Kodaira dimension of moduli of special cubic fourfolds

Anthony Várilly-Alvarado

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This is joint work with Sho Tanimoto.

1 Main Theorem

Definition 1 A (labelled) special cubic fourfold is a smooth cubic fourfold $x$, together with a rank two saturated lattice

$$K = \langle h^2, T \rangle \subset H^{2,2}(X) \cap H^4(X, \mathbb{Z})$$

where $h$ is the hyperplane class.

The discriminant of $X$ is the determinant of the lattice $K$.

Let $C_D$ denote the special cubic fourfolds with a labelling of discriminant $D$, and let $C$ the coarse moduli space of all cubic fourfolds.

Theorem 1 (Hassett) $C_D$ is an irreducible algebraic divisor in $C$, non-empty iff $D > 6$ and $D \equiv 0, 2 \pmod{6}$.

Examples:

1. If $T$ is a plane then $X \in C_8$ and $K_8$ is

   \[
   \begin{array}{c|cc}
   & h^2 & T \\
   \hline
   h^2 & 3 & 1 \\
   T & 1 & 3 \\
   \end{array}
   \]
2. \( D = 6n \) then
\[
K_D \simeq \begin{pmatrix} h^2 & T \\ h^2 & 3 & 0 \\ T & 0 & 2n \end{pmatrix}
\]

3. \( D = 6n + 2 \) then
\[
K_D \simeq \begin{pmatrix} h^2 & T \\ h^2 & 3 & 1 \\ T & 1 & 2n + 1 \end{pmatrix}
\]

For cubic fourfolds \( H^4(X, \mathbb{Z}) \simeq (+1)^{21} \oplus (-1)^{82} \) and
\[
K_D^\perp(-1) \simeq \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & \epsilon \\ 0 & \epsilon & 2n \end{pmatrix} \oplus U \oplus E_8(-1)^{82}
\]

where
\[
\epsilon = \begin{cases} 0 & \text{if } D \equiv 0 \mod 6 \\ 1 & \text{if } D \equiv 2 \mod 6 \end{cases}
\]

**Theorem 2 (Tanimoto-VA ’15)**

1. If \( D = 6n + 2 \)
   
   (a) If \( n > 18, n \not\in \{20, 21, 23\} \) then \( C_D \) is general type.
   
   (b) If \( n = 14, 18, 20, 21, 25 \) then \( \kappa(C_D) \geq 0 \).
   
   (c) If \( n \leq 7 \) then \( \kappa(C_D) < 0 \) (Nuer, Hassett).

2. If \( D = 6n \)
   
   (a) If \( n > 18, n \not\in \{20, 22, 23, 27, 32, 33\} \) then \( C_D \) is general type.
   
   (b) If \( n = 17, 23, 27, 33 \) then \( \kappa(C_D) \geq 0 \).
   
   (c) If \( n \leq 6 \) then \( \kappa(C_D) < 0 \) (Nuer, Hassett).

### 2 Arithmetic Motivation

Let \( X/\mathbb{Q} \) be a K3 surface. Skorobogatov/Zarhin ’08 showed that \( \text{Br}(X)/\text{Br}_0(X) \) is finite where
\[
\text{Br}_0(X) := \text{im}(\text{Br}(\mathbb{Q}) \to \text{Br}(X)).
\]

**Note:** \( \text{Br}(\bar{X}) = (\mathbb{Q}/\mathbb{Z})^{22 - \rho(X)} \), so “most” elements of the transcendental Brauer group do not descend to the ground field.
Question 1 (Uniform boundedness for Brauer groups) Let \( X/\mathbb{Q} \) be a K3 surface with \( \text{NS}(\bar{X}) \simeq L \) where \( L \) is a fixed lattice. Is there a constant \( c(L) \) such that \( |\text{Br}(X)/\text{Br}_0(X)| < c(L) ? \)

In joint work with McKinnie-Sawon-Tanimoto from 2014, we found the following picture: Let \( K_{2d} \) denote the coarse moduli space of K3 surfaces. Let \( K_{2d}(\langle \alpha \rangle) \) denote the moduli space of pairs of \( X \) and a subgroup

\[ 0 \neq \langle \alpha \rangle \subset \text{Br}(X)[p], \quad p \nmid d \]

where \( p \) is prime. (This description works when \( \text{Pic}(X) \) is cyclic.) This has three irreducible components:

- \( K_{2d}p^2 \)
- \( C_{2p^2} \) if \( d = 1, p \equiv 1 \pmod{3} \)
- mystery component

Sanity check: If the Brauer groups are to be controlled then the spaces should be of general type for \( p \gg 0 \).

