1 Introduction

Let $X$ be a smooth projective geometrically integral variety over $\mathbb{Q}$. We have an inclusion

$$
\phi : X(\mathbb{Q}) \hookrightarrow \prod_{p, \infty} X(\mathbb{Q}_p) = X(\mathbb{A}).
$$

Definition 1 $X$ satisfies weak approximation (WA) if $X(\mathbb{A}) \neq \emptyset$ and $\text{im}(\phi)$ is dense for the product of the $p$-adic topologies. A “dramatic” failure of WA is when $X(\mathbb{A}) \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$. Here we say the Hasse Principle (HP) fails.

Obstructions: (Manin) For any $S \subset \text{Br}(X)$ there exists an intermediate set

$$
\overline{X(\mathbb{Q})} \subset X(\mathbb{A})^S \subset X(\mathbb{A}).
$$

Thus elements of the Brauer groups can obstruct WA and HP–this is known as the Brauer-Manin (BM) obstruction.

Here we’ll consider the case of $X$ a K3 surface and $S$ a collection of transcendental elements of $\text{Br}(X)$.

2 Filtration of $\text{Br}(X)$

There is a filtration of the Brauer group

$$
\text{Br}_0(X) \subset \text{Br}_1(X) \subset \text{Br}(X)
$$
where \( \text{Br}_0(X) = \text{im}(\text{Br}(\mathbb{Q}) \to \text{Br}(X)) \) is the subgroup of ‘constant algebras’, which never yield obstructions (i.e., \( X(\mathbb{A})^S = X(\mathbb{A}) \) whenever \( S \subseteq \text{Br}_0(X) \) — this is class field theory). The next step \( \text{Br}_1(X) = \ker(\text{Br}(X) \to \text{Br}(\overline{X})) \) is the subgroup of ‘algebraic elements’. Finally, we have the transcendental elements \( \text{Br}(X) \setminus \text{Br}_1(X) \).

A ‘tool’ for studying algebraic elements is the Hochschild-Serre spectral sequence. The long exact sequence of low degree terms gives an isomorphism

\[
\text{Br}_1(X)/\text{Br}_0(X) \cong H^1(G_\mathbb{Q}, \text{Pic}(\overline{X})).
\]

Lifting elements can be tricky, in practice.

There is no such tool known for transcendental classes. These classes don’t exist for curves or surfaces of negative Kodaira dimension. Thus K3 surfaces are a good testing ground for these, especially since Skorobogatov-Zarhin have shown that \( \text{Br}(X)/\text{Br}_0(X) \) is finite for K3 surfaces over number fields.

3 History

Arithmetic applications of transcendental elements:

- Harari (1996): three-dimensional counterexample to HP;
- Skorobogatov-Swinnerton Dyer, Harari, Skorobogatov (2005);
- Ieronymou-Skorobogatov Zarhin: diagonal quartics;

In the examples above, the K3’s admit elliptic fibrations.

Our goal: To give transcendental obstructions on general K3 surfaces
4 Results

**Theorem 1 (Hassett, Varilly, VA (2011))** There exists an explicit K3 surface $X \subset \mathbb{P}(1,1,1,3)$ of degree two and an Azumaya algebra $\mathcal{A}$ in $\text{im}(\text{Br}(X) \hookrightarrow \text{Br}(k(X)))$, all defined over $\mathbb{Q}$, such that

i. $\text{Pic}(X) \simeq \mathbb{Z}$;

ii. $X(\mathcal{A}) \setminus X(\mathcal{A})^{\{A\}} \neq \emptyset$.

Thus $X$ has a transcendental obstruction to WA.

$(X, \mathcal{A})$ is associated to a cubic fourfold with a plane.

**Theorem 2 (Hassett, VA)** Ditto for HP except that

ii'. $X(\mathcal{A})^{\{A\}} = \emptyset$.

$(X, \mathcal{A})$ is associated to a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along type $(2, 2)$ divisor.

Note that since $\text{Pic}(X) \simeq \mathbb{Z}$, in both cases, the cohomology group $H^1(G_{\mathbb{Q}}, \text{Pic}(X))$ is trivial, so there are no non-constant algebraic elements, by the Hochschild-Serre spectral sequence. This way we guarantee that the obstructions are caused by a transcendental Brauer class. This also ensures our K3s have no elliptic fibrations.

