Derived Torelli and applications

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This is joint with Max Lieblich.

1 Statement of result

Let $k$ be field and $X$ a smooth projective scheme over $k$. Let $D(X)$ denote the bounded derived category of coherent sheaves. Let $K(X)_\mathbb{Q}$ denote the Grothendieck group of $D(X)$ tensored with $\mathbb{Q}$. The Riemann-Roch theorem gives an identification

$$K(X)_\mathbb{Q} \simeq A^*(X)_\mathbb{Q}$$

which gives a filtration by on $K(X)_\mathbb{Q}$ by codimension.

Thus each $X$ yields a pair $(D, F)$ where $D$ is a triangulated category and $F$ is a filtration on $K(D)_\mathbb{Q}$. Given two such pairs $(D, F)$ and $(D', F')$, an equivalence is an equivalence of triangulated categories $\sigma : D \sim D'$ such that $K(\sigma) : K(D)_\mathbb{Q} \sim K(D')_\mathbb{Q}$ respects the filtrations.

Example. Let $X$ be a K3 surface over $k = \bar{k}$. Then

$$F^2 = A^2(X)_\mathbb{Q}, \quad F^1 = NS(X)_\mathbb{Q} \oplus A^2(X)_\mathbb{Q}, \quad F^0 = A^*(X)_\mathbb{Q}.$$

Theorem 1 Let $X$ be a K3 surface over $k = \bar{k}$ and $Y/k$ a smooth projective scheme such that $(D(X), F_X) \simeq (D(Y), F_Y)$. Then $X \simeq Y$.

This is a well-known statement in characteristic zero; but in positive characteristics we need a mechanism that plays the role of the Hodge structure.

[Question: Does this yield a new proof of the Torelli Theorem? Answer: No, we have to lift to characteristic zero and use Torelli there to obtain our result.]
2 Fourier-Mukai transforms

Let $X, Y$ be smooth and projective over $k$ and $P \in D(X \times Y)$. Consider the Fourier-Mukai transform

$$\Phi_P : D(X) \to D(Y)$$

$$K \mapsto R_{p_2*}(L_{p_1*}K \otimes^{L} P)$$

**Theorem 2 (Orlov)** If $F : D(X) \sim D(Y)$ is an equivalence of triangulated categories then $F = \Phi_P$ for some $P \in D(X \times Y)$.

Write $A^\ast(X)_{\num} = A^\ast(X)_Q/\sim_{\num}$,

$$\beta(P) = \mathrm{ch}(P)\sqrt{Td_{X\times Y}} \in A^\ast(X \times Y)_{\num},$$

and consider

$$\Phi^P_{A^\ast} : A^\ast(X)_{\num} \to A^\ast(Y)_{\num}$$

using the same formula as above. Instead of the identification coming from Riemann–Roch, we will use the isomorphism

$$\sqrt{Td_X} \cdot \mathrm{ch}(-) : K(X)_Q \sim A^\ast(X)_Q.$$

For a surface $X$ we have a pairing on $A^\ast(X)_{\num}$

$$\langle (a, b, c), (a', b', c') \rangle = bb' - ac' - a'c$$

and $\Phi^P$ is compatible with this pairing.

**Corollary 1** $F^1_{A^\ast(X)_{\num}} = (F^2_{A^\ast})^\perp$ and $\Phi^P$ preserves the codimension filtration on $A^\ast(X)_{\num}$ if and only if $\Phi^P(F^2) = F^2$.

Alternate formulation:

**Theorem 3** Let $X$ and $Y$ be $K3$ surfaces over $k = \bar{k}$. Let $P \in D(X \times Y)$ be an object such that

$$\Phi^P : D(X) \sim D(Y)$$

and

$$\Phi^P_{A^\ast} : A^\ast(X)_{\num} \to A^\ast(Y)_{\num}$$

preserves the codimension filtration. Then $X \simeq Y$.  

