THE MINIMAL MODEL PROGRAM FOR THE HILBERT SCHEME OF POINTS ON $\mathbb{P}^2$
AND BRIDGELAND STABILITY

IZZET COSKUN

1. Introduction

This is joint work with Daniele Arcara, Aaron Bertram and Jack Huizenga. I will describe the stable base locus decomposition of the effective cone of the Hilbert scheme of points in $\mathbb{P}^2$.

Given a moduli space $M$, one can approach its birational geometry from three perspectives.

1. Run MMP
2. Vary the moduli functor
3. Vary the linearization in GIT (in case the moduli space is constructed via GIT)

In general, it is hard to compute stable base loci decompositions of the effective cone. For a moduli space, by playing these three perspectives against each other, one can make progress. I will demonstrate this with the Hilbert scheme of points in $\mathbb{P}^2$.

2. Preliminaries about the Hilbert scheme

Let $\mathbb{P}^{2[n]}$ denote the Hilbert scheme parameterizing zero-dimensional subschemes of $\mathbb{P}^2$ of length $n$. Let $\mathbb{P}^{2(n)}$ denote the symmetric product of $\mathbb{P}^2$. The Hilbert-Chow morphism plays an important role in understanding the geometry of $\mathbb{P}^{2[n]}$.

$$h : \mathbb{P}^{2[n]} \rightarrow \mathbb{P}^{2(n)}.$$ 

**Theorem 2.1** (Fogarty [F1]). The Hilbert scheme $\mathbb{P}^{2[n]}$ is a smooth, irreducible, projective variety of dimension $2n$. The Hilbert scheme $\mathbb{P}^{2[n]}$ admits a natural morphism to the symmetric product $\mathbb{P}^{2(n)}$ called the Hilbert-Chow morphism

$$h : \mathbb{P}^{2[n]} \rightarrow \mathbb{P}^{2(n)}.$$ 

The morphism $h$ is birational and gives a crepant desingularization of the symmetric product $\mathbb{P}^{2(n)}$.

Let $H = h^*(c_1(\mathcal{O}_{\mathbb{P}^{2(n)}}(1)))$ be the class of the pull-back of the ample generator from the symmetric product $\mathbb{P}^{2(n)}$. The exceptional locus of the Hilbert-Chow morphism is an irreducible divisor whose class we denote by $B$.

**Theorem 2.2** (Fogarty [F2]). The Picard group of the Hilbert scheme of points $\mathbb{P}^{2[n]}$ is the free abelian group generated by $\mathcal{O}_{\mathbb{P}^{2(n)}}(H)$ and $\mathcal{O}_{\mathbb{P}^{2(n)}}(B)$. In particular, the Neron-Severi space is spanned by the divisor classes $H$ and $B$.

The canonical class of $\mathbb{P}^{2[n]}$ is $-3H$. In particular, $\mathbb{P}^{2[n]}$ is a log Fano variety. Therefore, it is a Mori dream space.
3. The ample cone of $\mathbb{P}^2[n]$

The long exact sequence associated to the exact sequence of sheaves

$$0 \to \mathcal{I}_Z(k) \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{O}_Z(k) \to 0$$

gives rise to the inclusion

$$H^0(\mathbb{P}^2, \mathcal{I}_Z(k)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)).$$

This inclusion induces a rational map to the Grassmannian

$$\phi_k : \mathbb{P}^2[n] \dashrightarrow G_k = G\left(\begin{pmatrix}k+2 \\ 2\end{pmatrix} - n, \begin{pmatrix}k+2 \\ 2\end{pmatrix}\right).$$

Let $D_k(n) = \phi_k^*(\mathcal{O}_{G_k}(1))$ denote the pull-back of $\mathcal{O}_{G_k}(1)$ by $\phi_k$.

By results of Beltrametti, Sommese, Catanese and Göttsche ([BSG], [CG], [LQZ]), $D_k(n)$ is base point free for $k \geq n - 1$ and ample for $k \geq n$. $H$ and $D_{n-1}(n)$ are base point free, but not ample. $H$ contracts curves in the fibers of the Hilbert-Chow morphism. $D_{n-1}(n)$ contracts the locus of schemes supported on a fixed line. Therefore:

**Corollary 3.1** ([LQZ]). The nef cone of $\mathbb{P}^2[n]$ is the closed, convex cone bounded by the rays $H$ and $D_{n-1}(n) = (n - 1)H - \frac{B}{2}$. The nef cone of $\mathbb{P}^2[n]$ equals the base-point-free cone of $\mathbb{P}^2[n]$.

