This is joint work with V. Gritsenko. We work mostly over \( \mathbb{C} \) but there will be a few exceptions at the end.

1 Introduction

Let \( S \) be an Enriques surface over \( \mathbb{C} \), i.e., \( \omega_S \neq \mathcal{O}_S \) but \( \omega_S^2 = \mathcal{O}_S \) and \( q(S) = 0 \). Let \( X \to S \) denote the étale double cover by a K3 surface. We have \( \rho(S) = \frac{1}{2} e(X) = 12 \) and \( b_2(S) = 10 \). The Néron-Severi group of \( S \) is

\[
\text{NS}(S) \simeq U \oplus E_8(-1) \oplus \mathbb{Z}/2\mathbb{Z}
\]

where \( U \) is a hyperbolic plane and \( E_8 \) is associated with the corresponding root system. The \( \mathbb{Z}/2\mathbb{Z} \) is generated by the class of \( \omega_S \).

Consider a marking of \( S \)

\[
\varphi : H^2(S, \mathbb{Z})/\text{torsion} \simeq U \oplus E_8(-1).
\]

Recall

\[
H^2(X, \mathbb{Z}) \simeq 3U \oplus 2E_8(-1) =: L_{K3}
\]

with involution

\[
\varrho : L_{K3} \to L_{K3} \quad (x, y, z, u, v) \mapsto (z, -y, x, v, u)
\]

admitting eigenspaces

\[
\text{Eig}(\varrho)^+ = \{(x, 0, x, u, u), x \in U, u \in E_8(-1)\} \simeq U(2) \oplus E_8(-2) =: M
\]

and

\[
\text{Eig}(\varrho)^- = \{(x, y, -x, u, -u)\} \simeq U \oplus U(2) \oplus E_8(-2) =: N.
\]
The signature of $N$ is $(2, 10)$.

**Fact:** Given $\varphi$ we can find a marking

$$\tilde{\varphi} : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$$

such that $\varrho \circ \tilde{\varphi} = \tilde{\varphi} \circ \sigma^*$ where $\sigma$ is the involution on $X$. Note that $\tilde{\varphi}(p^*H^2(S, \mathbb{Z})) \simeq \text{Eig}(\varrho)^+ = M$. Since $\sigma$ acts on $H^{2,0}(X)$ by $-1$ we find

$$\tilde{\varphi}(\omega_X) \in N = \text{Eig}(\varrho)^-.$$

Then we can define a period domain

$$\Omega_N = \{[x] \in \mathbb{P}(N \otimes \mathbb{C}) : (x, x) = 0, (x, \bar{x}) > 0\} = D_N \sqcup D'_N$$

and we may assume $\tilde{\varphi}(\omega_X) \in D_N$.

Let $O(N)$ denote the integral orthogonal group and $O^+(N)$ those isometries preserving $D_N$. We take

$$M_{En} = O^+(N) \backslash D_N,$$

which is quasi-projective of dimension ten. For each root $\delta \in N$ (with $\delta^2 = -2$) we have a hyperplane

$$H_\delta := \{[x] : (x, \delta) = 0\} \rightarrow \Delta_{-2} \subset M_{En}.$$

**Theorem 1 (Horikawa, Namikawa)** Associating to an Enriques surface a period point as above gives a bijection between

$$M^0_{En} := M_{En} \backslash \Delta_{-2}$$

and Enriques surfaces up to isomorphism.

**Remark 1** This is not a moduli space in the sense of representing a functor of Enriques surfaces.

Liedtke has studied technical issues with defining this as a stack. A polarization or other additional data is needed.

**Theorem 2 (Kondo)** $M_{En}$ is rational.
2 Moduli spaces

Definition 1 A (semi-)polarized Enriques surface is a pair \((S, \mathcal{L})\) where \(\mathcal{L}\) is a (semi-)ample line bundle.

A numerically polarized Enriques surface is a pair \((S, \mathcal{L})\) where \(\mathcal{L}\) is ample and \(\mathcal{L} \in \text{Num}(S)\).

Viehweg has contructed quasi-projective coarse moduli spaces \(\mathcal{M}_{\text{En, } h}^a\) of polarized Enriques surfaces with a given type of polarization.

Definition 2 The type of a polarization is an \(O(M) = O(M(\frac{1}{2}))\)-orbit of a primitive vector \(h \in U \oplus E_8(-1) = M(\frac{1}{2})\).

Notation. We use \(\mathcal{M}\) to denote moduli spaces, and \(M\) to denote orthogonal modular varieties.