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3 Moduli spaces of sheaves

Let $X/k$ be a K3 surface and $E \in D(X)$ and consider the Mukai vector

$$v(E) = (\text{rk}(E), c_1(E), \text{rk}(E) + \frac{c_1(E)^2}{2} + c_2(E)) \in A^*(X)_\text{num}.$$ 

Let $h$ be a polarization on $X$ and $\mathcal{M}_h(v)$ the stack of Gieseker semistable sheaves on $X$ with Mukai vector $v$.

**Theorem 4 (Mukai)** For suitable $v$, every semistable sheaf is stable, and $\mathcal{M}_h(v)$ is a $\mathbb{G}_m$-gerbe over $M_h(v)$ and $X \times M_h(v)$ carries a universal family:

$$\begin{array}{ccc}
X \times M_h(v) & \leftarrow & X \\
& \searrow & \\
& & M_h(v)
\end{array}$$

**Theorem 5** Let $X, Y$ be K3 surfaces over $k = \bar{k}$ with $D(X) \simeq D(Y)$. Then after composing $D(X) \xrightarrow{\sim} D(M_h(v))$ then you can arrange for

$$D(Y) \simeq D(X) \simeq D(M_h(v))$$

to respect filtrations, whence $Y \simeq M_h(v)$.

4 Idea of proof

Given $X, Y, \Phi_P$ respecting filtrations:

1. We can arrange that $\Phi_P(1, 0, 0) = (1, 0, 0)$ and that $\Phi_P(\text{ample cone}) = \pm(\text{ample cone})$.

2. Consider the deformation functor

$$\text{Def}_X : \text{Art}_W \rightarrow \text{Set}$$

where the former is the category of Artinian local $W$-algebras with residue field $k$. Here $W = W(k)$ is the Witt vectors.

**Proposition 1** There is an isomorphism of deformation functors

$$\delta : \text{Def}_X \rightarrow \text{Def}_Y$$

such that for each $L \in \text{Pic}(X)$

$$\delta(\text{Def}_{(X,L)}) = \text{Def}_{(Y,\Phi(L))}.$$
Idea: Let $\mathcal{D}(X)$ denote the stack of perfect complexes on $X$ which are simple and universally gluable. Simple means $\text{Aut}(E) = \mathbb{G}_m$ and universally gluable means $\text{Ext}^i(E, E) = 0$ for $i < 0$. Without the latter condition, you will not get a stack structure. This was worked out by Lieblich previously.

We can think of $P \in \mathcal{D}(X \times Y)$ as $Y \to \mathcal{D}_X$ $y \mapsto P_y$

The fact that $P$ is an FM equivalence means the image lands in a special open set. 

**Picture:** $Y \to \mathcal{D}_X$ is an open immersion.

4 Choose ample and lift $X, Y, P$ to characteristic 0.

5 Realizations

Consider a Weil cohomology theory

$$H^*: (\text{sm. proj. var.}/k)^{op} \to \text{gr. v. spaces}/K$$

with $X$ even dimensional. Consider

$$\tilde{H}(X) = \bigoplus_{i=-\delta}^{\delta} H^{d+2i}(X)(i)$$

and assume it is pure. For example, when $X$ is a surface take

$$H^0(X)(-1) \oplus H^2(X) \oplus H^4(X)(1).$$

The transform $\Phi^P: D(X) \to D(Y)$ yields $\Phi^P_\tilde{H}: \tilde{H}(X) \xrightarrow{\sim} \tilde{H}(Y)$ with the same formula.
Corollary 2 (Huybrechts) Let $X$ and $Y$ be K3 surfaces over $\mathbb{F}_q$ with $D(X) \simeq D(Y)$. Then $\#X(\mathbb{F}_q) = \#Y(\mathbb{F}_q)$.

Two more results

1. If $X$ is a K3 surface over $k = \bar{k}$ then $X$ has only finitely many FM-partners.

2. If $X$ is supersingular and $D(X) \simeq D(Y)$ then $X \simeq Y$.

Katrina Honigs obtained analogous statements for abelian varieties over finite fields.