4. The effective cone of $\mathbb{P}^2[n]$

The effective cone is much harder to compute. We need to create more divisors.

A vector bundle $E$ of rank $r$ on $\mathbb{P}^2$ satisfies interpolation for $n$ points if the general $Z \in \mathbb{P}^2[n]$ imposes independent conditions on sections of $E$, i.e. if

$$h^0(E \otimes \mathcal{I}_Z) = h^0(E) - rn.$$ 

Assume $E$ satisfies interpolation for $n$ points. Let $W \subseteq H^0(E)$ be a general fixed subspace of dimension $rn$. A scheme $Z$ which imposes independent conditions on sections of $E$ will impose independent conditions on sections in $W$ if and only if the subspace $H^0(E \otimes \mathcal{I}_Z) \subseteq H^0(E)$ is transverse to $W$. Informally, we obtain a divisor $D_{E,W}(n)$ described as the locus of schemes which fail to impose independent conditions on sections in $W$. The class of $D_{E,W}(n)$ is

$$aH - \frac{r}{2}B,$$

where $c_1(E) = aL$. When we consider the complete linear system, we omit $W$ from the notation.

When interpreted in the same vector space spanned by $H$ and $B$, we have $\text{Eff}(\mathbb{P}^{2[n+1]}) \subseteq \text{Eff}(\mathbb{P}^{2[n]})$.

Let $\Phi$:

$$\Phi = \{\alpha \mid \alpha > \phi^{-1}\} \cup \left\{0, \frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \frac{55}{34}, \cdots\right\}, \quad \phi = 1 + \frac{\sqrt{5}}{2},$$

where $\phi$ is the golden ratio and the fractions are ratios of consecutive Fibonacci numbers. Then Jack’s theorem gives the following:

**Theorem 4.1.** ([Hui] Theorem 4.1) Let

$$n = \frac{r(r+1)}{2} + s, \quad s \geq 0.$$

Consider a general vector bundle $E$ given by the resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(r-2)^{\oplus k}s \to \mathcal{O}_{\mathbb{P}^2}(r-1)^{\oplus k(s+r)} \to E \to 0.$$
We conclude that the Riemann-Hurwitz formula implies that this degree is the degree of the ramification divisor of the map. It is easy to see that the first intersection number is clear since it equals the degree of the curve. The second intersection number can be computed using the Riemann-Hurwitz formula. The genus of the curve is contained in a smooth curve containing a general point of the smooth curve of degree r. The scheme Z defines a divisor DZ on the smooth curve C of degree n. The corresponding divisors of degree n on C induce a curve R in the Hilbert scheme P2[n]. We then have the following intersection numbers:

\[ R \cdot H = r, \quad R \cdot B = 2(r^2 - r + s). \]

The first intersection number is clear since it equals the degree of the curve C. The second intersection number can be computed using the Riemann-Hurwitz formula. It is easy to see that R · B is the degree of the ramification divisor of the map \( \phi_P : C \to \mathbb{P}^1 \) induced by the pencil \( P \subset H^0(C, \mathcal{O}_C(D_Z)) \). The Riemann-Hurwitz formula implies that this degree is

\[ 2n + (r - 1)(r - 2) - 2 = 2(r^2 - r + s). \]

We conclude that \( R \cdot D_E(n) = 0 \). Since Z was a general point of Z and we constructed a curve in the corresponding divisors of degree n on C. We deduce that the effective cone is equal to the cone spanned by B and \( H - \frac{r}{2(r^2 - r + s)} B \).

**Remark 4.3.** The proof of the theorem yields a Cayley-Bacharach type estimate on higher rank vector bundles:

**Corollary 4.4 (A Cayley-Bacharach Theorem for higher rank vector bundles on P2).** Let \( n = \frac{r(r+1)}{2} + s \) with \( 0 \leq s \leq r \).
(1) If $\frac{s}{r} \geq \frac{1}{2}$, then the effective cone of $\mathbb{P}^{2[n]}$ is contained in the cone generated by $H - \frac{r}{2(r^2 - r + s)} B$ and $B$.

