Integral Tate conjecture for cubic fourfolds

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Let \( F \) be a finite field of characteristic \( p \) and \( \ell \) a prime distinct from \( p \). Let \( \overline{F} \) be an algebraic closure with \( G = \text{Gal}(\overline{F}/F) \).

For \( X \) smooth and projective over \( F \), let

\[ c_i^\ell : \text{CH}^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}_{\text{et}}(X, \mathbb{Z}_\ell(i)) \]

be the cycle class map. Passing to algebraic closures we get maps:

\[ \overline{c}_i^\ell : \text{CH}^i(\overline{X}) \otimes \mathbb{Z}_\ell \to \bigcup_{H \subset G \text{ open}} H^{2i}_{\text{et}}(\overline{X}, \mathbb{Z}_\ell(i))^H. \]

**Remark 1** If \( X/\mathbb{C} \) smooth projective then we get maps

\[ c_i^i : \text{CH}^i(X) \to \text{Hdg}^{2i}(X, \mathbb{Z}). \]

**Facts:**

1. \( i = 1 \): The Kummer sequence gives

   \[ \text{coker}(c_1^\ell) \hookrightarrow T_\ell\text{Br}(X), \]

   where the right-hand-side has no torsion. Then the Tate conjecture implies the integral version.

2. If \( i = \dim(X) \) then the Lang-Weil estimates give a zero-cycle of degree 1.

3. If \( i = \dim(X) - 1 \) then Schoen shows \( \overline{c}_i^{\dim(X)-1} \) is surjective provided the Tate conjecture holds for divisors on surfaces.
We turn to $i = 2$: Here we have
\[ \text{coker}(c^2_\ell) = H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))/\text{max. div. subgroup} \]
here
\[ H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = \lim_{\rightarrow} H^3_{nr}(X, \mu^{\otimes 2} \ell). \]
which is conjecturally finite.

**Theorem 1 (Parimala-Suresh)** Let $S$ be a smooth projective surface over $\mathbb{F}$ and $X \to S$ a conic bundle. Then $H^3_{nr}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ for all $\ell$.

**Theorem 2 (Charles, P-)** Let $X \subset \mathbb{P}^5_\mathbb{F}$ be a smooth cubic with $p \geq 5$. Then $c^2_\ell$ is surjective.

It remains open whether $H^3_{nr}(X)$ is nonzero.

1. Voisin established the integral Hodge conjecture in this case, over $\mathbb{C}$, via normal functions of Zucker.

2. method for Theorem 2: algebraic normal functions

3. (F. Charles): the Tate conjecture holds for $X$.

**Idea:** Given $X/\mathbb{F}$ we consider the variety of lines $F$. Lift to characteristic zero, to get $\mathcal{X}$ and $\mathcal{F}$, where the latter is holomorphic symplectic of type $K3[2]$. Under some conditions, $F$, being a reduction of $\mathcal{F}$, satisfies the Tate conjecture. Using Beauville-Donagi results on the incidence correspondence, we deduce the conjecture for $X$.

**Proof:** (1) Take $\overline{X} \subset \mathbb{P}^5_\mathbb{F}$ and $\alpha \in H^4_{et}(\overline{X}, \mathbb{Z}_\ell(2))^H$. Take a Lefschetz pencil
\[ \overline{X} \leftarrow \text{Bl}_Y \overline{X} \supset Y \]
\[ \downarrow \quad \downarrow \]
\[ \mathbb{P}^1 \supset U \]
where $U$ is the smooth locus of the pencil and $S$ is a cubic surface.

**Lemma 1**

i. If $\alpha | Y = 0$ then $\alpha = 0$. 

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ii. One can assume up to replacing $\alpha$ by $\alpha + b[L \times \mathbb{P}^1]$ that $\alpha|Y_t = 0$ for all $t \in U$.

