DERIVED AUTOMORPHISM GROUPS OF K3 SURFACES OF PICARD RANK 1

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We dedicate this paper to Professor Mukai on the occasion of his 60th birthday.

Abstract. We give a complete description of the group of exact autoequivalences of the bounded derived category of coherent sheaves on a K3 surface of Picard rank 1. We do this by proving that a distinguished connected component of the space of stability conditions is preserved by all autoequivalences, and is contractible.

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1. Introduction

Let $X$ be a smooth complex projective variety. We denote by $D(X) = D^b \text{Coh}(X)$ the bounded derived category of coherent sheaves on $X$, and by $\text{Aut} D(X)$ the group of triangulated, $\mathbb{C}$-linear autoequivalences of $D(X)$, considered up to isomorphism of functors. There is a subgroup

$$\text{Aut}_{\text{st}} D(X) \cong \text{Aut} X \ltimes \text{Pic}(X) \times \mathbb{Z}$$

of $\text{Aut} D(X)$ whose elements are called standard autoequivalences: it is the subgroup generated by the operations of pulling back by automorphisms of $X$ and tensoring by line bundles, together with the shift functor.

The problem of computing the full group $\text{Aut} D(X)$ is usually rather difficult. Bondal and Orlov proved that when the canonical bundle $\omega_X$ or its inverse is ample, all autoequivalences are standard: $\text{Aut} D(X) = \text{Aut}_{\text{st}} D(X)$. The group $\text{Aut} D(X)$ is also known explicitly when $X$ is an abelian variety, due to work of Orlov [Orl02].

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Broomhead and Ploog [BP10] treated many rational surfaces (including most toric surfaces). However, no other examples are known to date.

The aim of this paper is to determine the group $\text{Aut} D(X)$ in the case when $X$ is a K3 surface of Picard rank 1.

**Mukai lattice.** For the rest of the paper $X$ will denote a complex algebraic K3 surface. In analogy to the strong Torelli theorem, which describes the group $\text{Aut} X$ via its action on $H^2(X, \mathbb{Z})$, one naturally starts studying $\text{Aut} D(X)$ via its action on cohomology. We will briefly review the relevant results, see [Huy06, Section 10] for more details.

The cohomology group $H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$, comes equipped with a polarized weight two Hodge structure, whose algebraic part is given by

$$\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}),$$

and whose polarization is given by the Mukai symmetric form

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

The lattice $H^*(X, \mathbb{Z})$ has signature $(4, 20)$, and the subgroup $\mathcal{N}(X)$ has signature $(2, \rho(X))$, where the Picard rank $\rho(X)$ is the rank of the Néron-Severi lattice $\text{NS}(X)$.

Any object of $D(X)$ has a Mukai vector $v(E) = \text{ch}(E) \sqrt{\text{td} X} \in \mathcal{N}(X)$, and Riemann-Roch takes the form

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}^i_X(E, F) = -(v(E), v(F)).$$

Since any autoequivalence is of Fourier-Mukai type, the Mukai vector of its kernel induces a correspondence; its action on cohomology preserves the Hodge filtration, the integral structure and the Mukai pairing. We thus get a map

$$\varpi: \text{Aut} D(X) \longrightarrow \text{Aut} H^*(X)$$

to the group of Hodge isometries.

The group $\text{Aut} H^*(X)$ contains an index 2 subgroup $\text{Aut}^+ H^*(X)$ of Hodge isometries preserving the orientation of positive definite 4-planes. Classical results due to Mukai and Orlov ([Muk87, Orl97]) imply that the image of $\varpi$ contains $\text{Aut}^+ H^*(X)$, see [HLOY04, Plo05]. A much more difficult recent result due to Huybrechts, Macrì and Stellari [HMS09] is that the image of $\varpi$ is contained in (and hence equal to) $\text{Aut}^+ H^*(X)$. Our results in this paper give an alternative, very different proof of this fact in the case when $X$ has Picard rank 1.

To determine the group $\text{Aut} D(X)$ it thus remains to study the kernel of $\varpi$, which we will denote by $\text{Aut}^0 D(X)$. This group is highly non-trivial due to the existence
of spherical twist functors. Recall that an object \( S \in D(X) \) is called spherical if

\[
\text{Hom}_{D(X)}(S, S[i]) = \begin{cases} 
\mathbb{C} & \text{if } i \in \{0, 2\}, \\
0 & \text{otherwise.}
\end{cases}
\]

Associated to any such object there is a corresponding twist or reflection functor \( \text{Tw}_S \in \text{Aut}(D(X)) \), which appeared implicitly already in \([\text{Muk87}]\), and which was studied in detail (and generalized) in \([\text{ST01}]\). The functor \( \text{Tw}_S \) acts on cohomology by a reflection in the hyperplane orthogonal to \( v(S) \), and hence its square \( \text{Tw}_S^2 \) defines an element of the group \( \text{Aut}^0 D(X) \).

**Stability conditions.** Following the approach introduced by the second author in \([\text{Bri08}]\), we study \( \text{Aut}^0 D(X) \) using a second group action, namely its action on the space of stability conditions.

We denote by \( \text{Stab}(X) \) the space of (full, locally-finite) numerical stability conditions \((Z, P)\) on \( D(X) \). This is a finite-dimensional complex manifold with a faithful action of the group \( \text{Aut}(D(X)) \). The central charge of a numerical stability condition takes the form

\[
Z(-) = (\Omega, v(-)) : K(D) \to \mathbb{C}
\]

for some \( \Omega \in \mathcal{N}(X) \otimes \mathbb{C} \), and the induced forgetful map \( \text{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C} \) is a local homeomorphism by \([\text{Bri07}]\).

Let \( \text{Stab}^\dagger(X) \subset \text{Stab}(X) \) be the connected component containing the set of geometric stability conditions, for which all skyscraper sheaves \( O_x \) are stable of the same phase. The main result of \([\text{Bri08}]\) is a description of this connected component, which we now review.

Recall that \( \mathcal{N}(X) \) has signature \((2, \rho(X))\). Define the open subset

\[
\mathcal{P}(X) \subset \mathcal{N}(X) \otimes \mathbb{C}
\]

consisting of vectors \( \Omega \in \mathcal{N}(X) \otimes \mathbb{C} \) whose real and imaginary parts span a positive definite 2-plane in \( \mathcal{N}(X) \otimes \mathbb{R} \). This subset has two connected components, distinguished by the orientation induced on this 2-plane; let \( \mathcal{P}^+(X) \) to be the component containing vectors of the form \((1, i\omega, -\frac{1}{2} \omega^2)\) for an ample class \( \omega \in \text{NS}(X) \otimes \mathbb{R} \).

Consider the root system

\[
\Delta(X) = \{ \delta \in \mathcal{N}(X) : (\delta, \delta) = -2 \},
\]

and the corresponding hyperplane complement

\[
\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.
\]

We note that \( \Delta(X) \) is precisely the set of Mukai vectors of spherical objects in \( D(X) \).
Theorem 1.1 ([Bri08 Theorem 1.1]). The forgetful map sending a stability condition to the associated vector $\Omega \in \mathcal{N}(X) \otimes \mathbb{C}$ induces a covering map

$$\pi: \text{Stab}^\dagger(X) \rightarrow \mathcal{P}^+_0(X),$$

The covering is normal, and the group of deck transformations can be identified with the subgroup of $\text{Aut}^0 D(X)$ which preserves the connected component $\text{Stab}^\dagger(X)$.

The Galois correspondence for the normal covering $\pi$ then gives a map

$$\pi_1(\mathcal{P}^+_0(X)) \rightarrow \text{Aut}^0 D(X),$$

which is injective if and only if $\text{Stab}^\dagger(X)$ is simply-connected, and surjective if and only if $\text{Stab}^\dagger(X)$ is preserved by $\text{Aut}^0 D(X)$. This suggests the following conjecture of the second author:

Conjecture 1.2 ([Bri08 Conjecture 1.2]). The group $\text{Aut} D(X)$ preserves the connected component $\text{Stab}^\dagger(X)$. Moreover, $\text{Stab}^\dagger(X)$ is simply-connected. Hence there is a short exact sequence of groups

$$1 \rightarrow \pi_1(\mathcal{P}^+_0(X)) \rightarrow \text{Aut} D(X) \xrightarrow{\pi} \text{Aut}^+ H^\ast(X) \rightarrow 1.$$

Let us write $\text{Stab}^\ast(X) \subset \text{Stab}(X)$ to denote the union of those connected components which are images of $\text{Stab}^\dagger(X)$ under an autoequivalence of $D(X)$. The content of Conjecture 1.2 is then that the space $\text{Stab}^\ast(X)$ should be connected and simply-connected.

Main result. The main result of this paper is the following:

Theorem 1.3. Assume that $X$ has Picard rank $\rho(X) = 1$. Then $\text{Stab}^\ast(X)$ is contractible. In particular, Conjecture 1.2 holds in this case.

As has been observed previously by Kawatani [Kaw12 Theorem 1.3], when combined with a description of the fundamental group of $\mathcal{P}^+_0(X)$, Theorem 1.3 implies:

Theorem 1.4. Assume that $X$ has Picard rank $\rho(X) = 1$. Then the group $\text{Aut}^0 D(X)$ is the product of $\mathbb{Z}$ (acting by even shifts) with the free group generated by the autoequivalences $\text{Tw}^\ast_{S}$ for all spherical vector bundles $S$.

As we will explain in Section 2, the assumption $\rho(X) = 1$ implies that any spherical coherent sheaf $S$ on $X$ is necessarily a $\mu$-stable vector bundle.

To prove Theorem 1.3, we start with the observation that the set of geometric stability conditions is contractible (this easily follows from the results in [Bri08 Section 10–11]). Now pick a point $x \in X$, and consider the width

$$w_{\mathcal{O}_x}(\sigma) = \phi^+(\mathcal{O}_x) - \phi^-(\mathcal{O}_x),$$

where $\phi^\pm(\mathcal{O}_x)$ is the maximal and minimal phase appearing in the Harder-Narasimhan filtration of $\mathcal{O}_x$; this defines a continuous function

$$w_{\mathcal{O}_x}: \text{Stab}^\ast(X) \rightarrow \mathbb{R}_{\geq 0}.$$
We then construct a flow on $\text{Stab}^* \ast (X)$ which decreases $w_{\mathcal{O}_x}$, and use it to contract $\text{Stab}^* (X)$ onto the subset $w_{\mathcal{O}_x}^{-1}(0)$ of geometric stability conditions.

**Remarks 1.5.**  
(a) We do not currently know how to generalize our methods to higher Picard rank. Note that in the general case it is not known whether the universal cover of $P^+_0 (X)$ is contractible, although this statement is implied by a special case of a conjecture of Allcock, see [All11, Conjecture 7.1].
(b) Many of the questions relevant to this article were first raised in [Sze01].
(c) There are various examples of derived categories of non-projective manifolds $Y$ for which it has been shown that a distinguished connected component of the space $\text{Stab}(Y)$ is simply-connected, see [IUU10, BT11, BM11, Qiu11, Sut13]. In each of these cases, the authors used the faithfulness of a specific group action on $D(Y)$ to deduce simply-connectedness of (a component of) $\text{Stab}(Y)$, whereas our logic runs in the opposite direction: a geometric proof of simply-connectedness implies the faithfulness of a group action.

**Relation to mirror symmetry.** We will briefly explain the relation of Conjecture 1.2 to mirror symmetry; the details can be found in Section 7. The reader is also referred to [Bri09] for more details on this. The basic point is that the group of autoequivalences of $D(X)$ as a Calabi-Yau category coincides with the fundamental group of a mirror family of K3 surfaces.

A stability condition $\sigma \in \text{Stab}^* (X)$ will be called *reduced* if the corresponding vector $\Omega \in \mathcal{N}(X) \otimes \mathbb{C}$ satisfies $(\Omega, \Omega) = 0$. This condition defines a complex submanifold

$$\text{Stab}^*_{\text{red}} (X) \subset \text{Stab}^* (X).$$

As explained in [Bri09], this is the first example of Hodge-theoretic conditions on stability conditions: it is not known how such a submanifold should be defined for higher-dimensional Calabi-Yau categories.

Define a subgroup

$$\text{Aut}^+_\text{CY} H^* (X) \subset \text{Aut}^+ H^* (X)$$

consisting of Hodge isometries whose complexification acts trivially on the complex line $H^{2,0}(X, \mathbb{C})$, and let

$$\text{Aut}_{\text{CY}} D(X) \subset \text{Aut} D(X)$$

denote the subgroup of autoequivalences $\Phi$ for which $\varpi (\Phi)$ lies in $\text{Aut}^+_\text{CY} H^* (X)$. Such autoequivalences are usually called *symplectic*, but we prefer the term *Calabi-Yau* since, as we explain in the Appendix, this condition is equivalent to the statement that $\Phi$ preserves all Serre duality pairings

$$\text{Hom}_X^i (E, F) \times \text{Hom}_X^{2-i} (F, E) \longrightarrow \mathbb{C}.$$ 

Let us now consider the orbifold quotient

$$\mathcal{L}_{\text{Kah}} (X) = \text{Stab}^*_{\text{red}} (X) / \text{Aut}_{\text{CY}} (X).$$
There is a free action of the group $\mathbb{C}$ on $\text{Stab}^*(X)$, given by rotating the central charge $Z$ and adjusting the phases of stable objects in the obvious way. The action of $2n \in \mathbb{Z} \subset \mathbb{C}$ coincides with the action of the shift functor $[2n] \in \text{Aut}_{\text{CY}} D(X)$. In this way we obtain an action of $\mathbb{C}^* = \mathbb{C}/2\mathbb{Z}$ on the space $\mathcal{L}_{\text{Kah}}(X)$, and we can also consider the quotient

$$\mathcal{M}_{\text{Kah}}(X) = \mathcal{L}_{\text{Kah}}(X)/\mathbb{C}^*.$$ 

We view this complex orbifold as a mathematical version of the stringy Kähler moduli space of the K3 surface $X$.

Using Theorem 1.1 one easily deduces the following more concrete description for this orbifold. Define period domains

$$\Omega^{\pm}(X) = \left\{ \Omega \in \mathcal{P}(\mathcal{N}(X) \otimes \mathbb{C}) : (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0 \right\},$$

$$\Omega^+_0(X) = \Omega^{\pm}(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp,$$

and let $\Omega^+_0(X) \subset \Omega_0(X)$ be the connected component containing classes $(1, i\omega, \frac{1}{2}\omega^2)$ for $\omega \in \text{NS}(X)$ ample. Then there is an identification

$$\mathcal{M}_{\text{Kah}}(X) = \Omega^+_0(X)/\text{Aut}_{\text{CY}} H^*(X).$$

The mirror phenomenon in this context is the fact that this orbifold also arises as the base of a family of lattice-polarized K3 surfaces. More precisely, under mild assumptions (which always hold when $X$ has Picard number $\rho(X) = 1$), the stack $\mathcal{M}_{\text{Kah}}(X)$ can be identified with the base of Dolgachev’s family of lattice-polarized K3 surfaces mirror to $X$ [Dol96].