(2) If $\frac{s}{r} < \frac{1}{2}$, then the effective cone of $\mathbb{P}^{2[n]}$ is contained in the cone generated by $H - \frac{r + 2}{2(r^2 + r + s - 1)} B$ and $B$.

Let $E$ be a vector bundle on $\mathbb{P}^2$ with rank $k$ and $c_1(E) = aL$. If

$$\frac{k}{a} > \frac{r}{r^2 - r + s} \quad \text{when} \quad \frac{s}{r} \geq \frac{1}{2}, \quad \text{or} \quad \frac{k}{a} > \frac{r + 2}{r^2 + r + s - 1} \quad \text{when} \quad \frac{s}{r} < \frac{1}{2},$$

then $E$ cannot satisfy interpolation for $n$ points. That is, every $Z \in \mathbb{P}^{2[n]}$ fails to impose independent conditions on sections of $E$.

5. Bridgeland stability conditions

Let $X$ be a smooth, projective variety. Let $\mathcal{D}$ denote the bounded, derived category of coherent sheaves on $X$.

**Definition 5.1.** A Bridgeland stability condition on $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ such that

- $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$
- $Z : K(\mathcal{D}) \to \mathbb{C}$ is a group homomorphism such that every non-zero object $E \in \mathcal{A}$ maps to the upper half plane. (This is the main positivity condition and is hard to achieve). $Z$ allows us to define a slope function.
- Every object in $\mathcal{D}$ has a Harder-Narasimhan filtration.

Bridgeland [Br1] has shown that the space of stability conditions (satisfying a certain finiteness condition) is a complex manifold. In fact, the map $(\mathcal{A}, Z) \mapsto Z$ is a local homeomorphism onto an open subset of a linear subspace of $\text{Hom}(K(\mathcal{D}), \mathbb{C})$.

Example: When $X$ is a curve, we can take $\mathcal{A}$ to be coherent sheaves and $Z = -d + ir$, where $d$ and $r$ denote degree and rank, respectively. This is a Bridgeland stability condition.

Non-example. Let $S$ be a surface and let $H$ be an ample divisor on $S$. Then if we take $\mathcal{A}$ to be the category of coherent sheaves and $Z = -d + ir$, where $d$ is the degree w.r.t. $H$ and $r$ is the rank, we do not get a Bridgeland stability condition. Torsion sheaves supported on points get mapped to zero by $Z$.

We need to modify the abelian category to make the example in the case of surfaces work.

**Definition 5.2.** Given $s \in \mathbb{R}$, define full subcategories $\mathcal{Q}_s$ and $\mathcal{F}_s$ of $\text{coh}(\mathbb{P}^2)$ by the following conditions on their objects:

- $Q \in \mathcal{Q}_s$ if $Q$ is torsion or if each $\mu_i > s$ in the Harder-Narasimhan filtration of $Q$.
- $F \in \mathcal{F}_s$ if $F$ is torsion-free, and each $\mu_i \leq s$ in the Harder-Narasimhan filtration of $F$.

Each pair $(\mathcal{F}_s, \mathcal{Q}_s)$ of full subcategories therefore satisfies [Br2, Lemma 6.1]:

(a) For all $F \in \mathcal{F}_s$ and $Q \in \mathcal{Q}_s$,

$$\text{Hom}(Q, F) = 0$$

(b) Every coherent sheaf $E$ fits in a short exact sequence:

$$0 \to Q \to E \to F \to 0,$$

where $Q \in \mathcal{Q}_s$, $F \in \mathcal{F}_s$ and the extension class are uniquely determined up to isomorphism.