Previous work in this direction is due to Kondo (for \( K_{2p^2} \) and \( p \equiv 2 \pmod{3} \)) and Gritsenko-Hulek-Sankaran (for \( K_d \) for \( d \gg 0 \)).

Question: (Charles) Fixing the ground field, are there just finitely many lattices that arise for the Picard group of a K3 surface?

3 Orthogonal modular varieties

Let \( L \) be an integral lattice of signature \((2, m)\) with \( m \geq 9 \), e.g., \( L = K_{D}(-1) \). Consider the (local) period domain

\[ \mathcal{D}_L = \{ x \in \mathbb{P}(L \otimes \mathbb{C}) : (x, x) = 0, (x, \bar{x}) > 0 \}^+, \]

which is a component of \( \mathcal{D}_L \cup \mathcal{D}'_L \). Let \( O^+(L) \subset O(L) \) denote the subgroup of the integral orthogonal group preserving \( \mathcal{D}_L \). We have an exact sequence

\[ 1 \to \hat{O}(L) \to O(L) \to O(D(L)) \to 1 \]

with kernel the stable orthogonal group. Here \( D(L) \) is the discriminant group. We set

\[ \hat{O}^+(L) = \hat{O}(L) \cap O^+(L). \]
For $\Gamma \subset O^+(L)$ of finite index let

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}_L,$$

be a modular variety of orthogonal type. It is quasi-projective (Baily-Borel) of dimension $m$.

Idea:

$$C_D \hookrightarrow \Gamma \backslash \mathcal{D}_{K_D}(-1)$$

for a suitable $\Gamma$. Modular forms for $\Gamma$ are roughly differential forms for $\mathcal{F}_L(\Gamma)$.

**Theorem 3 (Gritsenko-Hulek-Sankaran ’07) ‘low weight cusp form trick’**

The variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a non-zero $F_a \in S_a(\Gamma; \chi)$ for $\chi: \Gamma \to \mathbb{C}^*$ either the trivial or the determinant character, of weight $a < m$ that vanishes along the ramification divisor of the projection

$$\mathcal{D}_L \to \mathcal{F}_L(\Gamma).$$

Idea: $F_a^k M_{(m-a)k}(\Gamma) \subset \Gamma(\bar{Y}, \omega^{\otimes k}_\bar{Y})$ where $\bar{Y}$ is a desingularization of a compactification of $\mathcal{F}_L(\Gamma)$.

How do we produce an $F_a$?

**Theorem 4 (Borcherds)** For $L_{2,26} := U^{\oplus 2} \oplus E_8(-1)^{\oplus 3}$ there exists $0 \neq \Phi_{12} \in M_{12}(O^+(L_{2,26}), \det)$.

We leverage this form.

**Theorem 5 (Borcherds-Katzarkov-Pantev-Shepherd Barron; GHS)**

Let $L \hookrightarrow L_{2,26}$ be a primitive embedding giving $\mathcal{D}_L \hookrightarrow \mathcal{D}_{L_{2,26}}$. Let

$$R_{-2}(L) = \{ r \in L_{2,26} : r^2 = -2, (r, L) = 0 \}$$

and $N(L) = \#R_{-2}(L)/2$. Then

$$\Phi|_L = \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(L)/\pm 1} (Z, r)} \bigg|_{\mathcal{D}_L} \in M_{12 + N(L)}(\tilde{O}^+(L), \det)$$

and if $N_L > 0$ then $\Phi|_L$ is a cusp form.
We need $L := K_D^{-1}(-1) \hookrightarrow L_{2,26}$ such that $0 < N_L < 7$. This is OK if $D \equiv 2 \pmod{6}$. If $D \equiv 0 \pmod{6}$ then

$$C_D \hookrightarrow \Gamma_D^+ \backslash D_{K_D^{-1}(-1)}$$

but $\hat{O}^+(L) \subset \Gamma_D^+$ as an index two subgroup, and one has to show that the quasi-pullback $\Phi|_L$ is modular for the larger group $\Gamma_D^+$.

To get embeddings $K_D^1(-1) \hookrightarrow L_{2,26}$, set

$$B = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & \epsilon \\ 0 & \epsilon & 2n \end{pmatrix},$$

and consider embeddings $B \hookrightarrow U \oplus E_8(-1)$. This requires delicate analysis of theta-series and mass formulas, excluding elements where $N_L$ is too large. For $n \geq 150,000$ this all works. A computer search finds embeddings for $n \leq 4000$. The intermediate cases required some clever analysis.