Conjecture 1.2 is equivalent to the statement that the natural map

$$\pi_1^{\text{orb}}(\mathcal{M}_{\text{Kah}}(X)) \longrightarrow \text{Aut}_{\text{CY}} D(X)/[2]$$

is an isomorphism. Our verification of this Conjecture in the case $\rho(X) = 1$ thus gives a precise incarnation of Kontsevich’s general principle that the group of derived autoequivalences of a Calabi-Yau variety should be related to the fundamental group of the base of the mirror family.

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It is a pleasure to dedicate this article to Professor Mukai on the occasion of his 60th birthday. The intellectual debt this article owes to his work, starting with [Muk87], can’t be overstated.

2. Preliminaries

Let $X$ be a complex algebraic K3 surface.
Stability conditions. Recall that a numerical stability condition $\sigma$ on $X$ is a pair $(Z, \mathcal{P})$ where $Z: \mathcal{N}(X) \to \mathbb{C}$ is a group homomorphism, and

$$\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \subset D(X)$$

is a full subcategory. The homomorphism $Z$ is called the central charge, and the objects of the subcategory $\mathcal{P}(\phi)$ are said to be semistable of phase $\phi$. We refer to [Bri07] and [Bri08, Section 2] for a complete definition. Any object $E \in D(X)$ admits a unique Harder-Narasimhan (HN) filtration

$$0 = E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n = E$$

with $A_i \in \mathcal{P}(\phi_1)$ semistable, and $\phi_1 > \phi_2 > \cdots > \phi_n$. We refer to the objects $A_i$ as the semistable factors of $E$. We write $\phi_+(E) = \phi_1$ and $\phi_-(E) = \phi_n$ for the maximal and minimal phase appearing in the HN filtration respectively.

Using the non-degenerate Mukai pairing on $\mathcal{N}(X)$ we can write the central charge of any numerical stability condition in the form

$$Z(-) = (\Omega, v(-)): \mathcal{N}(X) \to \mathbb{C}$$

for some uniquely defined $\Omega \in \mathcal{N}(X) \otimes \mathbb{C}$. Fix a norm $\| \cdot \|$ on the finite-dimensional vector space $\mathcal{N}(X) \otimes \mathbb{R}$. A numerical stability condition $\sigma = (Z, \mathcal{P})$ is said to satisfy the support condition [KS08] if there is a constant $K > 0$ such that for any semistable object $E \in \mathcal{P}(\phi)$ there is an inequality

$$|Z(E)| \geq K \cdot \|E\|.$$

As shown in [BM11, Proposition B.4], this is equivalent to the condition that $\sigma$ be locally-finite [Bri07, Defn. 5.7] and full [Bri08, Defn. 4.2].

If the stability condition $\sigma = (Z, \mathcal{P})$ is locally-finite, each subcategory $\mathcal{P}(\phi)$ is a finite length abelian category; the simple objects of $\mathcal{P}(\phi)$ are said to be stable of phase $\phi$. Each semistable factor $A_i$ of a given object $E \in D(X)$ has a Jordan-Hölder filtration in $\mathcal{P}(\phi_i)$. Putting these together gives a (non-unique) filtration of $E$ whose factors $S_j$ are stable, with phases taken from the set $\{\phi_1, \cdots, \phi_n\}$. These objects $S_j$ are uniquely determined by $E$ (up to reordering and isomorphism); we refer to them as the stable factors of $E$.

We let $\text{Stab}(X)$ denote the set of all numerical stability conditions on $D(X)$ satisfying the support condition. This set has a natural topology induced by a (generalized) metric $d(-,-)$. We refer to [Bri07, Proposition 8.1] for the full definition, and simply list the following properties:

(a) For any object $E \in D(X)$, the functions $\phi^\pm(E): \text{Stab}(X) \to \mathbb{R}$ are continuous.
(b) Take $0 < \epsilon < 1$ and consider two stability conditions $\sigma_i = (P_i, Z_i)$ such that $d(\sigma_1, \sigma_2) < \epsilon$. Then if an object $E \in D(X)$ is semistable in one of the stability conditions $\sigma_i$, the arguments of the complex numbers $Z_i(E)$ differ by at most $\pi \epsilon$.

(c) The forgetful map $\text{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C}$ sending a stability condition to the vector $\Omega$ is a local homeomorphism [Bri07].

Let $\text{GL}^+_2(\mathbb{R})$ be the group of orientation-preserving automorphisms of $\mathbb{R}^2$. The universal cover $\tilde{\text{GL}}^+_2(\mathbb{R})$ of this group acts on $\text{Stab}(X)$ by post-composition on the central charge $Z: \mathcal{N}(X) \to \mathbb{C} \cong \mathbb{R}^2$ and a suitable relabelling of the phases (see [Bri07]). There is a subgroup $C \subset \tilde{\text{GL}}^+_2(\mathbb{R})$ which acts freely; explicitly this action is given by $\lambda \cdot (Z, P) = (Z', P')$ with $Z' = e^{\pi i \lambda} \cdot Z$ and $P'(\phi) = P(\phi - \text{Re } \lambda)$. There is also an action of the group $\text{Aut}(D(X))$ on $\text{Stab}(X)$ by $\Phi \cdot (Z, P) = (Z', P')$ with $Z' = Z \circ \varphi(\Phi)^{-1}$ and $P'(\phi) = \Phi(P(\phi))$.

**Period domains.** Recall the definitions of the open subsets

$$P^+_0(X) \subset P^+_1(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$$

from the introduction. Now consider the corresponding subsets

$$Q^\pm(X) = \{ \Omega \in \mathcal{N}(X) \otimes \mathbb{C}; \ (\Omega, \Omega) = 0, (\Omega, \Omega) > 0 \},$$

$$Q^+_0(X) = Q^\pm(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

These are invariant under the rescaling action of $\mathbb{C}^*$ on $\mathcal{N}(X) \otimes \mathbb{C}$. As with $P^\pm(X)$, the subset $Q^\pm(X)$ consists of two connected components, and we let $Q^+_1(X) = Q^\pm(X) \cap P^+_1(X)$ be the one containing classes $(1, i\omega, 0, 0)$ for $\omega \in \text{NS}(X)$ ample.

The normalization condition $(\Omega, \Omega) = 0$ is equivalent to the statement that $\text{Re } \Omega, \text{Im } \Omega$ are a conformal basis of the 2-plane in $\mathcal{N}(X) \otimes \mathbb{R}$ which they span:

$$(\text{Re } \Omega, \text{Im } \Omega) = 0, \quad (\text{Re } \Omega)^2 = (\text{Im } \Omega)^2 > 0.$$  

From this, one easily sees that each $\text{GL}_2(\mathbb{R})$-orbit in $P^\pm(X)$ intersects $Q^\pm(X)$ in a unique $\mathbb{C}^*$-orbit. It follows that

$$(2) \quad P^+_0(X)/\text{GL}^+_2(\mathbb{R}) = Q^+_0(X)/\mathbb{C}^*,$$

and further that $Q^+_0(X) \subset P^+_0(X)$ is a deformation retract.

Let $T^\pm(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ be the two components of the tube domain

$$T^\pm(X) = \{ \beta + i\omega: \beta, \omega \in \text{NS}(X) \otimes \mathbb{R}, (\omega, \omega) > 0 \},$$

where $T^+(X)$ denotes the component containing $i\omega$ for ample classes $\omega$. The map $\Omega = \exp(\beta + i\omega)$ defines an embedding

$$(3) \quad T^+(X) \hookrightarrow Q^+(X)$$
which gives a section of the $\mathbb{C}^*$-action on $\mathcal{Q}^+(X)$; this identifies $\mathcal{T}^+(X)$ with the quotients in (2). We set
\[ \mathcal{T}_0^+(X) = \mathcal{T}^+(X) \cap \mathcal{Q}_0^+(X) \]
for the corresponding hyperplane complement.

Consider the case when $X$ has Picard number $\rho(X) = 1$. The ample generator of $\text{NS}(X)$ allows us to identify $\mathcal{T}^+(X)$ canonically with the upper half plane $\mathcal{h}$. The hyperplane complement $\mathcal{T}_0^+(X) \subset \mathcal{T}^+(X)$ then corresponds to the open subset
\[ \mathcal{h}^0 = \mathcal{h} \setminus \{ \beta + i\omega \in \mathcal{h} : \langle \exp(\beta + i\omega), \delta \rangle \in \mathbb{R} \leq 0 \text{ when } \delta \in \Delta(X) \}. \]

We now recall briefly the description of the fundamental group $\pi_1(\mathcal{P}_0^+(X))$ given in [Kaw12, Prop. 2.14]. Since the root system $\Delta(X)$ is discrete in the projectivization $\mathcal{C}^+/\mathbb{R}^+$ of the negative cone, the holes $\mathcal{h} \setminus \mathcal{h}^0$ are discrete in the upper half plane. From this one deduces that the fundamental group $\pi_1(\mathcal{h}^0)$ is the free group with the obvious generators.\footnote{Let us briefly sketch a proof. For $\epsilon > 0$, let $\mathcal{h}^\epsilon = \mathcal{h}^0 \cap \{ \text{Im } z > \epsilon, |\text{Re } z| < 1/\epsilon \}$. Then $\mathcal{h}^\epsilon$ is homeomorphic to a disc with finitely many holes; by Seifert-van Kampen, its fundamental group is the free group with finitely many generators. On the other hand, using compactness of loops and homotopies one can show that $\pi_1(\mathcal{h}^\epsilon)$ is the union of the fundamental groups $\pi_1(\mathcal{h}^\epsilon)$ as $\epsilon \to 0$.}

Now $\mathcal{P}_0^+(X)$ is a $\text{GL}_2^+((\mathbb{R}))$-bundle over $\mathcal{h}^0$, and this bundle is trivial since the map (3) defines a section. This yields
\[ \pi_1(\mathcal{P}_0^+(X)) = \mathbb{Z} \times \pi_1(\mathcal{h}^0). \]

**Geometric stability conditions.** A stability condition in $\text{Stab}(X)$ is said to be **reduced** if the corresponding vector $\Omega \in \mathcal{N}(X) \otimes \mathbb{C}$ satisfies $(\Omega, \Omega) = 0$. The set of reduced stability conditions forms a complex submanifold
\[ \text{Stab}_{\text{red}}(X) \subset \text{Stab}(X). \]

Restricted to the components $\text{Stab}^\ast(X) \subset \text{Stab}(X)$, the reduced condition is precisely that the vector $\Omega$ lies in $\mathcal{Q}^+(X) \subset \mathcal{P}^+(X)$.

The starting point in the description of $\text{Stab}^\dagger(X)$ given in [Bri08] is a characterization of the set of stability conditions $U(X) \subset \text{Stab}_{\text{red}}(X)$, for which all skyscraper sheaves $\mathcal{O}_x$ of points $x \in X$ are stable of the same phase. Such stability conditions are called **geometric**. Note that the subset $U(X)$ is invariant under the $\mathbb{C}$-action on $\text{Stab}(X)$, and each orbit contains a unique stability condition for which the objects $\mathcal{O}_x$ are stable of phase 1.

To describe the set $U(X)$ we first note that the set $\Delta(X)$ of $(-2)$-classes splits as a disjoint union
\[ \Delta(X) = \Delta^+(X) \sqcup -\Delta^+(X), \]
where $\Delta^+(X) \subset \Delta(X)$ consists of Mukai vectors of spherical sheaves. Consider the open subset
\[ U(X) = \{ \beta + i\omega \in \mathcal{T}^+(X) : \delta \in \Delta^+(X) \implies \langle \exp(\beta + i\omega), \delta \rangle \notin \mathbb{R}_{\leq 0} \}. \]

The following result is proved in [Bri08, Sections 10-12].
Figure 1. The subsets $\mathcal{V}(X) \subset \mathfrak{h}^\rho \subset \mathfrak{h}$ of the upper half plane, for a generic K3 surface that is a double-cover of $\mathbb{P}^2$.

**Theorem 2.1.** The forgetful map $\text{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C}$ induces a bijection between the set of reduced, geometric stability conditions in which the objects $\mathcal{O}_x$ have phase 1, and the set of vectors of the form $\Omega = \exp(\beta + i\omega)$ with $\beta + i\omega \in \mathcal{V}(X)$. Thus there is an isomorphism

$$U(X) \cong \mathbb{C} \times \mathcal{V}(X).$$

Let us again consider the case when $X$ has Picard number $\rho(X) = 1$. The subset $\Delta^+(X)$ consists of $(-2)$-classes $(r, \Delta, s)$ with $r > 0$. The subset $\mathcal{V}(X) \subset \mathfrak{h}^0$ is obtained by removing the vertical line segment between the real line and each hole $\mathfrak{h} \setminus \mathfrak{h}^\rho$, see Figure 1. For each $\delta \in \Delta^+(X)$ there is a unique spherical sheaf $S_\delta \in \text{Coh} X$ with $v(S_\delta) = \delta$, and this sheaf $S_\delta$ is automatically a $\mu$-stable vector bundle.

The following result was proved by Kawatani [Kaw12, Prop. 5.4]. For the reader’s convenience we include a sketch proof here.

**Proposition 2.2.** Let $\delta \in \Delta^+(X)$ be a spherical class with positive rank, and $S_\delta$ the spherical vector bundle with Mukai vector $\delta$. The deck transformation of the

\[\text{2The existence is part of [Yos99, Theorem 0.1]. Mukai already proved that a spherical torsion-free sheaf is automatically locally free and $\mu$-stable see [Muk87, Prop. 3.3 and Prop. 3.14]; the torsion-freeness in the case $\rho(X) = 1$ follows with the same argument, see Remark 6.5. Finally, the uniqueness is elementary from stability, see [Muk87, Corollary 3.5].} \]
normal covering $\pi$ associated to the anti-clockwise loop in $h^o$ around $\delta^\perp$ is induced by the square $\text{Tw}_2^{\pm}S_\delta$ of twist functor associated to $S_\delta$.

Proof. Given the line segment in $h \setminus V(X)$ associated to such $\delta$, there are two corresponding walls $W_\delta^+$ and $W_\delta^-$ of the geometric chamber, depending on whether we approach the line segment from the left or the right, respectively. Let $S_\delta$ be the corresponding spherical vector bundle, with $Z(S_\delta) \in \mathbb{R}_{<0}$ for $(Z,P) \in W_\delta^\pm$. These walls are described by the cases $(A^+)$, $(A^-)$ of [Bri08, Theorem 12.1], respectively (where the vector bundle $A$ in the citation is exactly our vector bundle $S_\delta$).