As we’ll see below, we essentially explain a way to get the transcendental two-torsion of $\text{Br}(X)$ for a degree 2 K3 surface with $\text{Pic}(X) \simeq \mathbb{Z}$.

5 Hodge-theoretic motivation

Let $X$ be a complex projective K3 surface with transcendental lattice

$$T_X := \text{NS}(X)^+ \subset H^2(X, \mathbb{Z}).$$

The exponential sequence, together with some basic lattice theory gives a one-to-one correspondence

$$\{\alpha \in \text{Br}(X) \text{ of order } n \} \overset{1-1}{\longleftrightarrow} \{T_X \rightarrow \mathbb{Z}/n\mathbb{Z}\}.$$

So

$$\alpha \mapsto T_{(\alpha)} = \text{ker}(T_X \rightarrow \mathbb{Z}/n\mathbb{Z}) \subset T_X.$$
**Theorem 3 (van Geemen, Voisin, Mukai)** Let \( X \) be a K3 surface of degree two over \( \mathbb{C} \) with \( \text{Pic}(X) \cong \mathbb{Z} \). Let \( \alpha \in \text{Br}(X)[2] \). Exactly one of the following situations occur:

1. \( T_{(\alpha)} \hookrightarrow \Lambda_{K3} \) as the primitive cohomology of a K3 surface \( Y_{\alpha} \) of degree eight;

2. \( T_{(\alpha)} \cong \langle h^2, P \rangle^\perp \subset H^4(Y_{\alpha}, \mathbb{Z}) \) for a cubic fourfold \( Y_{\alpha} \) containing a plane \( P \);

3. \( T_{(\alpha)} \cong \text{primitive sublattice of } H^4(Y_{\alpha}, \mathbb{Z}) \) where \( Y_{\alpha} \) is a double cover of \( \mathbb{P}^2 \times \mathbb{P}^2 \) ramified along a \((2,2)\) divisor.

Idea: give geometric constructions “going backwards”, over arbitrary fields.

Use the auxiliary variety \( Y_{\alpha} \) to construct a bundle of quadrics \( \tilde{Y}_{\alpha} \to \mathbb{P}^2 \). Take \( r : \mathcal{W} \to \mathbb{P}^2 \) to be the variety parametrizing maximal isotropic subspaces. The Stein factorization

\[ \mathcal{W} \to X \to \mathbb{P}^2 \]

yields a K3 surface of degree two. The first arrow is a smooth \( \mathbb{P}^n \)-bundle, which represents our two-torsion element.

**Example:** In the case where \( Y_{\alpha} \) is a cubic fourfold containing a plane, we project away from the plane and blow it up to get a quadric surface bundle \( \tilde{Y}_{\alpha} \to \mathbb{P}^2 \). The discriminant locus on \( \mathbb{P}^2 \) is a sextic curve. If the fiber above a point \( t \in \mathbb{P}^2 \) is smooth, it has two families of lines, parametrized by two conics. These two conics form \( r^{-1}(t) \). If the fiber above \( t \) is a cone over a conic, then \( r^{-1}(t) \) is a single conic (the two conics of nearby fibers ‘come together’).

**Technical difficulties** in the computation:

- no guarantee that \( \text{Pic}(X) \cong \mathbb{Z} \); a classical trick is to embed \( \text{Pic}(X) \) into \( \text{Pic}(X \mod p) \) for multiple primes \( p \), and compare the Picard groups of the resulting reductions (e.g., van Luijk, Kloosterman);

- this involves serious point counting over finite fields, and we did geometry in characteristic two in order to use the prime two;

- use techniques of Elsenhans-Jahnel to count points at only one prime;
• for HP—which primes do we have to worry about? definitely the primes of bad reduction of $X$, which is not too bad in theory, but the discriminant can be huge (346 digit number) and thus hard to factor. The geometry of the two projections onto $\mathbb{P}^2$ share some primes of bad reduction, which can be computed using a GCD which turns out to be prime (171 digits);

• need to compute invariants, via recent work of Colliot-Thélène and Skorobogatov to control contributions of singular places, provided there are at most seven rational double points—the key is that the smooth locus be simply connected.

It turns out only the real place contributes.