Consider the incidence correspondence

\[ \{(L, x), x \in L\} \]
\[ \uparrow \]
\[ V \]
\[ p \]
\[ F \]
\[ \psi \]
\[ \downarrow \]
\[ U \]
\[ \pi \]
\[ \downarrow \]
\[ Y \]
\[ q \]

where $F_t$ is the Fano variety of lines of $Y_t$. The Leray spectral sequence gives

\[ H^1(U, \mathbb{R}^3\pi_*Z\ell(2)) \to H^4(Y, Z\ell(2)) \to H^0(U, \mathbb{R}^4\pi_*Z\ell(2)). \]

As $\alpha|Y_t = 0$ for all $t \in U$, the element $\alpha$ comes from $\beta \in H^1(U, \mathbb{R}^3\pi_*Z\ell(2))$, where this first term maps to

\[ H^1(U, \mathbb{R}^1\psi_*Z\ell(1)). \]

Let $\gamma$ be the image of $\beta$. Set $\mathcal{J} = \text{Pic}^0 F/U$. Then the Kummer sequence gives

\[ \mathcal{J}(U) \otimes \mathbb{Z}_\ell \xrightarrow{\eta} H^1(U, \mathbb{R}^1\psi_*Z\ell(1)) \to T\ell H^1(U, \mathcal{J}) \]

where the last term has no torsion. If $\alpha = c^2z$ for $z \in \text{CH}^2(X) \otimes \mathbb{Z}_\ell$ then $z' = p_*q^*z \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell$.

**Fact:** If $\alpha$ is algebraic, $\eta(z')$ coincides with the image of $\alpha$ along the map

\[ H^1(U, \mathbb{R}^3\pi_*Z\ell) \to H^1(U, \mathbb{R}^1\phi_*Z\ell(1)). \]

Next, for $\alpha$ a cohomology class as before, the ordinary Tate conjecture implies there exists $N > 0$ such that $N\alpha$ is algebraic. Thus $N\gamma \in \text{Im}(\eta)$ and $\gamma \in \text{Im}(\eta)$.

**Definition 1** An element $z' \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell$ such that $\eta(z') = \gamma$ is a normal function associated with $\alpha$.  

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(2) (Markushevich-Tikhomirov-Druel) Consider the moduli space \( \mathcal{M} \to U \) of semistable torsion-free rank-two sheaves on \( Y \) with \( c_2 = 0, c_1 = 2[L] \), defined in this generality by Langer. Thus we have

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow \\
\mathcal{J} \\
\downarrow \\
U
\end{array}
\]

such that for any \( t \in U \), \( \mathcal{M}_t \to \mathcal{J}_t \) induces a birational map from at least one component of \( \mathcal{M}_t \) to \( \mathcal{J}_t \). By (1) we get \( z' \in \mathcal{J}(U) \otimes \mathbb{Z}_\ell \). It is enough to assume \( z' \in \mathcal{J}(U) \), i.e., a section of \( \mathcal{J} \to U \).

Let \( K = \overline{\mathbb{F}}(U) \) and regard \( z'_K \in \mathcal{J}(K) \) which yields \( y \in \mathcal{M}(K) \). Realize \( \mathcal{M}_K = \text{Quot}/\text{GL} \) and let \( C \subset \text{Quot} \) denote the unique closed orbit above \( y \), i.e., a projective homogeneous space for the action of \( \text{GL} \). Using a result of Springer gives the rational point, i.e., \( C(K) \neq \emptyset \), whence a rank two sheaf \( \mathcal{F} \) on \( Y \), because the cohomological dimension of \( \overline{\mathbb{F}}(t) \) is 1. The class

\[
c_2(\mathcal{F}) - 2[L \times \mathbb{P}^1]
\]

is the desired cycle of class \( \alpha|Y \).

**Counterexamples**

1) Atiyah-Hirzebruch, Totaro, P-, Yagita: There exists \( \alpha \in \bigcup_H H^4(\overline{X}, \mathbb{Z}_\ell(2))^H \) such that \( \alpha \) is not algebraic but \( \ell \alpha = 0 \). Even modulo torsion it is possible to get examples. The method involves analyzing cycles on classifying spaces via approximations by quotients of linear spaces by algebraic groups.

2) Over \( \mathbb{C} \): Kollár: \( X_d \subset \mathbb{P}^4_d \) hypersurface with \( d \) sufficiently divisible (by \( \ell \)). Then any curve class is proportional to \( \ell \), thus the class of 1 is not represented by algebraic cycles. This should not be possible over \( \mathbb{F} \) by Schoen’s Theorem, however.

3) The map

\[
\text{CH}^2(X) \otimes \mathbb{Z}_\ell \to \text{CH}^2(\overline{X})^G \otimes \mathbb{Z}_\ell
\]

need not be surjective.