In the proof of [Bri08, Proposition 13.2], it is shown that crossing these walls leads into the image of the geometric chamber under the spherical twist $\text{Tw}_2^{\pm}S_\delta$. Now consider the loop $\gamma$ around the hole corresponding to $\delta$, and lift it to a path starting in the geometric chamber $U(X)$. It follows from the preceding discussion that its endpoint will lie in $\text{Tw}_2^{\pm}S_\delta U(X)$, with the sign depending on the orientation of the loop. □

Rigid and semirigid objects. The following important definition generalizes the notions of rigid and semirigid coherent sheaves from [Muk87]:

**Definition 2.3.** An object $E \in D(X)$ will be called rigid or semirigid if

(a) $\dim C\text{Hom}^1_X(E,E) = 0$ or 2, respectively, and
(b) $\text{Hom}^i_X(E,E) = 0$ for all $i < 0$.

We say that an object $E \in D(X)$ is (semi)rigid if it is either rigid or semirigid.

It follows from Riemann-Roch and Serre duality that if $E$ is (semi)rigid, then the Mukai vector $v(E)$ satisfies $(v(E), v(E)) \leq 0$, with strict inequality in the rigid case.

We will need a derived category version of [Muk87, Corollary 2.8]:

**Lemma 2.4.** Suppose $A \to E \to B$ is an exact triangle and $\text{Hom}_X(A,B) = 0$. Then

$$\dim C\text{Hom}^1_X(E,E) \geq \dim C\text{Hom}^1_X(A,A) + \dim C\text{Hom}^1_X(B,B).$$

**Proof.** Consider the space $V$ of maps of triangles

$$A \xrightarrow{f} E \xrightarrow{g} B \xrightarrow{h} A[1]
\quad \alpha \downarrow \quad \gamma \downarrow \quad \beta \downarrow \quad \alpha[1] \downarrow

There are obvious maps

$$F: V \to \text{Hom}^1_X(A,A) \oplus \text{Hom}^1_X(B,B), \quad G: V \to \text{Hom}^1_X(E,E).$$

The result follows from the two claims that $F$ is surjective and $G$ is injective.

For the first claim, note that given maps $\alpha: A \to A[1]$ and $\beta: B \to B[1]$ we obtain a map of triangles as above, because the difference

$$h[1] \circ \beta - \alpha[1] \circ h: B \to A[2]$$
vanishes by the assumption and Serre duality. For the second claim, a simple dia-
gram chase using the assumption $\text{Hom}_X(A, B) = 0$ shows that any map of triangles
as above in which $\gamma = 0$ is necessarily zero. \hfill \qed

The next result is a consequence of [HMS08, Proposition 2.9]. For the reader’s
convenience we include the easy proof here.

**Lemma 2.5.** Let $\sigma$ be a stability condition on $D(X)$, and $E$ an object of $D(X)$.

(a) If $E$ is rigid, then all stable factors of $E$ are rigid.

(b) If $E$ is semirigid, then all stable factors of $E$ are rigid or semirigid, and at
most one of them is semirigid.

**Proof.** First note that applying Lemma 2.4 repeatedly to the HN filtration of $E$
allows us to reduce to the case when the object $E \in \mathcal{P}(\varphi)$ is in fact semistable.

If $E$ has more than one non-isomorphic stable factor, then by taking a maximal
subobject whose Jordan-Hölder factors are all isomorphic, we can find a nontrivial
short exact sequence

$$0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$$

in $\mathcal{P}(\varphi)$ with $\text{Hom}_X(A, B) = 0$. Applying Lemma 2.4 shows that $A$ and $B$ are also
either rigid or semirigid, with at most one being semirigid, and we can then proceed
by induction on the length of $E$ in the finite length category $\mathcal{P}(\varphi)$.

Thus we can assume that all stable factors of $E$ are the same object $S$. If $E$ is
stable the claim is trivial. Otherwise there are non-identity maps $E \rightarrow E$ obtained
by factoring through a copy of $S$. In particular $\dim \text{Hom}_X(E, E) > 1$. Since $E$ is
semirigid it follows that $v(E)^2 < 0$. But $v(E)^2 = n^2 \cdot v(S)^2$ so also $v(S)^2 < 0$, and
it follows from Riemann-Roch and Serre duality that $S$ is rigid. \hfill \qed

**Wall-crossing.** Fix a stability condition $\sigma_0 = (Z_0, P_0) \in \text{Stab}(X)$ and a phase
$\phi \in \mathbb{R}$. Recall that the category $\mathcal{P}_0(\phi)$ of semistable objects of phase $\phi$ is a
finite length abelian category whose simple objects are the stable objects of phase $\phi$. Let
us now fix a Serre subcategory $\mathcal{A} \subset \mathcal{P}_0(\phi)$: this corresponds to choosing some subset
of the stable objects of phase $\phi$ and considering only those objects of $\mathcal{P}_0(\phi)$ whose
stable factors lie in this subset. Let

$$U = B_{\frac{1}{2}}(\sigma_0) \subset \text{Stab}(X)$$

be the ball of radius $\frac{1}{2}$ at $\sigma_0$ with respect to the standard metric on $\text{Stab}(X)$. Given
a stability condition $\sigma = (Z, \mathcal{P}) \in U$, the restriction of the central charge $Z$ defines
a (tilted) stability function on the abelian category $\mathcal{A}$. We then have two notions of
stability for objects $F \in \mathcal{A}$, namely stability with respect to the stability condition
$\sigma$, and stability with respect to the stability function $Z$. The following useful result
addresses the relationship between these two notions.

**Lemma 2.6.** For every $R > 0$ there is a neighbourhood $\sigma_0 \in U(R) \subset U$ with the
following property: if $\sigma = (Z, \mathcal{P}) \in U(R)$ and $F \in \mathcal{A}$ satisfies $|Z_0(F)| < R$, then $F$
is $\sigma$-semistable (resp. $\sigma$-stable) precisely if it is $Z$-semistable (resp. $Z$-stable).
Proof. We prove the result for semistability; the corresponding statement for stability follows in exactly the same way. First consider an arbitrary σ = (Z, P) ∈ U. If F ∈ A is Z-unstable, then there is a short exact sequence

\[ 0 \rightarrow A \rightarrow F \rightarrow B \rightarrow 0 \]

in A such that φ(A) > φ(F) > φ(B). Since σ ∈ U there is an inclusion A ⊂ P(φ − 1/2, φ + 1/2), and it follows easily that F is also σ-unstable (see [Bri07, Prop. 5.3]).

For the converse, take 0 < ϵ < 1/8 and consider triangles

\[ (A \rightarrow F \rightarrow B \rightarrow A)[1] \]

in D(X), all of whose objects lie in the subcategory P0(φ − 2ϵ, φ + 2ϵ), and such that F ∈ A satisfies |Z0(F)| < R. Semistability of F in σ0 ensures that for any such triangle there are inequalities

\[ \phi_0(A) \leq \phi_0(F) \leq \phi_0(B), \]

where φ0 denotes the phase function for the stability condition σ0. Moreover, it is easy to see that equality holds in (6) precisely if A, B ∈ A.

By the support property for σ0, the set of possible Mukai vectors v(A) and v(B) is finite. Thus we can choose U(R) small enough so that whenever the inequality (6) is strict, the same inequality of phases holds for all σ ∈ U(R). We can also assume that U(R) ⊂ Bϵ(σ0) is contained in the ϵ-ball centered at σ0.

Now suppose that F ∈ A satisfies |Z0(F)| < R and take a stability condition σ = (Z, P) ∈ U(R). Then F ∈ P(φ − ϵ, φ + ϵ). Suppose that F is not σ-semistable. Then we can find a triangle (5) with all objects lying in P(φ − ϵ, φ + ϵ), and such that φ(A) > φ(F) > φ(B). All objects of this triangle then lie in P0(φ − 2ϵ, φ + 2ϵ), so our assumption ensures that A, B ∈ A. It follows that F is not Z-semistable.

Remark 2.7. It follows immediately that for objects F ∈ A with |Z0(F)| < R and for stability conditions σ = (Z, P) ∈ U(R), the HN filtration of F with respect to σ coincides with the HN filtration of F in A with respect to the stability function Z (note that this latter filtration automatically exists because A ⊂ P(φ) is of finite length). A similar remark applies to Jordan-Hölder filtrations.

In studying wall-crossing behaviour, the following definition is often useful.

Definition 2.8. An object F ∈ D(X) is said to be quasistable in a stability condition σ if it is semistable, and all its stable factors have Mukai vectors lying on the same ray \( \mathbb{R}_{>0} \cdot v \subset N(X) \otimes \mathbb{R} \).

Note that if v(F) ∈ N(X) is primitive, then F is quasistable precisely if it is stable. The following result is a mild generalization of [Bri08, Prop. 9.4], and can be proved using the same argument given there. Instead we give an easy proof using Lemma 2.6.

Proposition 2.9. The set of points σ ∈ Stab(X) for which a given object F ∈ D(X) is stable (respectively quasistable) is open.
Proof. Let \( F \) be semistable in some stability condition \( \sigma_0 = (Z_0, P_0) \). Choose \( R > |Z_0(F)| \) and apply Lemma 2.6 with \( A \subset P(\phi) \) being the subcategory generated by the stable factors of \( F \). When \( F \) is quasistable all these stable factors have proportional Mukai vectors, so the stability functions on \( A \) induced by stability conditions in \( U(R) \) map \( K(A) \) onto a line in \( \mathbb{C} \). For such stability functions all objects of \( A \) are \( Z \)-semistable, and an object is \( Z \)-stable precisely if it is simple. The result therefore follows from Lemma 2.6. □

3. Walls and chambers

From now on, let \( X \) be a complex K3 surface of Picard rank 1. We shall also fix a (semi)rigid object \( E \in D(X) \) satisfying \( \text{Hom}_X(E, E) = \mathbb{C} \).

Eventually in Section 6 we will take \( E \) to be a skyscraper sheaf \( \mathcal{O}_x \).

Stable objects of the same phase. Let \( \sigma \in \text{Stab}(X) \) be a stability condition. We first gather some simple results about the relationship between (semi)rigid stable objects in \( \sigma \) of some fixed phase.

Lemma 3.1. Suppose that \( S_1, S_2 \) are non-isomorphic stable objects of the same phase, at least one of which is rigid. Then the Mukai vectors \( v(S_i) \) are linearly independent in \( N(X) \).

Proof. Since the \( S_i \) are stable of the same phase, \( \text{Hom}_X(S_i, S_j) = 0 \) for \( i \neq j \), and Serre duality and Riemann-Roch then show that \( (v_1, v_2) \geq 0 \). Suppose there is a non-trivial linear relation between the vectors \( v_i = v(S_i) \). Since the central charges \( Z(S_i) \) lie on the same ray it must take the form \( \lambda_1 v_1 = \lambda_2 v_2 \) with \( \lambda_1, \lambda_2 > 0 \). But then \( (v_i, v_i) \geq 0 \) which contradicts the assumption that one of the \( S_i \) is rigid. □

The Mukai vectors of rigid and semirigid objects are contained in the cone \( C = \{ v \in N(X) \otimes \mathbb{R} \setminus \{0\} : (v, v) \leq 0 \} \).

Since \( X \) has Picard number \( \rho(X) = 1 \), the lattice \( N(X) \) has signature \( (2, 1) \), and \( C \) is therefore a disjoint union of two connected components \( C^\pm \) exchanged by the inverse map \( v \mapsto -v \). By convention we take \( C^+ \) to be the component containing the class \( (0, 0, 1) \). The following elementary observation will be used frequently:

Lemma 3.2. Suppose \( \alpha, \beta \in C^+ \). Then \( (\alpha, \beta) \leq 0 \). Moreover
\[
(\alpha, \beta) = 0 \implies (\alpha, \alpha) = 0 = (\beta, \beta),
\]
in which case \( \alpha, \beta \) are proportional.

Proof. We can take co-ordinates \( (x, y, z) \) on \( N(X) \otimes \mathbb{R} \cong \mathbb{R}^3 \) so that the quadratic form associated to \( (-, -) \) is \( x^2 + y^2 - z^2 \). Then \( C \) is the set of nonzero vectors with \( x^2 + y^2 < z^2 \). This set has two connected components given by \( \pm z > 0 \). For signature reasons, if \( \alpha, \beta \in C \) satisfy \( (\alpha, \beta) = 0 \), then \( \alpha, \beta \) must be linearly dependent, and
so \((\alpha, \alpha) = 0 = (\beta, \beta)\). Suppose \(\alpha \in C^+\) satisfies \((\alpha, \alpha) < 0\). Since \((\alpha, \beta)\) varies continuously as \(\beta\) varies in \(C\), and is never zero, it follows that \((\alpha, \beta) \leq 0\) for all \(\beta\) in \(C^+\).

We note the following simple consequence:

**Lemma 3.3.** If there are three non-isomorphic stable (semi)rigid objects of the same phase, then at most one of them is rigid.

**Proof.** Denote the three objects by \(S_1, S_2, S_3\) and their Mukai vectors by \(v_i = v(S_i)\). As in the proof of Lemma 3.1 we have \((v_i, v_j) \geq 0\) for \(i \neq j\). We can assume that one of the objects, say \(S_1\), is rigid. Then by Lemma 3.2 we have \((v_1, v_i) > 0\) for \(i = 2, 3\) and we conclude that \(v_1\) lies in one component, say \(C^+\), and that \(v_2, v_3\) lie in the opposite component, \(C^-\). Suppose now that one of the objects \(S_2\) or \(S_3\) is also rigid. Then we can apply the same argument and conclude that \(v_2\) and \(v_3\) lie in opposite cones. This gives a contradiction. 

This leads to the following useful description of semistable (semi)rigid objects.

**Proposition 3.4.** Suppose that \(F\) is a semistable (semi)rigid object. Then exactly one of the following holds:

(a) \(F \cong S^\oplus k\) with \(S\) a stable spherical object and \(k \geq 1\);

(b) \(F\) is stable and semirigid;

(c) exactly 2 stable objects \(S_1, S_2\) occur as stable factors of \(F\), and their Mukai vectors \(v(S_i)\) are linearly independent in \(N(X)\).

**Proof.** Lemma 2.5 implies that \(F\) has at most one semirigid stable factor, the others being rigid. Therefore Lemma 3.3 shows that \(F\) has at most 2 stable factors. If there is just 1 then we are in cases (a) or (b) according to whether it is rigid or semirigid. If there are 2 then Lemma 3.1 shows that we are in case (c).

Note that in the situation of Prop. 3.4 the object \(F\) is quasistable in cases (a) and (b), but not in case (c).

**Codimension one walls.** Suppose that

\[\sigma_0 = (Z_0, P_0) \in \text{Stab}(X)\]

is a stability condition, and that \(F \in P_0(\phi)\) is a (semi)rigid semistable object which is not quasistable. Prop. 3.4 shows that \(F\) has exactly two stable factors \(S_1, S_2\) up to isomorphism, whose Mukai vectors \(v(S_i)\) are linearly independent. Lemma 2.6 shows that to understand stability of \(F\) near \(\sigma_0\) it is enough to consider stability functions on the abelian subcategory \(A \subset P_0(\phi)\) consisting of those objects all of whose stable factors are isomorphic to one of the \(S_i\). Note that the inclusion \(A \subset D(X)\) induces an injective group homomorphism \(\mathbb{Z}^2 \cong K(A) \rightarrow N(X)\), so we can identify \(K(A)\) with the sublattice of \(N(X)\) spanned by the \(v(S_i)\). For future reference we make the following observation:
Lemma 3.5. Suppose that $Z$ is a stability function on $A$ and let $\Theta \subset (0, 1]$ be the set of phases of $Z$-stable objects of $A$. Suppose that $F \in A$ is $Z$-stable and rigid. Then $\phi(F) \in \Theta$ is not an accumulation point.

