Artin fans

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Mikhalkin-Speyer: there is a tropical cubic curve $C$ of genus 1 in $TP^3$ which does not lift to an algebraic curve (Speyer, *Tropical Geometry*, Berkeley thesis 2005, Figure 5.1).

Figure 5.1: A Genus 1 Zero Tension Curve which is not Tropical
I want to understand this phenomenon.

**Principles:**
- Tropical curves $\square \rightarrow TP^3$ encode in detail degenerations of curves $C \rightarrow \mathbb{P}^3$.
- They encode **logarithmic stable maps** $C \rightarrow \mathbb{P}^3$.
- superabundance $\iff$ obstructedness
- I wish to describe a fairy-tale world in which this issue disappears, and is useful for geometers.
Logarithmic structures (Kato, Fontaine, Illusie)

Definition

A pre logarithmic structure is

\[ X = (X, M \xrightarrow{\alpha} \mathcal{O}_X) \quad \text{or just} \quad (X, M) \]

such that

- \( X \) is a scheme - the underlying scheme
- \( M \) is a sheaf of monoids on \( X \), and
- \( \alpha \) is a monoid homomorphism, where the monoid structure on \( \mathcal{O}_X \) is the multiplicative structure.

Definition

It is a logarithmic structure if \( \alpha : \alpha^{-1}\mathcal{O}_X^* \rightarrow \mathcal{O}_X^* \) is an isomorphism.
Examples

- \((X, \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X)\), the trivial logarithmic structure.
- Let \(X, D \subset X\) be a variety with a divisor. We define \(M_D \hookrightarrow \mathcal{O}_X\):
  \[
  M_D(U) = \left\{ f \in \mathcal{O}_X(U) \mid f_{U \setminus D} \in \mathcal{O}_X^*(U \setminus D) \right\}.
  \]
- Let \(k\) be a field, \(X = \text{Spec} \ k\), define the punctured point:
  \[
  \mathbb{N} \oplus k^\times \to k
  \]
  \[
  (n, z) \mapsto z \cdot 0^n
  \]
  defined by sending \(0 \mapsto 1\) and \(n \mapsto 0\) otherwise.
The magic of logarithmic geometry

Definition
Logarithmic smoothness = loc.fin. type + local lifting property.

Theorem (Kato)
$f : X \rightarrow Y$ is log smooth if étale locally it is the pullback of a toric morphism: locally on $Y$

$$
\begin{array}{c}
X \xrightarrow{\text{étale}} Y \times_{\text{Spec } R[M]} \text{Spec } R[N] \rightarrow \text{Spec } R[N]
\end{array}
$$

$Y \rightarrow \text{Spec } R[M]$ for a reasonable monoid homomorphism $M \rightarrow N$
Any toric or toroidal variety $X$ is logarithmically smooth over $\text{Spec } k$.

$$T_X \cong \mathcal{O}^{\dim X}_X.$$  

A nodal curve is logarithmically smooth over a punctured point.
Here be monsters!

Logarithmic obstructions to deforming a logarithmic map $C \to \mathbb{P}^3$ lie in sequence

$$H^1(C, T_C) \to H^1(C, \mathcal{O}_C^3) \to \text{Obs} \to 0.$$ 

These can be nonzero on a broken cubic curve! The example of Mikhalkin - Speyer is such.
Artin fans

Olsson:

\[
\{\text{Logarithmic structures } X \text{ on } X\} \quad \longleftrightarrow \quad \{X \to \text{Log}\}.
\]

*The stack Log is huge and does not specify combinatorial data.*

**Theorem (Wise; ℵ, Chen, Marcus)**

There is an initial factorization \( X \to \mathcal{A}_X \to \text{Log} \) such that \( \mathcal{A}_X \to \text{Log} \) is étale, representable, strict.

The stack \( \mathcal{A}_X \) is small, totally combinatorial.
The requirement “representable” is a compromise.
Example: \( \mathcal{A}_{\mathbb{A}^1} = [\mathbb{A}^1/\mathbb{G}_m] \). In general for toric \( X \), \( \mathcal{A}_X = [X/T] \).
\[ \mathbb{P}^3 = \left( \mathbb{A}^4 \setminus \{0\} \right)/\mathbb{G}_m. \]

So
\[ \{ C \to \mathbb{P}^3 \} \leftrightarrow \{ (\mathcal{L}, s_0, \ldots, s_3) | s_i \text{ do not vanish together} \}. \]

Now
\[ \mathcal{A}_{\mathbb{P}^3} = \left( \mathbb{A}^4 \setminus \{0\} \right)/\mathbb{G}_m^4. \]

So
\[ \{ C \to \mathcal{A}_{\mathbb{P}^3} \} \leftrightarrow \{ ((\mathcal{L}_0, s_0), \ldots, (\mathcal{L}_3, s_3)) | s_i \text{ do not vanish together} \}. \]
The monsters evaporate!

\[ T_{\mathbb{P}^3} = \mathcal{O}^3, \text{ but } T_{\mathcal{A}_{\mathbb{P}^3}} = 0. \]

Logarithmic obstructions to deforming a logarithmic map \( C \to \mathcal{A}_{\mathbb{P}^3} \) lie in a quotient of

\[ H^1(C, 0) = 0. \]

The obstructions are gone!
Theorem (ℵ-Wise)

If $Y \rightarrow X$ is a toroidal modification, then

Logarithmic Gromov–Witten invariants of $X$ coincide with those of $Y$.

Reason: $\mathcal{M}(\mathcal{A}_Y) \rightarrow \mathcal{M}(\mathcal{A}_X)$ is birational. So $\overline{\mathcal{M}}(Y) \rightarrow \overline{\mathcal{M}}(X)$ is virtually birational.
Nonarchimedean picture

\(X\) - log smooth over \(k = \bar{k}\) trivially valued.
Thuillier introduced:

\[
\begin{array}{ccc}
  X & \xrightarrow{\rho_X} & X^\infty \\
  \downarrow r_X & & \downarrow p_X \\
  X & \rightarrow & \Sigma_X
\end{array}
\]

Here \(X^\infty\) is the Berkovich analytic formal fiber and \(\Sigma_X\) its skeleton / extended cone complex.

Ulirsch introduced an analytification of \(X \rightarrow A\):

\[
\begin{array}{ccc}
  X & \xrightarrow{\Phi_X} & A_X \\
  \downarrow r_X & & \downarrow r_A \\
  X & \rightarrow & A_X
\end{array}
\]
I claim that $\mathcal{A}_X^{\simeq}$ is familiar to the audience:

So $\mathcal{A}_{A^1}^{\simeq}$ is homeomorphic to $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$, the skeleton of $A^1$. 
In general we have a homeomorphism $\mathcal{A}_X^\exists \sim \overline{\Sigma}_X$. The complete diagram is

$$
\begin{array}{cccc}
    X^\exists & \rightarrow & \mathcal{A}_X^\exists & \rightarrow & \overline{\Sigma}_X \\
    \rho_X & \downarrow & \rho_A & \downarrow & \rho_\Sigma \\
    X & \rightarrow & \mathcal{A}_X & \rightarrow & F_X \\
    r_X & \rightarrow & r_A & \rightarrow & r_\Sigma
\end{array}
$$

Here, at least when $X$ has Zariski charts,

- $F_X$ is the underlying monoidal space of $\mathcal{A}_X$, the Kato fan of $X$.
- The complex $\overline{\Sigma}_X$ can be identified as $F_X(\mathbb{R}_{\geq 0} \sqcup \{\infty\})$