (Tian)
\[ \dim S_0 \leq 1. \]

A sequence of $G_2$ instantons can develop:
- a “bubble” along an associative submanifold;
- a non-removable singularity along a 1-dim. submanifold of $M$. 

Question (Donaldson–Segal)

Is there a way to “compensate” for jumps of the number of $G_2$ instantons?
Degenerations of $G_2$ instantons

Conjecture (Tian)

$\dim S_0 \leq 1$.

A sequence of $G_2$ instantons can develop:

- a “bubble” along an associative submanifold;
- a non-removable singularity along a 1-dim. submanifold of $M$.

Expected behavior of $G_2$ instantons in 1-parameter families:
Degenerations of $G_2$ instantons

**Conjecture (Tian)**

$\dim S_0 \leq 1$.  

A sequence of $G_2$ instantons can develop:
- a “bubble” along an associative submanifold;
- a non-removable singularity along a 1-dim. submanifold of $M$.

Expected behavior of $G_2$ instantons in 1-parameter families:

**Question (Donaldson–Segal)**

Is there a way to “compensate” for jumps of the number of $G_2$ instantons?
Bubbles of $G_2$ instantons and Fueter sections

$\mathcal{M}_{k,n}$ framed moduli space of centered charge 1 $SU(n)$-instantons on $\mathbb{R}^4$; hyperKähler manifold, $(I_1, I_2, I_3)$ cx structures;

Ex. $\mathcal{M}_{1,2} = \mathbb{H} \setminus \{0\}/ \pm 1$. 

Definition

A map $u : \mathbb{R}^3 \to \mathcal{M}_{k,n}$ is called Fueter, if

$$I_1 \partial_1 u + I_2 \partial_2 u + I_3 \partial_3 u = 0.$$ 

Remark

Fueter sections can also be defined.

Ex. If $S \to M_3$ is a spinor bundle, $S/\pm 1$ can be thought of as a bundle with fibers $\mathcal{M}_{1,2}$. Fueter section $\equiv Z/2$ harmonic spinor.

Observation:

If $M \subset Y$ associative, then $S \to M$ is the normal bundle.

Conjecture

Let $A_i$ be any seq. of $G_2$ instantons such that $|F_{A_i}|^2$ "concentrates" near $\mathcal{M}_{k,n}$. Then there is a seq. $\epsilon_i \to 0$ s.t. $\rho^* \epsilon_i A_i$ converges to a Fueter section with values in $\mathcal{M}_{k,n}$. Here $\rho : S \to S$, $s \mapsto \epsilon^{-1}s$. 

Andriy Haydys (Freiburg University)
Bubbles of $G_2$ instantons and Fueter sections

$\mathring{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $SU(n)$-instantons on $\mathbb{R}^4$; hyperKähler manifold, $(l_1, l_2, l_3)$ cx structures;
Ex. $\mathring{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/ \pm 1$.

**Definition**

A map $u: \mathbb{R}^3 \to \mathring{\mathcal{M}}_{k,n}$ is called Fueter, if $l_1 \partial_1 u + l_2 \partial_2 u + l_3 \partial_3 u = 0$. 
Bubbles of $G_2$ instantons and Fueter sections

$\hat{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $SU(n)$-instantons on $\mathbb{R}^4$; hyperKähler manifold, $(l_1, l_2, l_3)$ cx structures;
Ex. $\hat{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/ \pm 1$.

**Definition**

A map $u: \mathbb{R}^3 \to \hat{\mathcal{M}}_{k,n}$ is called Fueter, if $l_1 \partial_1 u + l_2 \partial_2 u + l_3 \partial_3 u = 0$.

**Remark**

Fueter sections can also be defined.
Ex. If $\mathcal{S} \to \mathcal{M}^3$ is a spinor bundle, $\mathcal{S}/ \pm 1$ can be thought of as a bundle with fibers $\hat{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.
Bubbles of $G_2$ instantons and Fueter sections

$\hat{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $SU(n)$-instantons on $\mathbb{R}^4$; hyperKähler manifold, $(l_1, l_2, l_3)$ cx structures;

Ex. $\hat{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

**Definition**

A map $u: \mathbb{R}^3 \to \hat{\mathcal{M}}_{k,n}$ is called Fueter, if $l_1 \partial_1 u + l_2 \partial_2 u + l_3 \partial_3 u = 0$.

**Remark**

Fueter sections can also be defined.

Ex. If $\mathcal{S} \to M^3$ is a spinor bundle, $\mathcal{S}/\pm 1$ can be thought of as a bundle with fibers $\hat{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.