Proof. We can assume that $\text{Im} \ Z(S_1)/Z(S_2) \neq 0$ since otherwise $\Theta$ consists of a single point. Then $Z$ induces an isomorphism of real vector spaces $K(A) \otimes \mathbb{R} \cong \mathbb{C}$, and we can think of $\Theta$ as a subset of

$$S^1 = (K(A) \otimes \mathbb{R} \setminus \{0\})/\mathbb{R}_{>0}.$$ 

Suppose we have stable objects $F_n$ whose Mukai vectors $v_n \in K(A)$ define points $[v_n] \in S^1$ which converge to the point $[v]$ defined by $v = v(F)$. This means that there are positive real numbers $\lambda_n$ such that $\lambda_n \cdot v_n \to v$. In particular, $\lambda_n^2 \cdot (v_n, v_n) \to (v, v)$. Since $F$ is rigid we have $(v, v) = -2$, so omitting finitely many terms of the sequence we can assume that $(v_n, v_n) < 0$ for all $n$. But since the objects $F_n$ are stable this implies that $(v_n, v_n) = -2$ for all $n$. This then implies that $\lambda_n$ is a convergent sequence, and hence that the sequence $v_n$ also converges. But since the vectors $v_n$ lie in the integral lattice this means that the sequence must be eventually constant. \hfill \Box

Let us now consider the abstract situation where $A$ is a finite length abelian category with two simple objects $S_1$, $S_2$, and $Z : K(A) \to \mathbb{C}$ is a stability function. We note the following trivial statement.

Lemma 3.6. Suppose that

$$\text{Im} \ Z(S_1)/Z(S_2) \neq 0.$$ 

Then any $Z$-semistable object $A$ is automatically $Z$-quasistable.

Proof. Since $Z$ induces an isomorphism of real vector spaces $K(A) \otimes \mathbb{R} \cong \mathbb{C}$, two objects have the same phase precisely if their classes lie on a ray in $K(A)$. \hfill \Box

Later on we shall need the following more difficult result.

Lemma 3.7. Suppose $F_1, F_2 \in A$ are $Z$-stable and satisfy

$$\text{Hom}_X(F_1, F_2) = 0 = \text{Hom}_X(F_2, F_1).$$ 

Let $\Theta \subset (0, 1]$ be the set of phases of stable objects of $A$, and assume that at least one of the phases $\phi(F_i)$ is not an accumulation point of $\Theta$. Then, possibly after reordering the $F_i$, we have $F_i \cong S_i$.

Proof. The pair $(Z, A)$ induces a stability condition $(Z, P)$ on the bounded derived category $D = D^b(A)$ in the usual way. Set $\phi_i = \phi(F_i) \in (0, 1]$, and reorder the objects $F_i$ so that $\phi_1 \leq \phi_2$ We treat first the case when $\phi_1$ is not an accumulation point of $\Theta$.

Consider the heart $C = P(\lfloor \phi_1, \phi_1 + 1 \rfloor)$ of $D$. Note that $F_1, F_2 \in C$, and $F_1$ is a simple object of $C$. The assumption that $\phi_1$ is not an accumulation point of $\Theta$ implies that $C = P(\lfloor \phi_1, \phi_1 + 1 - \epsilon \rfloor)$ for some $\epsilon > 0$. Thus, the central charges $Z(E)$
for $E \in \mathcal{C}$ are contained in a strictly convex sector of the complex plane. Also note that $K(\mathcal{C}) = K(\mathcal{A}) = \mathbb{Z}^{\oplus 2}$, and thus the set of central charges $Z(E)$ for $E \in \mathcal{C}$ is discrete in this sector. It follows that $\mathcal{C}$ is of finite length, and that there is exactly one other simple object in $\mathcal{C}$ up to isomorphism, say $T$. The effective cone in $K(\mathcal{C})$ is then generated by the classes of the simple objects $F_1$ and $T$, and it follows that $\phi(F_1) \leq \phi(F_2) \leq \phi(T)$.

The assumption $\text{Hom}_{\mathcal{A}}(F_1, F_2) = 0$ shows that $F_1$ is not a subobject of $F_2$ in $\mathcal{C}$. It follows that $T$ is a subobject of $F_2$, and in particular there is a nonzero map $T \to F_2$. But since $F_2$ is $\mathbb{Z}$-stable this is only possible if $F_2 = T$. Then $\mathcal{C} = \mathcal{P}((\phi_1, \phi_2))$ is a subcategory of $\mathcal{A}$. But since $\mathcal{A}$ and $\mathcal{C}$ are both hearts in $D$, this implies that $\mathcal{A} = \mathcal{C}$, and therefore $F_1$ and $F_2$ are the two simple objects of $\mathcal{A}$ up to isomorphism.

We shall repeatedly use the following result.

**Lemma 3.8.** Assume that $w_E(\sigma) > 0$. Then

(a) the objects $A_\pm$ are both either rigid or semirigid, and at most one of them is semirigid, and

(b) $\text{Hom}^i_X(A_-, A_+) = 0$ unless $i \in \{1, 2\}$;

if we assume in addition that $w_E(\sigma) \not\in \mathbb{Z}$, we also have

(c) $\text{Hom}^1_X(A_-, A_+) = 0$ unless $i = 1$, and

(d) $(v(A_+), v(A_-)) > 0$.

**Proof.** Applying Lemma [2.4] repeatedly to the HN filtration of $E$ gives (a). The objects $A_\pm$ lie in the heart $\mathcal{A} = \mathcal{P}((\phi_+, \phi_+))$ on $D(X)$ and hence, using Serre duality, satisfy

$$\text{Hom}^i_X(A_\pm, A_\pm) = 0 \text{ unless } i \in \{0, 1, 2\}.$$
For (b) we must show that \( \text{Hom}_\mathcal{X}(A_-, A_+) = 0 \). Note that taking cohomology with respect to the above heart \( \mathcal{A} \) we have \( H^i_\mathcal{A}(E) = 0 \) unless \( 0 \leq i \leq n \). Moreover, there is an epimorphism \( H^n_\mathcal{A}(E) \to A_- \) and a monomorphism \( A_+ \to H^0_\mathcal{A}(E) \). Suppose there is a nonzero map \( f : A_- \to A_+ \). Using the spectral sequence
\[
E_2^{pq} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{X}(H^i_\mathcal{A}(E), H^{i+q}_\mathcal{A}(E)) \implies \text{Hom}_\mathcal{X}^{p+q}(E, E)
\]
it follows that there is a nonzero map \( E \to E[-n] \) which if \( n > 0 \) contradicts the fact that \( E \) is semirigid. In the case \( n = 0 \) we have that \( E \in \mathcal{A} \) and an epimorphism \( g : E \to A_- \) and a monomorphism \( h : A_+ \to E \). Then \( h \circ f \circ g \) is a nonzero map \( E \to E \) which by assumption must be a multiple of the identity. It follows that \( E \cong A_+ \cong A_- \) which contradicts the assumption that \( w_E(\sigma) > 0 \).

For (c), note first that since \( \phi(A_+) > \phi(A_-) \), and since the objects \( A_\pm \) are semistable, there are no nonzero maps \( A_+ \to A_- \); then apply Serre duality. The inequality of part (d) then follows by Riemann-Roch. Equality is impossible, by Lemma 3.2, since at least one of \( A_\pm \) is rigid.

\( \pm \)-walls. We say that a stability condition \( \sigma \in \text{Stab}(\mathcal{X}) \) is \((\pm)\)-generic if \( E_\pm \) is quasistable, and similarly \((\mp)\)-generic if \( E_\mp \) is quasistable.

**Lemma 3.9.** The subset of \((\pm)\)-generic stability conditions is the complement of a real, closed submanifold \( W_\pm \subset \text{Stab}(\mathcal{X}) \) of codimension 1. The object \( E_\pm \) is locally constant on \( W_\pm \), as well as on the complement of \( W_\pm \). Similarly for \((\mp)\)-generic stability conditions and \( E_\mp \).

**Proof.** The first claim is that being \((\pm)\)-generic is an open condition. Indeed, the first step in the HN filtration of \( E \) in a stability condition \( \sigma_0 = (Z_0, \mathcal{P}_0) \) is a triangle
\[
E_+ \to E \to F
\]
with \( E_+ \in \mathcal{P}_0(\phi_+) \) and \( F \in \mathcal{P}_0(< \phi_+) \). If \( E_+ \) is moreover quasistable Prop. 2.9 shows that \( E_+ \) remains semistable in a neighbourhood of \( \sigma_0 \). It follows that \( 9 \) remains the first step in the HN filtration of \( E \), so that the object \( E_+ \) is locally constant, and the claim follows then from Prop. 2.9.

Suppose now that \( \sigma_0 \in \text{Stab}(\mathcal{X}) \) is not \((\pm)\)-generic. By Prop. 3.4 there are then exactly two stable factors \( S_1, S_2 \) of \( E_+ \), whose Mukai vectors \( v(S_i) \) are linearly independent. These objects generate a Serre subcategory \( \mathcal{A} \subset \mathcal{P}_0(\phi_+) \) as in Section 3 Lemma 2.6 and the argument above show that for stability conditions \( \sigma = (Z, \mathcal{P}) \) in a neighbourhood of \( \sigma_0 \), the maximal HN factor of \( E \) in \( \sigma \) is precisely the maximal HN factor of the object \( E_+ \in \mathcal{A} \) with respect to the stability function \( Z \). The locus of non \((\pm)\)-generic stability conditions is therefore the set of points satisfying
\[
\text{Im} Z(S_1) / Z(S_2) = 0.
\]
Indeed, when this condition is satisfied, \( E_+ \) itself is semistable but not quasistable. On the other hand, when \( 10 \) does not hold, Lemma 3.6 shows that the maximal HN factor is automatically quasistable. \( \square \)
We refer to the connected components of the submanifold $W_+$ as (+)-walls. Similarly for (−)-walls. A connected component of the complement

\[(11) \quad \text{Stab}(X) \setminus (W_+ \cup W_-)\]

will be called a chamber. Lemma 3.9 shows that both objects $E_\pm$ are constant on every chamber. In fact, it is also smooth on the closure of each chamber: as $\sigma$ reaches a (−)-wall, the object $E_+$ may change, but the phases $\phi(E_+)$ of the two objects agree on the wall; thus $\phi(E_+)$ extends to a smooth function on the closure of the chamber.

**Remarks 3.10.**

(a) The union of the submanifolds $W_+$ and $W_-$ need not be a submanifold since (+) and (−) walls can intersect each other.

(b) The object $A_-$ need not be locally-constant on the complement (11) since its definition involves the function $\lfloor w_E \rfloor$ which is discontinuous at points of $w^{-1}(Z)$.

**Integral walls.** Consider now the subset

\[W_Z = \{ \sigma \in \text{Stab}(X) : w_\sigma(E) \in Z \text{ and } E \text{ is not } \sigma\text{-quasistable} \} \]

We call the connected components of $W_Z$ integral walls.

**Remark 3.11.** The condition that $E$ is not $\sigma$-quasistable is essential for the following result to hold. For example, when $E = \mathcal{O}_x$ is a skyscraper sheaf there is an open region $U$ where $E$ is stable (this coincides with the subset of geometric stability conditions, see Lemma 6.7): we do not want $W_Z$ to contain the closure of this subset, but rather its boundary (see also Prop. 3.15 below).

**Lemma 3.12.** The subset $W_Z$ is a real, closed submanifold of $\text{Stab}(X)$ of codimension 1.

**Proof.** Since $w$ is continuous, the subset $w^{-1}(Z)$ is closed. The fact that $W_Z$ is closed then follows from Prop. 2.9. If $\sigma \in \text{Stab}(X)$ satisfies $w_E(\sigma) = 0$ then $\sigma$ lies on $W_Z$ precisely if $\sigma$ is not (±)-generic, so for these points the result follows from Lemma 3.9. Thus we can work in a neighbourhood of a point $\sigma_0 \in W_Z$ for which $w_E(\sigma_0) > 0$.

Consider the stable factors $S_i$ of $A_\pm$. By Lemma 2.5 they are all (semi)rigid, and at most one is semirigid. Lemma 3.3 shows that there at most two of them. But if there is only one then there is a nonzero map $A_- \to A_+$ contradicting Lemma 3.8 (b). Hence the objects $A_\pm$ have exactly 2 stable factors $S_1, S_2$ between them, and by Lemma 3.1 the Mukai vectors $v(S_i)$ are linearly independent.

We claim that in a neighbourhood of $\sigma_0$ the closed subset $W_Z$ is cut out by the equation

\[(12) \quad \text{Im} Z(S_1)/Z(S_2) = 0.\]

Certainly, if this condition is satisfied, the stable factors of $A_\pm$ remain stable (by Prop. 2.9) and of equal phases; hence the objects $E_\pm$ remain semistable, and continue to be the extremal HN factors of $E$. Thus $w_E(\sigma)$ remains integral.
For the converse, we apply Lemma 2.6 to conclude that for any $\sigma = (Z, P)$ in a neighbourhood of $\sigma_0$, the extremal HN factors of $E$ (up to shift) are just given by the extremal HN factors of the objects $A_\pm \in \mathcal{A}$ with respect to the stability function $Z$. Call these objects $C_\pm$ and suppose that $\phi(C_+) = \phi(C_-)$. By Lemma 3.1 the Mukai vectors of the distinct stable factors of $\phi(C_\pm)$ are linearly independent in $\mathcal{N}(X)$. There must be more than one of them by Lemma 3.8(b). Since $Z$ maps these different stable factors onto a ray, condition (12) must hold.

Remarks 3.13. (a) It follows from the local descriptions given in Lemmas 3.9 and 3.12 that if a $(\pm)$-wall $W_1$ intersects an integral wall $W_2$ then in fact $W_1 = W_2$ is simultaneously a $(+)$-wall and an integral wall.

(b) It is easy to check that if a $(+)$-wall and a $(\mp)$-wall coincide then this wall is also an integral wall.

(c) The statements of Lemmas 3.9 and 3.12 continue to hold if we replace the manifold $\text{Stab}(\mathcal{X})$ by the complex submanifold $\text{Stab}^*_{\text{red}}(\mathcal{X})$ of reduced stability conditions. The point is that the walls, which are cut-out by equations of the form $(\Omega, v_1)/(\Omega, v_2) \in \mathbb{R}$, intersect the complex submanifold given by $(\Omega, \Omega) = 0$ transversely.

