Igusa integrals and volume asymptotics in analytic and adelic geometry

joint work with A. Chambert-Loir
Counting lattice points

Basic observation

\[ \text{# of lattice points} \sim \text{volume} + \text{error term} \]

Basic problems

- compute the volume
- prove that the error term is smaller than the main term

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Rational points on $\mathbb{P}^1$

$$\mathbb{P}^1(\mathbb{Q}) = \{ \mathbf{x} = (x_0, x_1) \in (\mathbb{Z}^2 \setminus 0)/\pm \mid \gcd(x_0, x_1) = 1 \}$$
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Height function

\[ H: \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{R}_{>0} \]

\[ x \mapsto \sqrt{x_0^2 + x_1^2} \]
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N(B) := \# \{ x \mid H(x) \leq B \} \sim \frac{1}{2} \cdot \frac{1}{\zeta(2)} \cdot \pi \cdot B^2, \quad B \rightarrow \infty
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Leading constant

\[ \frac{1}{\zeta(2)} = \prod_p \left( 1 + \frac{1}{p} \right) \cdot \left( 1 - \frac{1}{p} \right) \]

We will interpret this as a volume with respect to a natural regularized measure on the adelic space \( \mathcal{P}_1(A_{\text{fin}}, Q) \).
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We will interpret this as a volume with respect to a natural regularized measure on the adelic space \( \mathbb{P}^1(A_{\text{fin}}^\text{Q}) \).
Points of height $\leq 1000$ on the $E_6$ singular cubic surface $X \subset \mathbb{P}^3$

$$x_1x_2^2 + x_2x_0^2 + x_3^3 = 0,$$

with $x_0, x_2 > 0$.  

Introduction
Let $X^\circ := X \setminus \mathfrak{l}$, the unique line on $X$ given by $x_2 = x_3 = 0$.

**Derenthal (2005)**

$$N(X^\circ(\mathbb{Q}), B) \sim c \cdot B \log(B)^6, \quad B \to \infty.$$
Leading constant

\[ c = \alpha \cdot \beta \cdot \tau \]

where

\begin{itemize}
  \item \( \alpha = \frac{1}{6220800} \)
  \item \( \beta = 1 \)
  \item \( \tau = \prod_p \tau_p \cdot \tau_\infty \) with
    \[ \tau_p = \frac{(p^2 + 7p + 1)}{p^2} \cdot (1 - \frac{1}{p})^7 = \frac{\#X(\mathbb{F}_p)}{p^2} \cdot (1 - \frac{1}{p})^7 \]
    \[ \tau_\infty = 6 \int_{|tv^3| \leq 1, |t^2 + u^3| \leq 1, 0 \leq v \leq 1, |uv^4| \leq 1} dt du dv \]
\end{itemize}
Points of height $\leq 50$ on the Cayley cubic surface $(4A_1) \ X \subset \mathbb{P}^3$

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$
Cubic forms

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Many recent results on asymptotics of points of bounded height on cubic surfaces and other Del Pezzo surfaces (Batyrev-Tschinkel, Browning, Derenthal, de la Breteche, Fouvry, Heath-Brown, Moroz, Salberger, Swinnerton-Dyer, ...)

Introduction
The framework: Manin’s conjecture

Manin (1989)

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field $F$, in its anticanonical embedding.
The framework: Manin’s conjecture

**Manin (1989)**

Let $X \subset \mathbb{P}^n$ be a smooth projective Fano variety over a number field $F$, in its anticanonical embedding. Then there exists a Zariski open subset $X^\circ \subset X$ such that

$$N(X^\circ(F), B) \sim c \cdot B \log(B)^{b-1}, \quad B \to \infty,$$

where $b = \text{rk} \text{Pic}(X)$. 

**Introduction**
Data:

- $G$ a linear algebraic group over $F$
- $V$ a finite-dimensional vector space over $F$
- $\rho : G \to \text{End}(V)$ an algebraic representation
- fix $x \in V$ and consider the “flow” $\rho(G) \cdot x$
- $H : V(F) \to \mathbb{R}_{>0}$ - height
- $\{\gamma \in G(o_F) \mid H(\rho(\gamma) \cdot x) \leq B\}$
Algebraic flows

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One can consider a similar setup for projective representations and rational points.

**Arithmetic problem:**
Count \( \sigma_F \)-integral (or \( F \)-rational points) on \( G/H \), where \( H \) is the stabilizer of \( x \).
Some results

**Rational points:** (Franke-Manin-T.) $G/P$; (Strauch) twisted products of $G/P$; (Batyrev-T.) $X \supset T$; (Strauch-T.) $X \supset G/U$; (Chambert-Loir-T.) $X \supset \mathbb{G}_a^n$; (Shalika-T.) $X \supset U$ (bi-equivariant); (Shalika-Takloo-Bighash-T.) $X \supset G$, De Concini-Procesi varieties

In all cases, Manin’s conjecture, and its refinements by Batyrev-Manin, Peyre, Batyrev-T. hold.