**Observation:** If $M \subset Y$ associative, then $\mathcal{S} \to M$ is the normal bundle.
**Bubbles of $G_2$ instantons and Fueter sections**

$\hat{\mathcal{M}}_{k,n}$ framed moduli space of centered charge 1 $SU(n)$-instantons on $\mathbb{R}^4$; hyperKähler manifold, $(I_1, I_2, I_3)$ cx structures; Ex. $\hat{\mathcal{M}}_{1,2} = \mathbb{H} \setminus \{0\}/\pm 1$.

**Definition**

A map $u: \mathbb{R}^3 \to \hat{\mathcal{M}}_{k,n}$ is called Fueter, if $I_1 \partial_1 u + I_2 \partial_2 u + I_3 \partial_3 u = 0$.

**Remark**

Fueter sections can also be defined. Ex. If $\mathcal{S} \to \mathcal{M}$ is a spinor bundle, $\mathcal{S}/\pm 1$ can be thought of as a bundle with fibers $\hat{\mathcal{M}}_{1,2}$. Fueter section $\equiv \mathbb{Z}/2$ harmonic spinor.

**Observation:** If $\mathcal{M} \subset \mathcal{Y}$ associative, then $\mathcal{S} \to \mathcal{M}$ is the normal bundle.

**Conjecture**

Let $A_i$ be any seq. of $G_2$ instantons such that $|F_{A_i}|^2$ “concentrates” near an associative $\mathcal{M}$. Then there is a seq. $\varepsilon_i \to 0$ s.t. $\rho_{\varepsilon_i}^* A_i$ converges to a Fueter section with values in $\hat{\mathcal{M}}_{k,n}$. Here $\rho_{\varepsilon}: \mathcal{S} \to \mathcal{S}$, $s \mapsto \varepsilon^{-1}s$. 
The Seiberg–Witten equations with multiple spinors

$M$ closed oriented Riemannian three-manifold;
$L \to M$ Hermitian line bundle; $A \in \mathcal{A}(L)$;
$E \to M$ Hermitian vector bundle, $\text{rk } E = n$, $\Lambda^n E \cong \mathbb{C}$; $B \in \mathcal{A}(E)$;
The Seiberg–Witten equations with multiple spinors

\( M \) closed oriented Riemannian three-manifold;
\( L \to M \) Hermitian line bundle; \( A \in \mathcal{A}(L) \);
\( E \to M \) Hermitian vector bundle, \( \text{rk} \, E = n, \Lambda^n E \cong \mathbb{C} \); \( B \in \mathcal{A}(E) \);
\( \Psi \in \Gamma(\text{Hom}(E; \mathbb{S} \otimes L)) \);
\( \mu(\Psi) = \Psi \Psi^* - \frac{1}{2} |\Psi|^2 \) Hermitian endom. of \( \mathbb{S} \), i.e., \( \mu(\Psi) \in \Omega^1(M; i\mathbb{R}) \);
The Seiberg–Witten equations with multiple spinors

\( M \) closed oriented Riemannian three-manifold;
\( L \rightarrow M \) Hermitian line bundle; \( A \in \mathcal{A}(L) \);
\( E \rightarrow M \) Hermitian vector bundle, \( \text{rk } E = n, \wedge^n E \cong \mathbb{C} \); \( B \in \mathcal{A}(E) \);
\( \Psi \in \Gamma(\text{Hom}(E; \mathbb{R} \otimes L)) \);
\( \mu(\Psi) = \Psi \Psi^* - \frac{1}{2} |\Psi|^2 \) Hermitian endom. of \( \mathbb{R} \), i.e., \( \mu(\Psi) \in \Omega^1(M; i\mathbb{R}) \);

The Seiberg–Witten equations with \( n \) spinors:

\[ \mathcal{D}_{A,B} \Psi = 0, \quad \ast F_A = \mu(\Psi). \quad \text{(SW}_n\text{)} \]
The Seiberg–Witten equations with multiple spinors

\( M \) closed oriented Riemannian three-manifold;
\( L \rightarrow M \) Hermitian line bundle; \( A \in \mathcal{A}(L); \)
\( E \rightarrow M \) Hermitian vector bundle, \( \text{rk} \ E = n, \wedge^n E \cong \mathbb{C}; \ B \in \mathcal{A}(E); \)

\( \Psi \in \Gamma (\text{Hom}(E; \mathcal{S} \otimes L)); \)

\[ \mu(\Psi) = \Psi \Psi^* - \frac{1}{2} |\Psi|^2 \] Hermitian endom. of \( \mathcal{S} \), i.e., \( \mu(\Psi) \in \Omega^1(M; i\mathbb{R}); \)

The Seiberg–Witten equations with \( n \) spinors:

\[ \mathcal{D}_{A,B} \Psi = 0, \quad * F_A = \mu(\Psi). \] (SWn)

Remark

\( A \) is a variable in \( (\text{SWn}) \), whereas \( B \) is a parameter.
Theorem (H–Walpuski’15)

Let \((A_k, \Psi_k)\) be any sequence of the SW monopoles with \(n\) spinors.