No local minima. Using the action of the universal cover of $\text{GL}^+_2(\mathbb{R})$ on $\text{Stab}(\mathcal{X})$, it is easy to see that if the width function $w_E$ had a local minimum $\sigma_0 \in \text{Stab}(\mathcal{X})$, then this would have to satisfy $w_E(\sigma_0) \in \mathbb{Z}$. The following crucial result then shows that in fact the function $w_E$ has no positive local minima on $\text{Stab}(\mathcal{X})$.

Proposition 3.14. Let $\sigma_0 \in W_{\mathbb{Z}} \subset \text{Stab}(\mathcal{X})$ satisfy $w_E(\sigma_0) = n \in \mathbb{Z}_{>0}$. Locally near $\sigma_0$ the submanifold $W_{\mathbb{Z}}$ splits $\text{Stab}(\mathcal{X})$ into two connected components, with $w_E(\sigma) < n$ in one component and $w_E(\sigma) > n$ in the other.

Proof. As in the proof of Lemma 3.12, the objects $A_\pm$ have two stable factors $S_1, S_2$ between them; we again consider the finite length abelian subcategory $\mathcal{A} \subset \mathcal{P}_0(\phi)$ generated by $S_1$ and $S_2$. Considering the Jordan-Hölder filtration of $A_+$ and relabelling the objects $S_i$ if necessary, we can assume that there is a monomorphism $S_1 \rightarrow A_+$ in $\mathcal{A}$. It follows from Lemma 3.8(b) that $\text{Hom}_\mathcal{X}(A_-, S_1) = 0$, and hence there must be an epimorphism $A_- \rightarrow S_2$ in $\mathcal{A}$. Using Lemma 3.8(b) again, this in turn implies that $\text{Hom}_\mathcal{X}(S_2, A_+) = 0$.

Locally near $\sigma_0$ the submanifold $W_{\mathbb{Z}}$ is cut out by the equation $Z(S_1)/Z(S_2) \in \mathbb{R}_{>0}$. Note that there are no other relevant walls near $\sigma_0$, since any $(\pm)$-wall which intersects $W_{\mathbb{Z}}$ coincides with a component of $W_{\mathbb{Z}}$. Hence it makes sense to speak of the new objects $A_\pm$ for stability conditions $\sigma = (Z, \mathcal{P})$ on either side of the wall. By Lemma 2.6 the new $A_+$ is the maximal HN factor of $A_+$ with respect to the slope function $Z$, and, up to shift, the new $A_-$ is the minimal HN factor of $A_-$ with respect to $Z$.

On one side of the wall $\phi(S_1) > \phi(S_2)$. Then the new $A_+$ is in the subcategory generated by $S_1$, and the new $A_-$ is in the subcategory generated by $S_2$. The
width \( w(E) \) has increased to \( n + \phi(S_1) - \phi(S_2) \). On the other side of the wall \( \phi(S_1) < \phi(S_2) \). Write \( C_\pm \) for the new objects \( A_\pm \). Suppose that the width has also increased on this side. Then \( n = \lfloor w(E) \rfloor \) is constant near \( \sigma_0 \), and so \( C_- \) is precisely the minimal HN factor of \( A_- \) (rather than its shift).

By Lemma 3.6, the object \( C_+ \) is quasistable. Prop. 3.14 then shows that it has a single stable factor, call it \( T_+ \). Similarly, \( C_- \) has a single stable factor \( T_- \). By Lemma 3.8 we have \( \text{Hom}_X(C_+, C_-) = 0 \) unless \( k = 1 \), and it follows that

\[
\text{Hom}_A(T_-, T_+) = 0 = \text{Hom}_A(T_+, T_-).
\]

Applying Lemma 3.5 and Lemma 3.7 we conclude that \( \{T_-, T_+\} = \{S_1, S_2\} \). The assumption that the width has increased implies that in fact \( T_- = S_1 \) and \( T_+ = S_2 \). We thus get a chain of inclusions \( T_+ = S_2 \subset C_+ \subset A_+ \) in \( A \), in contradiction to \( \text{Hom}_X(S_2, A_+) = 0 \) observed above. \( \square \)

A similar result holds at points of \( W_\sigma \) of width zero.

**Proposition 3.15.** Let \( \sigma_0 \in W_\sigma \subset \text{Stab}(X) \) satisfy \( w(E) = 0 \). Locally near \( \sigma_0 \) the submanifold \( W_\sigma \) splits \( \text{Stab}(X) \) into two connected components. In one \( E \) is quasistable, and in the other \( w(E) > 0 \).

**Proof.** The proof is very similar to that of Prop. 3.14, and we just indicate the necessary modifications. In the first paragraph, note that \( A_+ = A_- = E \). Considering the Jordan-Hölder filtration of \( E \) we can assume that there is a monomorphism \( S_1 \to E \) as before. By the assumption that \( \sigma_0 \in W_\sigma \), this is not an isomorphism, as \( E \) cannot be stable. If there were a nonzero map \( E \to S_1 \) this would contradict the assumption that \( \text{Hom}_X(E, E) = \mathbb{C} \). It follows that there is an epimorphism \( E \to S_2 \) in \( A \). The rest of the proof of Prop. 3.14 applies without change and shows that on one side of the wall \( w(E) = 0 \) (i.e. \( E \) is semistable), and on the other \( w(E) > 0 \). Finally, note that on the first side we must in fact have \( E \) quasistable, since if it is not, \( \sigma \) lies on \( W_\sigma \). \( \square \)

4. Flow

Let us fix a complex K3 surface \( X \) and an object \( E \in D(X) \) with assumptions as in Section 3. In particular, the surface \( X \) has Picard rank \( \rho(X) = 1 \) and the object \( E \) is (semi)rigid and satisfies \( \text{Hom}_X(E, E) = \mathbb{C} \). The aim of this section is to construct a flow on the space \( \text{Stab}^\ast_{\text{red}}(X) \) that decreases the width \( w(E) \) to the nearest integer \( \lfloor w(E) \rfloor \).

**Construction.** Let \( \sigma = (Z, \mathcal{P}) \in \text{Stab}^\ast_{\text{red}}(X) \) be a reduced stability condition, and write \( Z(E) = (\Omega, v(E)) \) for some vector \( \Omega \in \mathcal{N}(X) \oplus \mathbb{C} \). Recall that \( \Omega \in \mathcal{Q}_0^+(X) \), so in particular

\[
(\Omega, \Omega) = 0, \quad (\Omega, \bar{\Omega}) = 2d > 0,
\]

and the orthogonal to the 2-plane in \( \mathcal{N}(X) \oplus \mathbb{R} \) spanned by the real and imaginary parts of \( \Omega \) is a negative definite line. Let \( \Theta = \Theta(\sigma) \in \mathcal{N}(X) \oplus \mathbb{R} \) be the unique
vector satisfying

\[(\Theta, \Omega) = 0, \quad -(\Theta, \Theta) = d = \frac{1}{2}(\Omega, \bar{\Omega}), \quad \Theta \in \mathbb{C}^+ .\]

For any stability condition \(\sigma\), we define a sign \(\epsilon = \epsilon(\sigma) \in \{\pm 1\}\) by the condition that \(v(E_+) = v(A_+)\) lies in \(\mathcal{C}(\sigma)\).

Lemma 4.1. *The sign \(\epsilon(\sigma)\) is locally constant on the complement of \(W_Z\).*

Proof. By Lemma 3.9, the object \(A_+\) is locally constant along any \((+)-\)wall, and on the complement of the set of \((+)-\)walls. By definition, it follows that \(\epsilon\) is locally constant along both types of strata.

Similarly, \(E_-\) is locally constant on \((-)-\)walls and their complement; if we restrict further to the complement of \(W_Z\), the same holds for \(A_-\). But due to Lemma 3.8 (d) and Lemma 3.2, the objects \(A_\pm\) lie in different connected components of the cone \(\mathcal{C}\); thus the object \(A_-\) also determines the sign \(\epsilon(\sigma)\), which is therefore also locally constant along these strata.

Now recall from Remark 3.13 (b) that a \((+)-\)wall can only coincide with a \((-)-\)wall if they are contained in \(W_Z\). This implies that \(\epsilon(\sigma)\) is locally constant on \(\text{Stab}(X) \setminus W_Z\). \(\square\)

Define a complex number of unit modulus by

\[(\zeta(\sigma) = i \cdot \exp \left(\frac{i\pi}{2}(\phi(A_+) + \phi(A_-))\right).\]

Finally, define a nonzero vector

\[(v(\sigma) = \epsilon(\sigma) \cdot \zeta(\sigma) \cdot \Theta(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}.\]

Lemma 4.2. *The flow* 

\[\frac{d}{dt} \Omega = v(\sigma)\]

*of the vector field \(v(\sigma)\) exists locally uniquely on \(\text{Stab}^{\ast}_{\text{red}}(X) \setminus \text{w}_{E}^{-1}(\mathbb{Z})\), the space of reduced stability conditions of non-integral width. It preserves the positive real number \(2d = (\Omega, \bar{\Omega})\).*

Proof. The vector \(v(\sigma)\) varies continuously on \(\text{Stab}^{\ast}_{\text{red}}(X)\) by the above Lemma. Since the set of \((\pm)-\)walls is locally finite, the resulting vector field is Lipschitz continuous on every compact subset; by the Picard-Lindelöf Theorem, the flow then exists locally and is unique.

From \((\Omega, \Theta) = 0 = (\bar{\Omega}, \Theta)\) one obtains \(\frac{d}{dt}(\Omega, \Omega) = \frac{d}{dt}(\Omega, \bar{\Omega}) = 0\); thus the condition of being reduced is preserved, and \((\Omega, \bar{\Omega})\) is constant. \(\square\)
Flow decreases width. Simple sign observations show that the flow defined in the last subsection moves $Z(A_{\pm})$ in the direction $\mp \zeta(\sigma)$ and hence decreases the width, see Figure 2. To make this observation precise, we first point out that since $w_E$ is smooth on the closure of each chamber, the function $w_E(\sigma(t))$ will be piecewise differentiable. Thus we can define $\frac{dw_E}{dt}(\sigma(t))$ at time $t$ to be the derivative of $w_E$ restricted to the interval $[t, t + \epsilon)$.

**Lemma 4.3.** Under the flow of $v(\sigma)$, the functions $\phi(A^+)$ and $\phi(A^-)$ are decreasing and increasing, respectively, and the derivative $\frac{d}{dt} w_E(\sigma(t))$ is negative. Moreover, setting $\theta = w_E(\sigma) - \lfloor w_E(\sigma) \rfloor$, one has

$$-\frac{d}{dt} w_E(\sigma) \geq \frac{2}{\pi} \cos \left( \frac{\pi \theta}{2} \right) > 0.$$  

**Proof.** Note that

$$\epsilon(\sigma) \cdot (\Theta, A_+) < 0, \quad \epsilon(\sigma) \cdot (\Theta, A_-) > 0.$$  

Indeed, since $\Theta \in C^+$ we have $(\Theta, A_+) \in \epsilon(\sigma) \cdot \mathbb{R}_{<0}$, and the objects $A_{\pm}$ have classes in opposite cones by Lemma 3.8. Thus the flow has the effect of adding negative multiples of the vector $\zeta$ to $Z(A_+)$ and positive multiples to $Z(A_-)$. It is then clear that this decreases $\phi(A_+)$ and increases $\phi(A_-)$.

Writing $\Omega = X + iY$ and $\Theta = W$, the vectors $(W, X, Y)$ form an orthogonal basis for $N(X) \otimes \mathbb{R}$ such that

$$(X, X) = (Y, Y) = -(W, W) = d > 0.$$  

It follows that for any vector $v \in N(X) \otimes \mathbb{R}$ one has

$$|(\Omega, v)|^2 - (\Theta, v)^2 = d(v, v).$$  

In particular, if $v = v(E)$ is the Mukai vector of a (semi)rigid object, then $(v, v) \leq 0$ gives

$$(16) \quad |(\Theta, v)| \geq |Z(E)|.$$  

We will prove the inequality at time $t = 0$. Consider a stability condition $\sigma$ with width $w$. Set $[w] = n$ and put $\theta = w - [w]$. Rotating by a fixed scalar $z \in \mathbb{C}$,
we can assume that at time $t = 0$, we have $\zeta = -1$ and $\phi(A_\pm) = (1 \pm \theta)/2$. Set $Z(A_\pm) = x_\pm + iy_\pm$. As we flow, there are angles $0 \leq \alpha_\pm < \pi/2$ such that

$$
\frac{x_\pm}{y_\pm} = \tan(\alpha_\pm), \quad \frac{y_\pm}{|Z(A_\pm)|} = \cos(\alpha_\pm),
$$

see also Figure 3. At time $t = 0$, we have $\alpha_\pm = \pi \theta/2$.

Since $\zeta(0)$ is real, the derivatives $\frac{d}{dt} y_\pm$ vanish at $t = 0$, and so

$$
\mp \sec^2(\alpha_\pm) \left. \frac{d\alpha_\pm}{dt} \right|_{t=0} = \frac{1}{\cos(\alpha_\pm) \cdot |Z(A_\pm)|} \left. \frac{dx_\pm}{dt} \right|_{t=0}.
$$

But $\frac{d}{dt} x_\pm|_{t=0} = \pm |(\Theta, A_\pm)|$, so by the above inequality (16) we get

$$
0 \geq -\cos \left( \frac{\pi \theta}{2} \right) \geq \left. \frac{d\alpha_\pm}{dt} \right|_{t=0}.
$$

Writing $\alpha_+ + \alpha_- = \pi(w - n)$ gives the result. \hfill \Box

In particular, unless the flow ceases to exist at an earlier point in time, it takes a point with non-integral width $w$ to a point of width $\lfloor w \rfloor$ in finite time less than $\frac{\pi}{2} \left( \cos \left( \frac{1}{2} \pi \theta \right) \right)^{-1}$.

