The period-index problem for Severi-Brauer varieties: recent progress and future prospects

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The simplest case to consider in the problem of rational points is forms of $\mathbb{P}^{n-1}$.

**Definition 1** Given a field $K$, a Severi-Brauer variety over $K$ is a scheme $X/K$ such that $X \otimes K \simeq \mathbb{P}^{n-1}_K$ for some $n \geq 1$.

**Example 1** A non-split conic $(x^2 + y^2 + z^2 = 0) \subset \mathbb{P}^2_{\mathbb{R}}$

There are various measures of twistedness for SB varieties:

1. Min. pos. degree of a 0-cycle on $X$ [= min. degree of a closed point]; this is called the *index* of $X$, written $\text{ind}(X)$;

2. Min. degree of a hypersurface in $X$, which is called the *period* of $X$, written $\text{per}(X)$; note that $\text{per}(X)\mid n$ because the anticanonical divisor exists; this is also called the exponent of $X$.

Abramovich: Does anyone use degrees of cycles in the middle? Colliot-Thélène: No systematic terminology for this.

- $\text{per}(X)\mid \text{ind}(X)$ because $\text{ind}(X) = \text{ind}(X^\vee)$, where $X^\vee$ is the dual variety; note that $\text{per}(X) = \text{per}(X^\vee)$ as well;

- $\text{ind}(X)$ and $\text{per}(X)$ have the same prime factors;

Thus $\text{ind}(X)\mid \text{per}(X)^d$ for some $d$. 
Question 1 (Period-index problem) Understand the d’s that arise in terms of $K$.

Rephrase: $\text{Aut}(\mathbb{P}^{n-1}) = \text{PGL}_n$ so we have

$$[X] \in H^1_{et}(\text{Spec } K, \text{PGL}_n).$$

The diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \mu_n & \rightarrow & \text{SL}_n & \rightarrow & \text{PGL}_n & \rightarrow & 1 \\
\downarrow & & \downarrow & & \| & & \| \\
1 & \rightarrow & \mathbb{G}_m & \rightarrow & \text{GL}_n & \rightarrow & \text{PGL}_n & \rightarrow & 1
\end{array}
$$

yields compatible maps

$$H^1(\text{Spec } K, \text{PGL}_n) \rightarrow H^2(\text{Spec } K, \mu_n) \downarrow \rightarrow H^2(\text{Spec } K, \mathbb{G}_m).$$

The group $H^2(\text{Spec } K, \mathbb{G}_m) = \text{Br}(K)$ is called the Brauer group. A SB variety $X$ yields

$$\text{cl}(X) \in \text{Br}(K)[n].$$

Given $\alpha \in \text{Br}(K)$

- the period of $\alpha$, $\text{per}(\alpha)$, is the order of $\alpha \in \text{Br}(K)$;
- the index of $\alpha$ in the minimum $[L : K]$ such that $\alpha|L = 0$.

Basic theorem: Given $X$, $\text{per}(X) = \text{per}(\text{cl}(X))$ and $\text{ind}(X) = \text{ind}(\text{cl}(X))$.

Note that

1. May introduce per, ind terminology for any $H^i_{et}(\text{Spec } K, \mathcal{F})$ where $\mathcal{F}$ is an abelian étale sheaf. Famously for $\mathcal{F} = E$ an elliptic curve and $i = 1$, e.g., Tate, Lichtenbaum, O’Neil, Clark, . . .

2. Over $k(a_1, \ldots, a)$ of characteristic $\neq 2$, we can make a SB variety $X$ of period 2 and arbitrarily high index. Let $C_i = (x^2 + a_{2i-1}y^2 + a_{2i}z^2 = 0)$ and consider Segre embedding

$$C_1 \times \cdots \times C_n \hookrightarrow X.$$ 

Then $\text{per}(X) = 2$ and $\text{ind}(X) = 2^n$. 

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<table>
<thead>
<tr>
<th>Field</th>
<th>People</th>
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<tbody>
<tr>
<td>( k = k )</td>
<td>Hilbert (NSS) ?</td>
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<tr>
<td>( \bar{k}(C), \mathbb{F}_q )</td>
<td>Tsen, Wedderburn, Châtelet</td>
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<td>( C ) curve</td>
<td></td>
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<tr>
<td>( \bar{k}(S), \mathbb{F}_q(S) )</td>
<td>de Jong, Starr, L-</td>
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<td>( \mathbb{Q} )</td>
<td>Brauer-Hasse-Noether</td>
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<td>( \mathbb{Q}_p(C) )</td>
<td>Saltman, L-, Harbater-Hartman-Krashen</td>
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<td>( d )-local field</td>
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<td>( d )-local field(( C ))</td>
<td>( d + 1^* ) L-</td>
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Here * means prime-to-\( p \)

**Question 2** Does there exist a bound for any threefold /\( \mathbb{C} \), e.g., \( \mathbb{C}(x,y,z) \)?

**Conjecture 1 (Colliot-Thélène)** If \( X \) is a \( d \)-fold over \( \bar{k} \) or a \((d-1)\)-fold over \( \mathbb{F}_q \) then for all \( \alpha \in \text{Br}(k(X)) \) we have \( \text{ind}(\alpha) \mid \text{per}(\alpha)^{d-1} \).