(a) If \(\|\Psi_k\|_{L^2} \leq C\), then a subsequence of \((A_k, \Psi_k)\) converges to a solution of \((SWn)\);

(b) If \(\lim_{k} \|\Psi_k\|_{L^2} = \infty\), then there is a closed nowhere dense \(Z \subset M\) and a subsequence of \((A_k, \Psi_k)\), which converges to some \((A, \Psi)\) over \(M \setminus Z\); Moreover,

- \((A, \Psi)\) solves \(\frac{1}{2} \nabla A, B \Psi = 0\), \(\mu(\Psi) = 0\) on \(M \setminus Z\);
- \(\|\Psi\|\) extends as a \(C^0\)-function on \(M\) and \(Z = |\Psi|^{-1}(0)\).

Remark \((SWn) \iff \frac{1}{2} \nabla A, B \hat{\Psi} = 0\), \(\varepsilon^2_* F_A = \mu(\hat{\Psi})\), \(\|\hat{\Psi}\|_{L^2} = 1\).
Theorem (H–Walpuski’15)

Let \((A_k, \Psi_k)\) be any sequence of the SW monopoles with \(n\) spinors.

(a) If \(\|\Psi_k\|_{L^2} \leq C\), then a subsequence of \((A_k, \Psi_k)\) converges to a solution of \((SW_n)\);

(b) If \(\lim_k \|\Psi_k\|_{L^2} = \infty\), then there is a closed nowhere dense \(Z \subset M\) and a subsequence of \((A_k, \|\Psi_k\|_{L^2}^{-1}\Psi_k)\), which converges to some \((A, \Psi)\) over \(M \setminus Z\); Moreover,

- \((A, \Psi)\) solves \(\Box_{A,B} \Psi = 0, \quad \mu(\Psi) = 0\) on \(M \setminus Z\);

- \(|\Psi|\) extends as a \(C^0\)-function on \(M\) and \(Z = |\Psi|^{-1}(0)\).
Theorem (H–Walpuski’15)

Let \((A_k, \Psi_k)\) be any sequence of the SW monopoles with \(n\) spinors.

(a) If \(\|\Psi_k\|_{L^2} \leq C\), then a subsequence of \((A_k, \Psi_k)\) converges to a solution of \((\text{SW}_n)\);

(b) If \(\lim_k \|\Psi_k\|_{L^2} = \infty\), then there is a closed nowhere dense \(Z \subset M\) and a subsequence of \((A_k, \|\Psi_k\|_{L^2}^{-1}\Psi_k)\), which converges to some \((A, \Psi)\) over \(M \setminus Z\); Moreover,

- \((A, \Psi)\) solves \(\mathcal{D}_{A,B} \Psi = 0, \ \mu(\Psi) = 0\) on \(M \setminus Z\);

- \(|\Psi|\) extends as a \(C^0\)-function on \(M\) and \(Z = |\Psi|^{-1}(0)\).

Remark

\((\text{SW}_n) \iff \mathcal{D}_{A,B} \hat{\Psi} = 0, \ \varepsilon^2 * F_A = \mu(\hat{\Psi}), \ \|\hat{\Psi}\|_{L^2} = 1.\)
Theorem (H–Walpuski’15)

Let \((A_k, \Psi_k)\) be any sequence of the SW monopoles with \(n\) spinors.

(a) If \(\|\Psi_k\|_{L^2} \leq C\), then a subsequence of \((A_k, \Psi_k)\) converges to a solution of \((\text{SW}_n)\);

(b) If \(\lim_k \|\Psi_k\|_{L^2} = \infty\), then there is a closed nowhere dense \(Z \subset M\) and a subsequence of \((A_k, \|\Psi_k\|_{L^2}^{-1} \Psi_k)\), which converges to some \((A, \Psi)\) over \(M \setminus Z\); Moreover,

- \((A, \Psi)\) solves \(\mathcal{D}_{A,B} \Psi = 0, \quad \mu(\Psi) = 0\) on \(M \setminus Z\);

- \(|\Psi|\) extends as a \(C^0\)-function on \(M\) and \(Z = |\Psi|^{-1}(0)\).

Remark

\[(\text{SW}_n) \iff \mathcal{D}_{A,B} \hat{\Psi} = 0, \quad \varepsilon^2 \ast F_A = \mu(\hat{\Psi}), \quad \|\hat{\Psi}\|_{L^2} = 1.\]

Theorem (Taubes)

\(\dim Z \leq 1.\)
The limit \((A, \Psi)\)

Fix \(m \in M \setminus Z\).

\[
\text{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/U(1).
\]
The limit \((A, \Psi)\)

Fix \(m \in M \setminus Z\).

\[
\text{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/\mathrm{U}(1).
\]

Fact (ADHM construction of instantons on \(\mathbb{R}^4\))

\(\mu^{-1}(0)/\mathrm{U}(1)\) is isometric to \(\hat{\mathcal{M}}_{1,n}\).
The limit \((A, \Psi)\)

Fix \(m \in M \setminus Z\).

\[
\text{Hom}\left(C^n, C^2\right) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/U(1).
\]

Fact (ADHM construction of instantons on \(\mathbb{R}^4\))

\(\mu^{-1}(0)/U(1)\) is isometric to \(\hat{\mathcal{M}}_{1,n}\).