**Global properties.** We now study the flow $\sigma(t)$ defined above in more detail. Let $\sigma \in \text{Stab}_\text{red}^*(X)$ be a stability condition with $w_E(\sigma) \in (n, n+1)$, and let $I \subset [0, \infty)$ be the maximal interval of definition of the flow in $w_E^{-1}(n, n+1)$ starting at $\sigma$. By Lemma 4.3, this interval is necessarily finite. Moreover $I$ must be of the form $I = [0, t_0)$ since the flow can always be extended in the neighbourhood of any given stability condition. Thus we have a flow

$$
(17) \quad \sigma: [0, t_0) \rightarrow \text{Stab}_\text{red}^*(X) \setminus w^{-1}(\mathbb{Z}).
$$

Let $\Omega(t) \in \mathcal{Q}^+(X)$ be the underlying flow of central charges, and let us shorten notation by writing $\Theta(t) := \Theta(\sigma(t)) \in \mathcal{C}^+$ and $\zeta(t) = \zeta(\sigma(t)) \in \mathbb{C}$ for the quantities defined above in (13) and (14); in addition, recall from Lemma 4.1 that $\epsilon = \epsilon(\sigma(t))$ is constant along the flow line.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Computing the derivative $v(\sigma).w_E$}
\end{figure}
Lemma 4.4. The vector $\Theta(t)$ satisfies $(\Theta(t), \Theta(t)) = -d$ for all $t$, and obeys the differential equation

$$\frac{d\Theta}{dt}(t) = \epsilon \text{Re } \zeta(t) \text{Re } \Omega(t) + \epsilon \text{Im } \zeta(t) \text{Im } \Omega(t).$$

Proof. The fact that $d$ is constant under the flow was already proved in Lemma 4.2. Write $\Psi(t)$ for the right-hand side of the above equation. It is sufficient to show that $\frac{d\Theta}{dt}(t)$ and $\Psi(t)$ have the same pairing with each vector in the orthogonal basis $\Theta(t), \text{Re } \Omega(t)$ and $\text{Im } \Omega(t)$ of $\text{NS}(X) \otimes \mathbb{R}$. This follows from

$$\left( \frac{d\Theta}{dt}(t), \Theta(t) \right) = \frac{1}{2} \frac{d}{dt} (\Theta(t), \Theta(t)) = 0 = \left( \Psi(t), \Theta(t) \right)$$

and

$$0 = \frac{d}{dt} (\Theta(t), \Omega(t)) = \left( \frac{d\Theta}{dt}(t), \Omega(t) \right) + \epsilon \zeta (\Theta(t), \Theta(t))$$

$$= \left( \frac{d\Theta}{dt}(t), \Omega(t) \right) - \epsilon \zeta d = \left( \frac{d\Theta}{dt}(t), \Omega(t) \right) - \left( \Psi(t), \Omega(t) \right).$$

Note that $(\xi, \xi) = -d$ defines the Minkowski model of the hyperbolic plane as a subset $\mathcal{H} \subset \mathbb{C}^+$. Up to rescaling by $\frac{1}{\sqrt{d}}$, the standard invariant metric on $\mathcal{H}$ is induced by the quadratic form on $\mathcal{N}(X) \otimes \mathbb{R}$. In particular, the vectors $\text{Re } \Omega, \text{Im } \Omega$ form an orthonormal basis of the tangent space to $\mathcal{H}$ at $\Theta$. Since $|\zeta| = 1$, the vector $\Theta(t)$ is moving in $\mathcal{H}$ with constant speed. Since $\mathcal{H}$ is complete, the limit $\lim_{t \to t_0} \Theta(t)$ exists, and $\Theta$ extends to a continuous function on the closed interval $[0, t_0]$.

It follows that $\Omega(t)$ also extends to a continuous function on $[0, t_0]$, as it is the integral of the continuous function $\epsilon \cdot \zeta(t) \cdot \Theta(t)$. Since $(\Omega(t), \overline{\Omega(t)})$ is constant, we also have $\Omega_0 := \Omega(t_0) \in \mathcal{Q}^+(X)$. If $\Omega_0$ lies in the subset $\mathcal{Q}_0^+(X) \subset \mathcal{Q}^+(X)$, then by Theorem 1.1, the path $\Omega(t)$ lifts to a continuous path

$$\sigma: [0, t_0] \to \text{Stab}_{\text{red}}^*(X).$$

By the maximality of the interval $[0, t_0]$ it follows that $w_E(\sigma(t_0)) = n$, as desired. The only other possibility is $(\Omega_0, \delta) = 0$ for some root $\delta \in \Delta(X)$; we prove that this cannot happen in the next section.

Later on we shall need the following simple statement about the behaviour of the flow near an integral wall:

Lemma 4.5. The vector field $v(\sigma)$ on the open set $U = w_E^{-1}(n, n+1)$ extends continuously to the closure $\overline{U}$ of $U$ in $w_E^{-1}([n, n+1])$, and is transversal to the boundary wall where $w_E = n$.

Proof. While $A_+$ and $A_-$ may change as $w_E(\sigma)$ approaches $n$ from above, their phases extend continuously (becoming equal at the wall $w_E(\sigma) = n$). Therefore,
locally we can extend equations (14) and (15) continuously to $\mathcal{U}$. Locally, the equation of the wall is given by $\text{Im} Z(A_+/Z(A_-) = 0$. Since $\zeta$ becomes orthogonal to the ray spanned by $Z(A_\pm)$ at the wall, it follows from the same considerations as used in the proof of Lemma 4.3 that the derivative of the equation with respect to $v(\sigma)$ does not vanish. In other words, the vector field is transversal to the wall. □

We will also use the following consequence:

**Lemma 4.6.** With the same notation as in the previous lemma, let $W \subset w^{-1}_E(n)$ be an integral wall bordering $\mathcal{U}$, and let $\sigma \in W$. Then there is a neighborhood $V \subset \mathcal{U}$ of $\sigma$ with a homeomorphism $V \to (V \cap W) \times (0, \epsilon)$ that identifies the flow on $V$ with the flow on the right-hand side induced by $-\frac{d}{dt}$ on $[0, \epsilon)$.

**Proof.** We choose a neighborhood $V' \subset W$ of $\sigma$ small enough such that the inverse flow, associated to $-v$, exists until time $\epsilon$ for all $\sigma' \in V'$. Since $v$ is transversal to the wall, and since the flow is locally unique, the flow induces an injective local homeomorphism $V' \times (0, \epsilon) \hookrightarrow \mathcal{U}$. The result follows by taking $V$ to be the image of this map. □

5. **Avoiding holes**

We continue to assume that $X$ is a K3 surface of Picard rank $\rho(X) = 1$, and that $E$ is a (semi)rigid object with $\text{Hom}(E, E) = \mathbb{C}$. In this section we will prove that the flow constructed in the previous section cannot fall down any of the holes $Q^+(X) \setminus Q^+_0(X)$. More precisely, suppose, as before, that the vector field $v(\sigma)$ gives rise to a flow (17) defined on an interval $[0, t_0)$. As we proved in the last section, the underlying flow of central charges $\Omega(t)$ extends to a flow

$$\Omega(t) : [0, t_0) \to Q^+(X).$$

In this section we prove that in fact $\Omega_0 = \Omega(t_0) \in Q^+_0(X)$.

**Lemma 5.1.** There exists $0 < t_1 < t_0$ with the following property: for all $t \in [t_1, t_0)$, the vector $\zeta(t)$ lies in the interior of the convex cone spanned by $Z(A_+)$ and $-Z(A_-)$ at time $t = t_1$.

**Proof.** We write $\tilde{\phi}^\pm(t)$ for the phases $\phi(A_\pm)$ as a function of $t$, and recall that $\tilde{\phi}^+ < \tilde{\phi}^- + 1$. We have to show that $t_1$ can be chosen so that for all $t \in [T_1, t_0)$ an appropriate branch of

$$\psi(t) := \frac{1}{\pi} \arg \zeta(t)$$

lies strictly between $\tilde{\phi}^+(t_1)$ and $\tilde{\phi}^-(t_1) + 1$, see also Figure 4. By Lemma 4.3 the functions $\tilde{\phi}^+(t)$ and $\tilde{\phi}^-(t)$ are bounded monotone decreasing and increasing,
Figure 4. Restraining $\zeta(t)$ as $t$ approaches a boundary point

respectively, and thus extend to continuous function on $[0, t_0]$. By definition of $\zeta(t)$, we have

$$\tilde{\phi}^+(t) < \psi(t) = \frac{1}{2} (\tilde{\phi}^+(t) + \tilde{\phi}^-(t) + 1) < \tilde{\phi}^-(t) + 1$$

for all $0 \leq t \leq t_0$. The claim then follows by continuity.

Assume for a contradiction that $(\Omega_0, \delta) = 0$ for some root $\delta \in \Delta(X)$. Write $Z_0$ for the central charge $Z(-) = (\Omega_0, -)$. By definition of $\zeta(t)$, we have

$$\tilde{\phi}^+(t) < \psi(t) = \frac{1}{2} (\tilde{\phi}^+(t) + \tilde{\phi}^-(t) + 1) < \tilde{\phi}^-(t) + 1$$

for all $0 \leq t \leq t_0$. The claim then follows by continuity.

Lemma 5.2. Consider a sufficiently small neighbourhood $V$ of $Z_0$ in $\mathcal{P}^+(X)$, and let $U \subset \text{Stab}_{\text{red}}(X)$ be a connected component of its inverse image. Then there is an object $S$ of class $v(S) = \delta$ that is stable for all stability conditions in $U$.

Proof. For a generic stability condition, there is a unique stable object of class $\delta$, up to shift. Therefore, it is enough to choose $V$ such that its preimage does not intersect any walls for the class $\delta$.

Any stable factor of $S$ would have to be rigid, and hence have Mukai vector a spherical class $\delta'$. But a simple geometric argument in $\mathcal{N}(X)$ shows that for $Z$ sufficiently close to $Z_0$, we have $|Z(\delta)| < |Z(\delta')|$ for every class $\delta' \in \Delta(X)$, a contradiction.

For general reasons (see [Bri08, Prop. 9.3]), any compact subset of $\text{Stab}(X)$ meets only finitely many walls for our fixed object $E$. Now, however, we need a similar result for certain non-compact subsets of $\text{Stab}(X)$ whose images in $\text{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C})$ contain $Z_0$ in their closure.

Lemma 5.3. Let $[\phi_1, \phi_2]$ be an interval of length less than 2, and $\| \cdot \|$ some norm on $\mathcal{N}(X) \otimes \mathbb{R}$. Then there exists $\epsilon > 0$ with the following property: if $V_{e, \phi_1, \phi_2} \subset \mathcal{P}^+_0(X)$ is the subset of central charges $Z$ satisfying

$$\|Z - Z_0\| \leq \epsilon \quad \text{and} \quad Z(\delta) \in \mathbb{R}_{>0} \cdot e^{i\pi \phi} \text{ for some } \phi \in [\phi_1, \phi_2],$$

and if $U_{e, \phi_1, \phi_2} \subset \text{Stab}^+(X)$ is any connected component of the preimage of $V_{e, \phi_1, \phi_2}$, then there are only finitely many ($\pm$) or integral walls intersecting $U_{e, \phi_1, \phi_2}$.

Here $\| \cdot \|$ also denotes the induced operator norm on $\text{Hom}_{\mathbb{Z}}(\mathcal{N}(X), \mathbb{C})$. 

Proof. From the proof of the support property for $\text{Stab}(X)$, \cite[Lemma 8.1]{Bri08}, we can deduce that there exists a constant $K$ such that

\begin{equation}
|Z(v)| \geq K \|v\|
\end{equation}

for all $v \in \mathcal{N}(X)$ with $(v, v) \geq -2$ with $v \neq \pm \delta$, and all $Z \in V_{\epsilon, \phi_1, \phi_2}$.

Given a stability condition $\sigma$, recall that the mass of our object $E$ with respect to $\sigma$ is defined by

\[ m_E(\sigma) := \sum_{i} |Z(A_i)|. \]

This is a continuous function on $\text{Stab}(X)$ (\cite[Proposition 8.1]{Bri07}).

Claim: The mass $m_E(\sigma)$ is bounded on $U_{\epsilon, \phi_1, \phi_2}$.

To prove the claim, first note that by compactness, it evidently holds on the intersection $U_N$ of $U_{\epsilon, \phi_1, \phi_2}$ with the set defined by $|Z(\delta)| \geq \frac{1}{N}$ for any $N > 0$. We choose $N$ large enough such that $U_N$ contains central charges $Z$ such that $|Z(\delta)|$ attains any possible phase $\phi \in [\phi_1, \phi_2]$.

Let $\sigma_i = (\mathcal{P}_i, Z_i) \in U_N$ be two stability conditions that are contained in the same chamber for $E$, and that satisfy $|Z_2(\delta)| \leq |Z_1(\delta)|$. Here, by chamber, we mean a connected component of the complement of all walls in $U_N$ for the object $E$. Thus the stable factors $A_i$ of $E$ are constant in the interior of the chamber, and so

\[ m_E(\sigma_2) = \sum_{i=1}^{m} |Z_2(A_i)| \leq \sum_{i} |Z_1(A_i)| + \sum_{i: v(A_i) \neq \pm \delta} |(Z_2 - Z_1)(A_i)| \]

\[ \leq m_E(\sigma_1) + \sum_{i: v(A_i) \neq \pm \delta} \|Z_2 - Z_1\| \cdot \|v(A_i)\| \]

\[ \leq m_E(\sigma_1) + \|Z_2 - Z_1\| \cdot \|Z_2 - Z_1\| \cdot \frac{1}{K} m_E(\sigma_1) \leq m_E(\sigma_1) e^{\frac{\|Z_2 - Z_1\|}{K}} \]

It follows by continuity and induction on the number of chambers traversed that if stability conditions $\sigma_1, \sigma_2 \in U_N$ can be connected by path of length $D$ along which $|Z(\delta)|$ is decreasing, then

\[ m_E(\sigma_2) \leq m_E(\sigma_1) e^{\frac{D}{K}}. \]

Taking the limit as $N \to \infty$, the same result holds for all $\sigma_1, \sigma_2 \in U_{\epsilon, \phi_1, \phi_2}$.

Let $M$ be the maximum of $m_E$ on the compact set $U_N$. Any point in $\sigma' \in U_{\epsilon, \phi_1, \phi_2}$ can be reached from a point in $U_N$ along a path of bounded length $\leq 2\epsilon$ along which $|Z(\delta)|$ is decreasing: indeed, the subset of $U_{\epsilon, \phi_1, \phi_2}$ where $Z(S)$ has constant phase is connected, and (by assumption on $N$) contains a point $\sigma''$ that is also in $U_N$; then we can just use the straight line segment between the central charges $Z''$ and $Z'$ of $\sigma''$ and $\sigma'$, respectively. It follows that $m_E$ is bounded on $U_{\epsilon, \phi_1, \phi_2}$ by $Me^{\frac{2\epsilon}{K}}$, proving the claim.

There are only finitely many $v \in \mathcal{N}(X)$ with $(v, v) \geq -2$, such that $|Z(v)| \leq Me^{\frac{D}{K}}$ can hold for some $Z \in V_{\epsilon, \phi_1, \phi_2}$. Therefore, there are only finitely many classes that
can appear as the Mukai vector of a stable factor of $E$ for any stability condition in $U_{\epsilon,\phi_1,\phi_2}$. The loci where pairs of these classes have equal phase defines a finite set of walls.

**Proposition 5.4.** Consider a stability condition $\sigma$ with $w_E(\sigma) \notin \mathbb{Z}$. Then the flow of the vector field $v(\sigma)$ starting at $\sigma$ ends at a stability condition of integral width $\lfloor w_E(\sigma) \rfloor$.

**Proof.** Assume otherwise. By the results of Section 4, this means that the path $\Omega(t) \in \mathbb{Q}^+(X)$ leads to a point $\Omega_0$ as above. By Lemma 5.2, there is a spherical object $S$ with $Z_0(S) = 0$ that is $\sigma(t)$-stable for $t$ sufficiently close to $t_0$. Up to shift, we may assume that $(\Theta, v(S)) > 0$ along the path.