Various techniques involved in the search for bounds:

1. Rat’l curve, higher rat’l connectedness (de Jong, Starr);
2. Stacks (L-);
3. Formal algebraic geometry and ‘patching’ (Harbater-Hartmann-Krashen);

Starr: Does ‘topology’ mean motivic cohomology/\( \mathbb{A}^1 \) homotopy? Lieblich: No, normal topology like spectral sequences, spectra, classifying spaces, etc.

A natural question is to what extent one can soup these up to motivic statements.

**Illustration:**

I. \( \alpha \in \text{Br}(\bar{k}(S)) \) \( \Rightarrow \) \( \text{ind}(\alpha) = \text{per}(\alpha) \);

II. \( \alpha \in \text{Br}(\mathbb{F}_q(S)) \) \( \Rightarrow \) \( \text{ind}(\alpha) \mid \text{per}(\alpha)^2 \);

Similar initial set-up:
a. Make a proper model of $\alpha, K$: given $\alpha \in \text{Br}(k(S))$, there exists a proper smooth Deligne-Mumford stack $S/k$ such that $k(S) = k(S)$ and a lift $\tilde{\alpha} \in \text{Br}(S)$ restricting to $\alpha$ over the function field; geometrically, this says that the SB variety extends to a $\mathbb{P}^{n-1}$-bundle $X \to S$.

b. Easy case: $X = \mathbb{P}^{n-1} = \mathbb{P}(E)$

Universal reduction to this case: there exists a $\mu_n$-gerbe $Y \to S$ and a twisted vector bundle $E$ on $Y$ such that

$$X \times Y \simeq \mathbb{P}(E) \to Y.$$

Here, twisted means that the $\mu_n$ stabilizer at every point coming from the gerbe structure acts on $E$ via multiplication.

c.  

I. First, fiber by curves

$$\tilde{S} \to S$$

and fix section $p: \mathbb{P}^1 \to \tilde{S}$; let $\tilde{Y} = Y \times_s \tilde{S}$. Consider the relative moduli space

$$\mathcal{M}_{Y/\mathbb{P}^1}(n, \mathcal{O}_Y(p)) \to \mathbb{P}^1,$$

stable twisted vector bundle of rank $n$ and determinant $\mathcal{O}(p)$. Here $n = \text{per}(\alpha)$. Our fibration is a RC fibration, so Graber-Harris-Starr implies there exists object over $k(t)$. Thus $\mathcal{V}/\mathcal{Y}$ twisted vector bundle of rank $n$, whence $\text{per}(\alpha) = \text{ind}(\alpha)$.

II. Have $\mathcal{Y} \to S$ and make $\mathcal{M}_Y$, the stack of simple $\mathcal{Y}$-twisted sheaves of rank $n^2$. Then we have

**Theorem 1 (L-)**  
- $\mathcal{M}_Y \neq \emptyset$;
- there exists a geometrically integral locally closed substack $\mathcal{Z} \subset \mathcal{M}_Y$.

Then the Lang-Weil estimates imply existence of 0-cycle of degree 1.

An inspirational theorem from topology:

**Theorem 2 (Antieau)** Let $X$ be a finite CW complex of cohomological dimension $d$, $\ell$ a prime such that $2\ell > d + 1$, and given $\alpha \in \text{Br}(X)$ such that $\text{per}(\alpha) = \ell^k$. Then

$$\text{ind}(\alpha) | \text{per}(\alpha)^{\lfloor d/2 \rfloor}.$$
While the results properly belong to topology (and, in particular, the exponent in the relation is too small in certain cases, like Stein spaces, to come from algebraic geometry), one hopes to be able to important some of the techniques into algebraic geometry (although this might require hard work). In addition, Antieau has suggested potential counterexamples to CT’s conjecture arising from classifying spaces. Might one use one of Totaro’s approximations and study these classes as algebro-geometric candidates?

A related question: given a curve $C$ over an algebraically closed field, consider the finite étale covers $C' \to C$. There are associated pullback maps $M_n(C) \to M_n(C')$, where $M$ denotes the moduli space of stable sheaves of a fixed rank $n$ and trivial determinant. There are also rank increasing maps $M_n(C') \to M_{nm}(c')$. Does the homotopy colimit of the $M(C')$ over all coverings and ranks have especially nice structure? (E.g., is it contractible?) This is another kind of topological result that might be relevant for understanding the period-index problem in higher dimensions.