We have an involution

\[
\iota : \mathcal{M}_{\text{En, } h}^a \to \mathcal{M}_{\text{En, } h}^a,
\]

\[
(S, \mathcal{L}) \mapsto (S, \mathcal{L} \otimes \omega_S)
\]

and set

\[
\mathcal{M}_{\text{En, } h}^{\text{num}} := \mathcal{M}_{\text{En, } h}^a / \langle \iota \rangle.
\]

3 Groups and orthogonal modular varieties

Given a lattice \(L\), let \(D(L)\) denote the discriminant group with its standard quadratic form. Note that \(D(M) = D(N) = \mathbb{F}_2^{10}\) and

\[
O(D(M)) = O(D(N)) = O^+(\mathbb{F}_2^{10})
\]

is the orthogonal group of even type which has order \(2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31\).

Fix \(h \in U \oplus E_8(-1) = M(\frac{1}{2})\) primitive and

\[
O(M, h) = \{g \in O(M), g(h) = h\}.
\]

Set

\[
\hat{O}(N) \subset \Gamma_h = \pi^{-1}_N(\pi_M(O(M, h))) \subset O(N)
\]

so that \(\Gamma_h^+ = \Gamma_h \cap O^+(N)\) acts on \(D_N\). Here \(\pi_M : O(M) \to O(D(M))\) and similarly for \(\pi_N\).
Let

\[ M_{En,h} = \Gamma^+_h \backslash D_N \to M_{En} \cup \cup \Delta_{-2,h} \to \Delta_{-2} \]

where the hypersurface \( \Delta_{-2,h} \) is the inverse image of \( \Delta_{-2} \). It is irreducible.

An element \( v \in N, v^2 = -4 \) belongs to one of two \( O^+(N) \) orbits. It is even or odd respectively depending on whether \( (v,N) \) is \( 2\mathbb{Z} \) or \( \mathbb{Z} \). Even \((-4)\)-vectors are of the form \( (\delta' , -\delta) \) for \( \delta' \in U \oplus E_8(-1) \) when thinking of the embedding of \( U \oplus E_8(-1) \) into \( \text{Eig}(\rho)^- \).

The \((-4)\)-vectors define a hypersurface

\[ \Delta_{ev,-4} \subset M_{En} \]

the nodal Enriques surfaces, with preimage

\[ \Delta_{ev,-4,h} \subset M_{En,h}. \]

Let

\[ \Delta_{ev,-4,h}^\perp \subset \Delta_{ev,-4,h} \]

consist of those irreducible components where \( \delta' \cdot h = 0 \). We have spaces

\[ M_{En,h}^{\text{num}} = M_{En,h} \setminus (\Delta_{-2,h} \cup \Delta_{ev,-4,h}^\perp). \]

**Theorem 3** There is a degree 2 étale double cover

\[ \mathcal{M}_{Eh,h}^a \to M_{En,h}^{\text{num}} \]

which identifies \( M_{En,h}^{\text{num}} \) with \( \mathcal{M}_{Eh,h}^{\text{num}} \).

**Remark 2**

1. The open part \( M_{En,h}^{\text{num}} \) which is given by removing all of \( \Delta_{ev,-4,h} \) corresponds to non-nodal polarized Enriques surfaces.

2. The points on \( \Delta_{ev,-4,h}^\perp \setminus \Delta_{-2} \) can be interpreted as semi-polarized Enriques surfaces.

**Question 1**

1. Is \( \mathcal{M}_{Eh,h}^a \) always connected?

2. Is \( \mathcal{M}_{Eh,h}^a \) an orthogonal modular variety?

**Corollary 1** There are only finitely many isomorphism classes of moduli spaces of (primitively) polarized Enriques surfaces.
Ingredients of the proof:

1. There are only finitely many groups
   \[ \tilde{O}(N) \subset \Gamma_h \subset O(N) . \]

2. \( \Delta^e_{4,h} \) has finitely many components.

3. \( M_{\text{num}}^{\text{En},h} \) has only finitely many étale double covers since this is a finite CW complex and thus \( H^1(M_{\text{num}}^{\text{En},h}, \mathbb{Z}/2\mathbb{Z}) \) is finite.

**Corollary 2** The Kodaira dimension of the moduli space of numerically polarized Enriques surfaces is negative for \( h^2 < 32 \).

**Corollary 3** There are moduli space of numerically polarized Enriques surface birational to the moduli space with 2-level structure

\[ M_{\text{En}}(2) = \tilde{O}^+(N) \backslash D_N \]

which is of general type (Gritsenko, in preparation).

### 4 Arithmetic questions

**Question 2** Over which fields can we construct these moduli spaces?

For the Cossec-Verra presentation, Liedtke showed the moduli space is defined over \( \mathbb{Q} \). Also, Kondo has constructed a projective model of \( M_{\text{En}}(2) \subset \mathbb{P}^{185} \) given by \( 2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 \) quartic equations. This embedding is given via automorphic forms and the relation can be understood on terms of representation theory.

**Question 3** Over which field is \( M_{\text{En}}(2) \) defined?

This is a projective variety of general type which might be of arithmetic interest.