Take $0 < t_1 < t_0$ as in Lemma 5.1 and put $\sigma(t_1) = (Z_1, P_1)$. Under the assumptions, $Z_0(S) - Z_1(S) = -Z_1(S)$ is an integral of a positive multiple of $\zeta(t)$; the Lemma thus implies that $Z_1(S)$ lies in the interior of the cone spanned by $-Z_1(A_+)$ and $Z_1(A_-)$, see Figure 5. Since $Z(S) \to 0$ along the flow, it follows from the definition of the sign $\epsilon(\sigma)$ that $v(A_-)$ and $v(S)$ lie in the same component $C^\pm$, and $v(A_+)$ in the opposite one.

Take $\epsilon > 0$ and let the phases of $A_+[-1]$ and $A_-$ in the stability condition $\sigma_1$ be $\phi_1$ and $\phi_2$ respectively. Since $Z(S) \to 0$ under the flow, we can increase $t_1$ if necessary so that for all $t \in [t_1, t_0]$ the central charge of the stability condition $\sigma(t)$ lies in the open subset $V = V_{\epsilon,\phi_1,\phi_2}$ of Lemma 5.3. Thus, choosing the connected component $U = U_{\epsilon,\phi_1,\phi_2}$ appropriately we can assume that $\sigma(t) \in U$ for $t \in [t_1, t_0]$.

If we take $\epsilon > 0$ small enough, we can assume that $S$ is stable for all $\sigma \in U$. Applying Lemma 5.3 we can also assume that the ($\pm$) and integral walls contained in $U$ all have images in $V$ whose closures contain $Z_0$. Note that any ($\pm$) or integral wall is given locally by $\text{Im} \, Z(S_1)/Z(S_2) \in \mathbb{R}_{>0}$ for stable (semi)rigid objects $S_1, S_2$. This is equivalent to the condition that the (always one-dimensional) kernel of $Z$ is contained in the span of $v(S_1), v(S_2)$. If the closure of the image of $W$ contains $Z_0$ (whose kernel is spanned by $v(S)$), it follows that $v(S)$ is a linear combination of the $v(S_i)$, and hence $Z(S)$ is also aligned with $Z(A_{\pm})$ along the wall.

Consider moving $\sigma$ on a path in the open subset $U$ obtained by rotating $Z(S)$ anticlockwise around the hole. The support condition (18) implies that for $Z, Z' \in V$.
and $(v, v) \geq -2$ with $v \neq \pm \delta$

\[
\frac{|Z'(v) - Z(v)|}{|Z(v)|} \leq \frac{\|Z - Z'\|}{K|v|} \leq 2\epsilon/K.
\]

Thus for $\epsilon$ small enough we can always arrange that as we rotate around the hole, the phase of $Z(S) = Z(\delta)$ increases arbitrarily fast compared to the phases of $Z(A_{\pm})$.

Suppose that this path hits an integral wall $W$. By Lemma 3.8 and Lemma 3.7 the stable factors of $A_{\pm}$ in the stability conditions before we hit the wall are the two stable objects defining the wall; since $v(S) \neq v(A_{\pm})$ this contradicts the argument above. Thus we can prolong our path until we meet a $(\pm)$-wall, where as above $Z(S)$ and $Z(A_{-})$ align. There are now two possibilities: if $S$ and $A_{-}$ are both stable, then $(v(S), v(A_{-})) \geq 0$ implies that $v(S)$ and $v(A_{-})$ lie in opposite components $\mathcal{C}^{\pm}$, a contradiction. Otherwise, we are on a $( - )$-wall, and there is a short exact sequence

$0 \to S^{\oplus k} \to A_{-} \to T \to 0$

with $T$ quasistable. After crossing this wall, $T$ will be the new $A_{-}$, so Lemma 3.4 gives $(v(T), v(A_{+})) > 0$. Since we also have $(v(T), v(S)) > 0$ this implies that $v(A_{+})$ and $v(S)$ lie in the same component of $\mathcal{C}^{\pm}$, which gives another contradiction. \hfill \square

6. Conclusion of the proof

In this section we complete the proofs of our main Theorems. Take assumptions as in the previous sections: thus $X$ is a complex K3 surface of Picard rank $\rho(X) = 1$, and $E \in D(X)$ is a (semi)rigid object satisfying $\text{Hom}_{X}(E, E) = \mathbb{C}$.

Retraction onto width 0. The aim of this section is to combine the flows of Proposition 5.4 defined on $w_{E}^{-1}(n, n + 1)$ for all $n$ to retract $\text{Stab}_{\text{red}}(X)$ onto a subset of the geometric chamber.

We write $W_{\leq n}$, $W_{= n}$, $W_{< n}$ and $W_{(n, n+1)}$ for the subsets of $\text{Stab}_{\text{red}}(X)$ defined by $w_{E} \leq n$, etc. (Note that $W_{\mathbb{Z}}$ is a strict subset of $\cup_{n \in \mathbb{Z}} W_{= n}$, as we make no assumptions about quasistability of $E$ in $W_{\leq 0}$.)

Lemma 6.1. The inclusion $W_{\leq n} \subset W_{< n+1}$ is a deformation retract.

Proof. Consider the map $W_{(n, n+1)} \to W_{= n}$ sending a stability condition $\sigma$ with $w_{E}(\sigma) \in (n, n + 1)$ to the endpoint of the flow along the vector field $v$ starting at $\sigma$. Lemma 4.6 shows that the identity on $W_{\leq n}$ extends this map continuously to give a retract $r_{n}: W_{< n+1} \to W_{\leq n}$.

A homotopy between the identity on $W_{< n+1}$ and $r_{n}$ is given by the normalized flow map

$\text{Flow}': W_{< n+1} \times [0, 1] \to W_{\leq n}$,

for which $\text{Flow}'(\sigma, t)$ is the point on the flow line of $\sigma$ with width $w_{E}(\sigma) - t$ if $0 \leq t < w_{E}(\sigma) - n$, and given by $r_{n}(\sigma) \in W_{\leq n}$ if $t \geq w_{E}(\sigma) - n$. 

We now make this construction rigorous. Consider the union $U \subset W_{(n,n+1)} \times \mathbb{R}_{\geq 0}$ of all maximal (closed) intervals of definition of the flow of $v(\sigma)$, and let

$$\text{Flow}: U \to W_{[n,n+1]}$$

be the induced continuous map: $\text{Flow}(\sigma,t)$ is the position of the flow line starting at $\sigma$ after time $t$. It follows from construction and Lemma 4.3 that the map

$$\Gamma: U \to W_{(n,n+1)} \times [0,1), (\sigma,t) \mapsto (\sigma,w_E(\sigma) - w_E(\text{Flow}(\sigma,t)))$$

is a homeomorphism onto its image

$$V = \{(\sigma,s): 0 \leq s \leq w_E(\sigma) - n\} \subset W_{(n,n+1)} \times [0,1).$$

Thus, on $W_{(n,n+1)} \times [0,1]$, we can define $\text{Flow}'$ as follows: first set $\text{Flow}'|_V = \text{Flow} \circ \Gamma^{-1}$, and then extend it continuously by $\text{Flow}'(\sigma,s) = r_n(s)$ for $(\sigma,s) \notin V$.

This is a homotopy between the inclusion $W_{(n,n+1)} \hookrightarrow \text{Stab}_{\text{red}}(X)$ and $r_n$.

Another application of Lemma 4.6 shows that this homotopy extends continuously to $W_{<n+1}$, as desired. □

**Lemma 6.2.** For each $n \geq 0$, there is a subset $U \subset W_{<n+1}$, containing $W_{\leq n}$, such that $U$ is a deformation retract of $W_{\leq n+1}$.

**Proof.** The set $W_{=n+1}$ is a locally finite set of walls. Each wall is bordering $W_{\leq n+1}$ on one side by Lemma 3.14. In other words, $W_{\leq n+1}$ is a manifold with boundary, and the boundary is $W_{=n+1}$. Therefore, we can find a tubular neighbourhood homeomorphic to $B \times [0,\epsilon)$ for each component $B \subset W_{=n+1}$, such that they are pairwise disjoint, and let $U$ be the complement of $\bigcup B \times [0,\frac{1}{2}\epsilon)$.

Combining Lemmas 6.1 and 6.2, we obtain a retraction

$$R_n: W_{\leq n+1} \to W_{\leq n},$$

and a homotopy

$$H_n: [0,1] \times W_{\leq n+1} \to W_{\leq n+1}$$

between the identity and $R_n$.

**Lemma 6.3.** The subset $W_{=0} \subset \text{Stab}_{\text{red}}(X)$ is a deformation retract.

**Proof.** We define a left inverse $R_\infty$ to the inclusion by the infinite composition

$$R_\infty = R_0 \circ R_1 \circ R_2 \circ \ldots : \text{Stab}_{\text{red}}(X) \to W_{=0}.$$

Of course, on each $W_{<n+1}$, this map is a finite composition and continuous. It follows that it is well-defined and continuous on $\text{Stab}_{\text{red}}(X)$.

Similarly, to define the homotopy choose any infinite decreasing sequence $t_0 = 1 > t_1 > t_2 > \cdots > 0$; we can define a homotopy $H_\infty$ between the identity and $R_\infty$ as the infinite composition of the homotopies that apply $H_n$ in the interval $[t_{n+1}, t_n]$. The same argument as before shows that $H_\infty$ is well-defined and continuous after restriction to $[0,1] \times W_{\leq n+1}$, and therefore well-defined and continuous on all of $[0,1] \times \text{Stab}_{\text{red}}(X)$. □
**Geometric stability conditions.** We now fix our object $E$ to be some skyscraper sheaf $E = O_x$. To prove Theorem 1.3 it remains to prove that $W_{=0}$ is connected and contractible. In fact, given our assumption that the Picard rank $\rho(X) = 1$, the interior of the subset $W_{=0}$ coincides with the subset of geometric stability conditions, as we now demonstrate.

**Lemma 6.4.** Assume that $F \in \text{Coh}(X)$ is a semirigid sheaf. Then either $F \cong O_x$ for some $x \in X$, or the support of $F$ is all of $X$.

**Proof.** If neither of these alternatives hold, then $F$ has one-dimensional support, and so $v(F) = (0, D, s)$ with $D$ an effective curve class. But this is impossible, since $\text{Pic}(X) = \mathbb{Z}$ implies that $(v(F), v(F)) = D^2 > 0$ contradicting the assumption that $F$ is semirigid. \hfill \Box

**Remark 6.5.** Combined with Lemma 2.4 this immediately shows that any rigid sheaf is automatically torsion-free.

The following is due to Hartmann.

**Lemma 6.6.** Given $\phi \in \text{Aut}^+ H^*(X, \mathbb{Z})$, there exists an autoequivalence $\Phi \in \text{Aut} D(X)$ whose induced map in cohomology is given by $\phi$, and such that $\Phi$ preserves the connected component $\text{Stab}^\dagger(X)$.

**Proof.** See [Har12, Proposition 7.9]. \hfill \Box

We can now prove the claimed characterisation of geometric stability conditions.

**Lemma 6.7.** Suppose $\sigma \in \text{Stab}^*(X)$ is such that some $O_x$ is stable. Then all skyscraper sheaves are stable of the same phase.

**Proof.** We first assume that $\sigma$ is not on a wall with respect to the class $v(O_x)$.

By definition of $\text{Stab}^*(X)$, there is an autoequivalence $\Phi \in \text{Aut} D(X)$ such that $\Phi(\sigma) \in \text{Stab}^\dagger(X)$. By the previous Lemma, we may assume that $\Phi$ acts as the identity on cohomology; in particular, it leaves the class $v(O_x)$ invariant. Composing with squares of spherical twists, we may further assume that $\Phi(\sigma)$ is in the geometric chamber. The stable objects in $\sigma$ of class $v(O_x)$ and the correct phase are exactly the objects $E_y = \Phi^{-1}(O_y)$ for $y \in X$.

The objects $E_y$ are pairwise orthogonal. By assumption one of these objects is $O_x$; hence all the others have support disjoint from $x$. Using Lemma 2.4 repeatedly, it follows that each cohomology sheaf of $E_y$ is rigid or semirigid, and at most one is semirigid. By Lemma 6.4 this is only possible if $E_y$ is the shift of a skyscraper sheaf: $E_y \cong O_z[n]$ for some $z \in X$ and $n \in \mathbb{Z}$. Using the representability of $\Phi$ as a Fourier-Mukai transform, standard arguments (see [Huy06, Corollary 5.23 and Corollary 6.14]) show that $n$ is independent of $y$, and that $\Phi$ is the composition of an automorphism of $X$ with a shift. In particular, all $O_y$ are stable of the same phase.
Finally, if $\sigma$ is on a wall, then by perturbing it a bit and applying the previous argument we can conclude that $\sigma$ is in the boundary of the geometric chamber. However, it follows from [Bri07, Theorem 12.1] that in the case of Picard rank one, every wall of the geometric chamber destabilizes every skyscraper sheaf $\mathcal{O}_x$. □

**Final steps.** We can now complete the proofs of our main Theorems.

**Lemma 6.8.** Let $E = \mathcal{O}_x$ be a skyscraper sheaf and consider the width function $w = w_E$. Then $W = 0$ is contractible.

**Proof.** By the previous Lemma, the set $W = 0$ coincides with the closure of the geometric chamber. Recall from Theorem [2.1] that its interior is homeomorphic to $\mathbb{C} \times \mathcal{V}(X)$. It is immediate to see that $\mathcal{V}(X)$ is contractible. With arguments as in Lemma [6.2] one also shows that there is an open subset of the geometric chamber that is a deformation retract of its closure $W = 0$. □

**Proof of Theorem 1.3.** The result is immediate from Lemmas 6.8 and 6.3. □

**Proof of Theorem 1.4.** As discussed in the introduction, since $\text{Stab}^\ast(X)$ is simply-connected, there is an isomorphism

$$\pi_1(P_0^+(X)) \xrightarrow{\sim} \text{Aut}^0 D(X).$$

Recall that $\pi_1(P_0^+(X))$ is the product of $\pi_1(\text{GL}_2^+(\mathbb{R})) \cong \mathbb{Z}$ with the fundamental group of $h^0 \subset h$, which in turn is a free group generated by loops around the holes $\delta$ (see equation (4) and the surrounding discussion).

By Proposition 2.2 these loops act by squares of spherical twists. Finally, from the definition of the action of the universal cover of $\text{GL}_2^+(\mathbb{R})$ action, it is obvious that the generator of $\pi_1(\text{GL}_2^+(\mathbb{R}))$ acts by an even shift. □

To conclude, we point out two consequences of our results for spherical objects; they may be of independent interest, e.g. in relation to [Huy12].

**Corollary 6.9.** Let $S \in D(X)$ be a spherical object. Then there exists a stability condition $\sigma \in \text{Stab}^\dagger(X)$ such that $S$ is $\sigma$-stable.

**Proof.** This is immediate from Lemma 6.3 applied to $E = S$. □

**Remark 6.10.** Based on the above Corollary, one can also prove the following statement: $\text{Aut}^0 D(X)$ acts transitively on the set of spherical objects $S$ with fixed Mukai vector $v(S) = \delta$.