A pair of full subcategories $(\mathcal{F}, \mathcal{Q})$ of an abelian category $\mathcal{A}$ satisfying conditions (a) and (b) is called a torsion pair. A torsion pair $(\mathcal{F}, \mathcal{Q})$ defines a t-structure on $\mathcal{D}^b(\mathcal{A})$ with:

$$\mathcal{D}^{\geq 0} = \{ \text{complexes } E \mid H^{-1}(E) \in \mathcal{F} \text{ and } H^i(E) = 0 \text{ for } i < -1 \}$$
The heart of the $t$-structure defined by a torsion pair consists of:

$$\{ E \mid H^{-1}(E) \in \mathcal{F}, H^0(E) \in \mathcal{Q}, \text{ and } H^i(E) = 0 \text{ otherwise} \}.$$ 

The natural exact sequence:

$$0 \rightarrow H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \rightarrow 0$$

for such an object of $D^b(A)$ implies that the objects of the heart are all given by pairs of objects $F \in \mathcal{F}$ and $Q \in \mathcal{Q}$ together with an extension class in $\text{Ext}^2_A(Q, F)$.

**Definition 5.3.** Let $A_s$ be the heart of the $t$-structure on $D^b(\text{coh}(\mathbb{P}^2))$ obtained from the torsion-pair $(\mathcal{F}_s, \mathcal{Q}_s)$ defined in Definition 5.2.

If we define

$$Z_{s,t}(E) = -\int_{\mathbb{P}^2} e^{-(s+it)L} \text{ch}(E),$$

then the pair $(A_{s,t}, Z_{s,t})$ is a Bridgeland stability condition. Abramovich and Polishchuk [AP] have constructed moduli spaces of Bridgeland stable objects with fixed invariants. We get a chamber decomposition of the $(s,t)$-plane depending on the isomorphism type of the corresponding moduli space.

Equating the slopes of objects, we see that the potential walls in the $(s,t)$-plane are nested semi-circles.

**Remark 5.4.** Specializing to the case of the ideal sheaf of points, the moduli spaces are projective. They can be constructed by GIT as moduli spaces of quiver representations.

### 6. Explicit examples

**Notation for the examples:**

- Let $C_k(n)$ be the curve class in $\mathbb{P}^{2[n]}$ given by fixing $k-1$ points on a line, $n-k$ points off the line, and allowing an $n$th point to vary along the line. We have
  $$C_k(n) \cdot H = 1 \quad C_k(n) \cdot B = 2(k-1).$$

- Let $A_{2,k}(n)$ be the curve class in $\mathbb{P}^{2[n]}$ given by fixing $k-1$ points on a conic curve, fixing $n-k$ points off the curve, and allowing an $n$th point to move along the conic. We have
  $$A_{2,k}(n) \cdot H = 2 \quad A_{2,k}(n) \cdot B = 2(k-1).$$

- Let $L_k(n)$ be the locus of schemes of length $n$ with a linear subscheme of length at least $k$. Observe that $L_k(n)$ is swept out by irreducible curves of class $C_k(n)$.

- Let $Q_k(n)$ be the locus of schemes of length $n$ which have a subscheme of length $k$ contained in a conic curve. Clearly $Q_k(n)$ is swept out by $A_{2,k}(n)$.

#### 6.1. The walls for $\mathbb{P}^{2[2]}$.

<table>
<thead>
<tr>
<th>Divisor class</th>
<th>Divisor description</th>
<th>Dual curves</th>
<th>Stable base locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$B$</td>
<td>$C_1(2)$</td>
<td>$B$</td>
</tr>
<tr>
<td>$H$</td>
<td>$H$</td>
<td>$C(2)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$H - \frac{1}{2}B$</td>
<td>$D_1(2)$</td>
<td>$C_2(2)$</td>
<td></td>
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</tbody>
</table>

The Bridgeland walls of $\mathbb{P}^{2[2]}$ are described as follows.
• There is a unique semi-circular Bridgeland wall with center $x = -\frac{5}{2}$ and radius $\frac{3}{2}$ in the $(s,t)$-plane with $s < 0$ and $t > 0$ corresponding to the destabilizing object $\mathcal{O}_{\mathbb{P}^2}(-1)$.