Theorem (H’12)

For any solution of

\[
\mathcal{D}_{A,B} \Psi = 0, \quad \mu(\Psi) = 0 \quad \text{on} \ M \setminus Z
\]

the projection of \(\Psi\) is a Fueter section.
The limit \((A, \Psi)\)

Fix \(m \in M \setminus Z\).

\[
\text{Hom}(\mathbb{C}^n, \mathbb{C}^2) \supset \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/U(1).
\]

**Fact (ADHM construction of instantons on \(\mathbb{R}^4\))**

\(\mu^{-1}(0)/U(1)\) is isometric to \(\mathring{\mathcal{M}}_{1,n}\).

**Theorem (H’12)**

For any solution of

\[
\mathcal{D}_{A,B} \Psi = 0, \quad \mu(\Psi) = 0 \quad \text{on } M \setminus Z
\]

the projection of \(\Psi\) is a Fueter section.

**Proposal (H–Walpuski’15)**

Count \(G_2\) instantons together with the Seiberg–Witten monopoles on associative submanifolds \(M \subset Y\).
What we need to understand better

- Unremovable singularities of $G_2$ instantons;

- The role of reducible Seiberg–Witten monopoles;

- Singularities of associative submanifolds.
What we need to understand better

- Unremovable singularities of $G_2$ instantons;
- $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of $\mathbb{Z}$;
What we need to understand better

- Unremovable singularities of $G_2$ instantons;
- $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of $\mathbb{Z}$;
- The rôle of reducible Seiberg–Witten monopoles;
What we need to understand better

- Unremovable singularities of $G_2$ instantons;
- $\mathbb{Z}/2$ harmonic spinors, in particular properties and the rôle of $\mathbb{Z}$;
- The rôle of reducible Seiberg–Witten monopoles;
- Singularities of associative submanifolds.
Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

\begin{itemize}
  \item $Z \neq \emptyset$ provided $c_1(L^2) \neq 0$.
\end{itemize}
Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

\[ Z \neq \emptyset \] provided $c_1(L^2) \neq 0$.

Assume $Z = \emptyset$ and $c_1(L^2) \neq 0$. Then

\[ \Psi \Psi^* = \frac{1}{2} |\Psi|^2 \neq 0 \implies \ker \Psi^* = \{0\} \implies \Psi \text{ epi} \implies \Psi \text{ iso}; \]
Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

\( Z \neq \emptyset \) provided $c_1(L^2) \neq 0$.

Assume $Z = \emptyset$ and $c_1(L^2) \neq 0$. Then

\[
\Psi \Psi^* = \frac{1}{2} |\Psi|^2 \neq 0 \implies \ker \Psi^* = \{0\} \implies \Psi \text{ epi} \implies \Psi \text{ iso};
\]

\[
\Psi : E \xrightarrow{\cong} \mathcal{S} \otimes L \implies \Lambda^2 E \cong \Lambda^2 (\mathcal{S} \otimes L) \cong L^2,
\]

which is a contradiction.
Some properties of $\mathbb{Z}/2$ harmonic spinors

Assume $n = 2$.

$\diamond$ $Z \neq \emptyset$ provided $c_1(L^2) \neq 0$.

Assume $Z = \emptyset$ and $c_1(L^2) \neq 0$. Then

$$\Psi \Psi^* = \frac{1}{2} |\Psi|^2 \neq 0 \quad \implies \quad \ker \Psi^* = \{0\} \quad \implies \quad \Psi \text{ epi} \quad \implies \quad \Psi \text{ iso};$$

$$\Psi : E \xrightarrow{\cong} \mathcal{S} \otimes L \quad \implies \quad \Lambda^2 E \cong \Lambda^2 (\mathcal{S} \otimes L) \cong L^2,$$

which is a contradiction.

**Theorem (H’16)**

Assume $(A, \Psi)$ solves $\mathcal{D}_{A,B} \Psi = 0, \mu(\Psi) = 0$ over $M \setminus Z$. Then there is an extra infinitesimal structure $(\theta, \text{or})$ on $Z$ such that $[Z, \theta, \text{or}] \in H_1(M, \mathbb{Z})$ is well-defined. Moreover,

$$[Z, \theta, \text{or}] = \text{PD}(c_1(L^2)).$$