(As a consequence, $\text{Aut}^0 D(X)$ acts transitively on the set of spherical objects in $D(X)$ if and only if $\text{Aut}^+ H^\ast(X, \mathbb{Z})$ acts transitively on the set $\Delta(X)$ of spherical classes. Similar results have been proved for spherical vector bundles and mutations in [Kul89].)

By the above Corollary, our claim follows if for any wall $\mathcal{W}$ where stability for the class $\delta$ changes, we can prove that the two stable objects $E^+, E^-$ of class $v(T^\pm) = \delta$ on either side of the wall are related by $T^- = \Phi(T^+)$, for some autoequivalence in
Φ ∈ Aut^0 D(X). This can, for example, be shown with the same arguments as in the proof of [BM13 Proposition 6.8].

7. Relation with mirror symmetry

Return to the case of a general algebraic K3 surface X. We will describe a precise relation between the group of autoequivalences and the monodromy group of the mirror family implied by Conjecture 1.2.

Stringy Kähler moduli space. We start by reviewing the construction of an interesting subgroup of Aut^0 D(X), which we learnt about from Daniel Huybrechts. Let us write

$$\text{Aut}^+_\text{CY} H^*(X) \subset \text{Aut}^+ H^*(X)$$

for the subgroup of Hodge isometries φ whose complexification acts trivially on the line $H^{2,0}(\mathbb{C}) \subset H^*(X, \mathbb{C})$. This is equivalent to the statement that φ acts trivially on the transcendental lattice $T(X) := \mathcal{N}(X) \subset H^*(X, \mathbb{Z})$: for any integral class $\alpha \in T(X)$, the difference $\phi(\alpha) - \alpha$ is integral, and in the orthogonal complement of both $H^{2,0}(\mathbb{C})$ and $\mathcal{N}(X)$, and thus equals zero. In particular,

$$\text{Aut}^+_\text{CY} H^*(X) \subset \text{Aut} \mathcal{N}(X)$$

is the subgroup of index two preserving orientations of positive definite two-planes.

Definition 7.1. We call an autoequivalence $\Phi \in \text{Aut} D(X)$ Calabi-Yau if the induced Hodge isometry $\varpi(\Phi)$ lies in the subgroup Aut^+_\text{CY} H^*(X).

Write Aut^+_\text{CY} D(X) ⊂ Aut D(X) for the group of Calabi-Yau autoequivalences. In the Appendix we explain how the Calabi-Yau condition can be interpreted as meaning that $\Phi$ is an autoequivalence of the category $D(X)$ as a Calabi-Yau category.

Now consider the quotient stack

$$\mathcal{L}_{\text{Kah}}(X) = \text{Stab}^*_\text{red}(X)/\text{Aut}_{\text{CY}}(X).$$

The action of $\mathbb{C}$ on Stab^*(X) induces an action of $\mathbb{C}^*$ on $\mathcal{L}_{\text{Kah}}(X)$ and we also consider the quotient

$$\mathcal{M}_{\text{Kah}}(X) = \mathcal{L}_{\text{Kah}}(X)/\mathbb{C}^* \cong (\text{Stab}^*_\text{red}(X)/\mathbb{C})/(\text{Aut}_{\text{CY}}(X)/[2]).$$

which we view as a mathematical version of the stringy Kähler moduli space of the K3 surface X.

Remark 7.2. If Conjecture 1.2 holds then Stab^*_\text{red}(X) is the universal cover of $\mathcal{L}_{\text{red}}(X)$, and there are isomorphisms

$$\pi_1^{\text{orb}}(\mathcal{L}_{\text{Kah}}(X)) \cong \text{Aut}_{\text{CY}} D(X), \quad \pi_1^{\text{orb}}(\mathcal{M}_{\text{Kah}}(X)) \cong \text{Aut}_{\text{CY}} D(X)/[2].$$

In particular, by Theorem 1.3 this is the case whenever $\rho(X) = 1$.

We have the following more concrete descriptions of these stacks
**Proposition 7.3.** There are isomorphisms
\[ \mathcal{L}_{\text{Kah}}(X) \cong Q^+_0(X)/\text{Aut}_{\text{CY}}^+ H^*(X), \quad \mathcal{M}_{\text{Kah}}(X) \cong \Omega^+_0(X)/\text{Aut}_{\text{CY}}^+ H^*(X). \]
Moreover, these stacks are Deligne-Mumford and have quasi-projective coarse moduli spaces.

**Proof.** There is a short exact sequence
\[ 1 \to \text{Aut}^0 D(X) \to \text{Aut}_{\text{CY}} D(X) \to \text{Aut}_{\text{CY}}^+ H^*(X) \to 1. \]
Together with Theorem 1.1 this leads to the given isomorphisms. The stabilizer of \( \Omega \in Q^+_0(X) \) acts faithfully on \( \Omega^\perp \cap \mathcal{N}(X) \); since \( \text{Re} \Omega, \text{Im} \Omega \) span a positive definite 2-plane in \( \mathcal{N}(X) \otimes \mathbb{R} \), this lattice is negative definite, and thus has finite automorphism groups. Therefore, the stabilizer of \( \Omega \in Q^+_0(X) \) is finite. Also recall from above that \( \text{Aut}_{\text{CY}}^+ H^*(X) \) has finite index two in \( \text{Aut} \mathcal{N}(X) \). Thus, the Baily-Borel theorem applies, and the above stacks have quasi-projective coarse moduli spaces. \( \square \)

**Mirror families.** We now relate the space \( \mathcal{M}_{\text{Kah}}(X) \) to moduli spaces of lattice-polarised K3 surfaces. For this we need to make

**Assumption 7.4.** Suppose that the transcendental lattice \( T(X) \) contains a sublattice isomorphic to the hyperbolic plane.

This condition is automatic if \( \rho(X) = 1 \) cf. [Dol96, Section 7]. Note that the lattice \( \mathcal{N}(X) \) contains a canonical sublattice \( H = H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \) isomorphic to the hyperbolic plane. Given the assumption, we can choose another such sublattice \( H' \subset T(X) \). Then there are orthogonal direct sums
\[ \mathcal{N}(X) = H \oplus M, \quad T(X) = H' \oplus M', \]
where \( M = \text{NS}(X) \) and \( M' \) is some lattice of signature \( (1, 18 - \rho) \).

Now recall the notion of an ample \( M' \)-polarized K3 surface from [Dol96, Section 1]; this includes the data of a K3 surface \( \hat{X} \) together with a primitive isometric embedding \( \rho: M' \to \text{NS}(\hat{X}) \) whose image contains an ample class. (The notion depends on additional choices of data; different choices are equivalent up to precomposing the embedding \( \rho \) with an isometry of \( M' \).)

**Remark 7.5.** There is no separated moduli stack of ample \( M' \)-polarized K3 surfaces for the following well-known reason: Consider a smooth family \( Y \to B \) of K3 surfaces over a one-dimensional base \( B \), and assume that a special fiber \( b \in B \) contains a \((-2)\)-curve \( C \) that does not deform as an algebraic class. Then flopping at \( C \) produces a non-isomorphic family \( \hat{Y} \to B \) that is isomorphic to \( Y \) after restricting to the complement of \( b \). Note that the central fibers are isomorphic as K3 surfaces, but not isomorphic as ample \( M' \)-polarized K3 surfaces.

**Lemma 7.6.** The orbifold \( \mathcal{M}_{\text{Kah}}(X) \) admits a family of \( M' \)-polarized K3 surfaces, and its coarse moduli space is the coarse moduli space \( M' \)-polarized K3 surfaces.
Proof. Consider the orthogonal complement $L = H'^\perp \subset H^* (X, \mathbb{Z})$. Note that $L$ is isomorphic to the K3 lattice $H^2 (K3, \mathbb{Z})$. We have an orthogonal decomposition (not necessarily a direct sum)

$$L = H \oplus M \perp M^\vee.$$ 

In particular, inside $L$ we have $(M^\vee)^\perp = \mathcal{N} (X)$. Note also that $\text{Aut}_{\text{CY}} H^* (X, \mathbb{Z})$ can be identified with the group of automorphisms of $L$ which fix every element of $M^\vee$.

This is exactly the situation considered by Dolgachev in [Dol96, Section 6]: in terms of his notation $\mathcal{N} (X)$ becomes identified with the lattice $N$, the space $\mathbb{P} Q^0 (X)$ becomes $D^0_M$, and the group $\text{Aut}_{\text{CY}} H^* (X, \mathbb{Z})$ becomes $\Gamma (M^\vee)$. In particular, the statement regarding coarse moduli is proved there. In order to construct a family, let us choose in addition a class $l \in M^\vee$ with $l^2 > 0$ such that $l$ is not orthogonal to any spherical class $\delta \in L \setminus \mathcal{N} (X)$; requiring $l$ to be ample avoids the non-Hausdorff issue explained above.

Now we use the strong global Torelli theorem: given $\Omega \in \mathbb{Q}^+_0 (X)$, there exists a unique K3 surfaces $\tilde{X}$ with a marking $L \sim H^2 (\tilde{X}, \mathbb{Z})$ such that $H^{2,0} (X, \mathbb{C})$ is spanned by $\Omega$, and such that $\tilde{l}$ is an ample class. These fit together into a family over the period domain, on which $\text{Aut}_{\text{CY}} H^* (X, \mathbb{Z})$ acts. Taking the quotient by this action produces a family over $\mathcal{M} (X^\vee)$, and remembers the marking by the sublattice $M^\vee$ as claimed.

Following Dolgachev, we consider this family of $M^\vee$-polarised K3 surfaces as a mirror family to the family of (ample) $M$-polarized K3 surfaces, of which our surface $X$ is a member. Thus in the case $\rho = 1$ we can conclude that the group $\text{Aut}_{\text{CY}} D (X) / [2]$ is isomorphic to the fundamental group of the base of the mirror family. Alternatively, note that the full group $\text{Aut}_{\text{CY}} D (X)$ is isomorphic to the fundamental group of this augmented mirror moduli space $L_{\text{Kah}} (X)$ parameterizing pairs consisting of an ample $M^\vee$-polarized K3 surfaces together with a choice of nonzero holomorphic 2-form.

Remark 7.7. The lattice $M^\vee$ and its embedding in the K3 lattice $L$ depends on our choice $H' \subset T (X)$. Different choices lead to different equivalence classes of embeddings and hence different families of $M^\vee$-polarised K3 surfaces. The bases of these families are all identified with the space $\mathcal{M}_{\text{Kah}} (X)$, but as families of $M^\vee$-polarised K3 surfaces they are different. All should be considered as mirror families of $X$. It is easy to check that the families of derived categories given by these different mirror families are all the same.

Remark 7.8. Finally, we want make to make explicit a relation of Conjecture 1.2 to Homological Mirror Symmetry. Given an ample $M^\vee$-polarized K3 surface $\tilde{X}$, choose a Kähler form $\omega$ with a very general class $[\omega] \in M^\vee \otimes \mathbb{R}$. Consider $(\tilde{X}, \omega)$ as a symplectic manifold, and denote its symplectic mapping class group by $\text{Map} (\tilde{X}, \omega)$. 


The monodromy representation induces a map
$$\pi_1(\mathcal{L}(X^\vee)) \to \text{Map}(\hat{X}, \omega).$$
Moreover, if \( \omega \) is sufficiently generic, then we can identify \( \text{Map}(\hat{X}, \omega) \) with the mapping class group of diffeomorphism preserving the embedding \( M^\vee \hookrightarrow H^2(\hat{X}) \); it follows that the above map is an isomorphism of groups.

The symplectic mapping class group acts on the Fukaya category of \((\hat{X}, \omega)\) via autoequivalences. If both Conjecture 1.2 and Homological Mirror Symmetry hold, then \( \text{Map}(\hat{X}, \omega) \) is equal to the group of CY-autoequivalences of the Fukaya category.

**Appendix A. Calabi-Yau autoequivalences**

In this Appendix we explain that an autoequivalence \( \Phi \in \text{Aut } D(X) \) is Calabi-Yau in the sense used above precisely if \( \Phi \) respects the Serre duality pairings

\[
\text{Hom}^i(E, F) \times \text{Hom}^{-i}(F, E[2]) \to \mathbb{C}
\]

induced by a choice of holomorphic volume form \( \omega \in H^{2,0}(X, \mathbb{C}) \).

A proof would be more natural in the setting of dg-categories; thus we restrict ourselves to a sketch of the arguments. First we recall some basic definitions and results on Serre functors from [BO01, Section 1]:

- A graded autoequivalence is an autoequivalence \( \Phi \) together with a natural transformation \( \Phi \circ [1] \Rightarrow [1] \circ \Phi \). Any exact autoequivalence has the structure of a graded autoequivalence.
- A Serre functor is a graded autoequivalence \( S \) together with functorial isomorphisms
  \[
  \text{Hom}(A, B) \cong \text{Hom}(B, S(A))^*
  \]
satisfying an extra condition.
- A Serre functor is unique up to a canonical graded natural transformation.
- Given an equivalence \( \Phi \), there is a canonical natural isomorphism
  \[
  \Phi \circ S \Rightarrow S \circ \Phi.
  \]

Let us define a Calabi-Yau-2 category to be a triangulated category together with natural isomorphism \( \tau : [2] \Rightarrow S \), where \( S \) is a Serre functor. By the canonical uniqueness of Serre functors, this is equivalent to specifying functorial Serre duality pairings as in [19].

A graded autoequivalence acts on the set of natural transformations \( [2] \Rightarrow S \) via conjugation, and the given natural transformations \( \Phi \circ [2] \Rightarrow [2] \circ \Phi \) and \( \Phi \circ S \Rightarrow S \circ \Phi \). From the construction of the latter transformation, it follows that \( \Phi \) leaves \( \tau \) invariant if and only if it respects the Serre duality pairings [19].

On the other hand, the induced actions of \( \Phi \) on the cohomology \( H^*(X, \mathbb{C}) \) of \( X \) and the Hochschild homology of \( D(X) \) are compatible with the HKR isomorphism,
see [MS09, Theorem 1.2]. Therefore, $\Phi$ is Calabi-Yau in the sense of Definition 7.1 if and only if it leaves the second Hochschild homology group

\begin{equation}
\text{HH}_2(X) = \text{Hom}_{D(X \times X)}(\mathcal{O}_\Delta[2], \omega_\Delta[2])
\end{equation}

invariant.

Passing to Fourier-Mukai transforms induces a natural map

\[ \text{Hom}_{D(X \times X)}(\mathcal{O}_\Delta[2], \omega_\Delta[2]) \to \text{Hom}(\mathbb{C}, S), \]

compatible with the action of $\Phi$. While it is not true (without passing to dg-categories) for any pair of kernels that the above map is an isomorphism, it does hold in our situation: both sides are (non-canonically) isomorphic to scalars $\mathbb{C}$, and the map is non-trivial. Thus, $\Phi$ is Calabi-Yau in the sense of Definition 7.1 if and only if it respects the natural transformation $[2] \Rightarrow S$, or, equivalently, respects the Serre duality pairings (19).

References


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