<table>
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</thead>
<tbody>
<tr>
<td>$B$</td>
<td>$B$</td>
<td>$C_1(4)$</td>
<td>$B$</td>
</tr>
<tr>
<td>$H$</td>
<td>$H$</td>
<td>$C(4)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$H - \frac{1}{6}B$</td>
<td>$D_5(4)$</td>
<td>$C_4(4)$</td>
<td>$L_4(4)$</td>
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<td>$H - \frac{1}{4}B$</td>
<td>$D_2(4)$</td>
<td>$C_3(4)$</td>
<td>$L_3(4) = E_1(4)$</td>
</tr>
<tr>
<td>$H - \frac{1}{4}B$</td>
<td>$E_1(4)$</td>
<td>$A_{2,4}(4)$</td>
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</table>

The Bridgeland walls in the $(s,t)$-plane with $s < 0$ and $t > 0$ are the following three semi-circles $W_x$ with center $(x,0)$ and radius $\sqrt{x^2 - 8}$:

• The rank one wall $W_{-\frac{2}{3}}$ corresponding to the destabilizing object $\mathcal{O}_{\mathbb{P}^2}(-1)$.
• The rank one wall $W_{-\frac{3}{4}}$ corresponding to the destabilizing object $\mathcal{I}_p(-1)$.
• The rank one wall $W_{-3}$ corresponding to the destabilizing object $\mathcal{O}_{\mathbb{P}^2}(-2)$. This wall also arises as a rank two wall corresponding to the destabilizing object $\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}$.


<table>
<thead>
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<tbody>
<tr>
<td>$B$</td>
<td>$B$</td>
<td>$C_1(7)$</td>
<td>$B$</td>
</tr>
<tr>
<td>$H$</td>
<td>$H$</td>
<td>$C(7)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$H - \frac{1}{12}B$</td>
<td>$D_6(7)$</td>
<td>$C_7(7)$</td>
<td>$L_7(7)$</td>
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<tr>
<td>$H - \frac{1}{10}B$</td>
<td>$D_5(7)$</td>
<td>$C_6(7)$</td>
<td>$L_6(7)$</td>
</tr>
<tr>
<td>$H - \frac{1}{8}B$</td>
<td>$D_4(7)$</td>
<td>$C_5(7)$</td>
<td>$L_5(7)$</td>
</tr>
<tr>
<td>$H - \frac{1}{6}B$</td>
<td>$D_3(7)$</td>
<td>$C_4(7), A_{2,7}(7)$</td>
<td>$L_4(7) \cup Q_7(7)$</td>
</tr>
<tr>
<td>$H - \frac{1}{5}B$</td>
<td>$D_{T_{22}(1)}(7)$</td>
<td>$A_{2,6}(7)$</td>
<td>$Q_6(7) = E_2(7)$</td>
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<tr>
<td>$H - \frac{5}{21}B$</td>
<td>$E_2(7)$</td>
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The case $n = 7$ is interesting because we see the first example of a collapsing rank two wall. The Bridgeland walls in the $(s,t)$-plane are the following six semi-circles $W_x$ with center at $(x,0)$ and radius $\sqrt{x^2 - 14}$.

• The rank one wall $W_{-\frac{12}{5}}$ corresponding to the destabilizing object $\mathcal{O}_{\mathbb{P}^2}(-1)$. 

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• The rank one wall $W_{-13}$ corresponding to the destabilizing object $I_p(-1)$.
• The rank one wall $W_{-11}$ corresponding to the destabilizing object $I_{Z'}(-1)$, where $Z'$ is a scheme of length two.
• The two coinciding rank one walls $W_{-9}$ corresponding to the destabilizing objects $O_{p^2}(-2)$ and $I_{Z'}(-1)$, where $Z'$ is a scheme of length 3.
• The rank one wall $W_{-4}$ corresponding to the destabilizing object $I_p(-2)$.
• The rank two wall $W_{-39}$ corresponding to the destabilizing object $T_{p^2}(-4)$.

7. Change of variables

Let $W_x$ be a Bridgeland wall with center at $x$ and radius $\sqrt{x^2-2n}$. Let the divisor $H + \frac{1}{2y}B$ span a wall of the Mori decomposition.

**Conjecture 7.1.** The change of variables formula

$$x = y - \frac{3}{2}$$

gives a one-to-one correspondence between the Bridgeland walls and Mori walls.

We have verified this for $n \leq 9$. It also holds for all $n$ in a large part of the cone. Even in ranges we have not proved the change of variables formula, it still has non-trivial predictive value for classical problems about base loci.

**References**