Integral points on $G/H$: Duke-Rudnick-Sarnak; Eskin-McMullen; Eskin-Mozes-Shah; Borovoi-Rudnick; Gorodnik, Maucourant, Oh, Shah, Nevo, Weiss

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Comparison with volume asymptotics

In many, but not all, cases the number of rational / integral points (lattice points) is asymptotic to the volume of height balls.
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Nevertheless, one has to address the following

**Problem**

Compute these volumes.
Consider the set $V_P(\mathbb{Z})$ of integral $2 \times 2$-matrices $M$ with characteristic polynomial

$$P(X) := X^2 + 1.$$ 

Put

$$\|M\| = \| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$ 

The volume of the “height ball” is given by $c \cdot B$, where

$$c = \zeta^*_\mathbb{Q}(\sqrt{-1})(1) \cdot \frac{\pi^{1/2}}{\Gamma(3/2)} \cdot \frac{\pi}{\Gamma(2/2)\zeta(2)}.$$
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The number of integral matrices in the ball of radius $B$ converges to the volume.
Matrices with fixed characteristic polynomial


For general

\[ V_P := \{ M \in \text{Mat}_n \mid \det(X \cdot \text{Id} - M) = P(X) \}, \]

where \( P \) has \( n \) distinct roots, one has

\[ \#\{ M \in V_P(\mathbb{Z}) \mid \|M\| \leq B \} \sim c_P \cdot B^m, \quad m = n(n-1)/2, \]

where

\[ c_P = \frac{2^{r_1}(2\pi)^{r_2} hR}{w \sqrt{D}} \cdot \frac{\pi^{m/2}/\Gamma(1 + (m/2))}{\prod_{j=2}^{n} \pi^{-j/2}\Gamma(j/2)\zeta(j)}. \]
Let $G$ be a semi-simple (real) Lie group with trivial character, $\mu$ a Haar measure on $G$, $V$ a finite-dimensional vector space over $\mathbb{R}$, and $\rho: G \to V$ a faithful representation. Let $\| \cdot \|$ be a norm on $V$. Then

$$\text{vol}(B) = \mu(\{g \in G \mid \|\rho(g)\| \leq B\}) \sim c \cdot B^a \log(B)^{b-1}, \quad B \to \infty,$$

where $a, b$ are defined in terms of the relative root system of $G$ and the weights of $\rho$, and $1 \leq b \leq \text{rank}_\mathbb{R}(G)$. 

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where $a, b$ are defined in terms of the relative root system of $G$ and the weights of $\rho$, and $1 \leq b \leq \text{rank}_{\mathbb{R}}(G)$. Moreover,

$$\text{vol}(B)^{-1} \cdot \int_{\|\rho(g)\| \leq B} f(\rho(g))d\mu(g) \to \int_{\mathcal{P}\text{End}(V)} f(\rho(g))d\mu_\infty(g),$$

where the limit measure $\mu_\infty$ is supported on a $G$ bi-invariant submanifold of $\mathcal{P}\text{End}(V)$. 

**Maucourant (2004)**
Volume asymptotics

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$$\text{vol}(B)^{-1} \cdot \int_{\|\rho(g)\| \leq B} f(\rho(g))d\mu(g) \to \int_{\text{PEnd}(V)} f(\rho(g))d\mu_\infty(g),$$

where the limit measure $\mu_\infty$ is supported on a $G$ bi-invariant submanifold of $\text{PEnd}(V)$.

The proof uses the $Ka^+K$-decomposition and integration formula.
The computation of asymptotics of volumes of adelic “height balls” was an open problem, in many cases.
Develop a geometric framework which is
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  - applicable to cubic surfaces and algebraic groups,
Goal

Develop a geometric framework which is

- applicable in the analytic and adelic setup,
- applicable to cubic surfaces and algebraic groups,
- applicable in the study of rational and integral points.
Heights

- $F/\mathbb{Q}$ number field
- $X = X_F$ projective algebraic variety over $F$
- $X(F)$ its $F$-rational points
- $\mathcal{L} = (L, (\| \cdot \|_v))$ adelicly metrized very ample line bundle
- $H_{\mathcal{L}} : X(F) \to \mathbb{R}_{>0}$ associated height, depends on the metrization (choice of norms)
- $H_{\mathcal{L}}$ is not invariant with respect to field extensions
- $H_{\mathcal{L}+\mathcal{L}'} = H_{\mathcal{L}} \cdot H_{\mathcal{L}'}$ (height formalism)
Let $X$ be a smooth projective Fano variety of dimension $d$ over a number field $F$. Assume that $-K_X$ is equipped with an adelic metrization.

For $x \in X(F_v)$ choose local analytic coordinates $x_1, \ldots, x_d$, in a neighborhood $U_x$. In $U_x$, a section of the canonical line bundle has the form $s := dx_1 \wedge \ldots \wedge dx_d$. Put

$$\omega_{K_X,v} := \|s\|_v dx_1 \cdots dx_d,$$

where $dx_1 \cdots dx_d$ is the standard normalized Haar measure on $F_v^d$. This local measure globalizes to $X(F_v)$. 
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where $dx_1 \cdots dx_d$ is the standard normalized Haar measure on $F_v^d$. This local measure globalizes to $X(F_v)$. For almost all $v$,

$$\int_{X(F_v)} \omega_{K_X,v} = \frac{X(\mathbb{F}_q)}{q^d}.$$
Choose a finite set of places $S$, and put

$$\omega_{K_X} := L_S^*(1, \mathrm{Pic}(\bar{X})) \cdot |\text{disc}(F)|^{-1} \cdot \prod_v \lambda_v \omega_{K_X, v},$$

with $\lambda_v = L_v(1, \mathrm{Pic}(\bar{X}))^{-1}$ for $v \notin S$ and $\lambda_v = 1$, otherwise. Put

$$\tau(K_X) := \int_{X(F) \subset X(\mathbb{A}_F)} \omega_{K_X}.$$
Choose a finite set of places $S$, and put

$$\omega_{\mathcal{K}_X} := L^*_S(1, \text{Pic}(\tilde{X})) \cdot |\text{disc}(F)|^{-1} \cdot \prod_{\nu} \lambda_{\nu} \omega_{\mathcal{K}_X,\nu},$$

with $\lambda_{\nu} = L_{\nu}(1, \text{Pic}(\tilde{X}))^{-1}$ for $\nu \notin S$ and $\lambda_{\nu} = 1$, otherwise. Put

$$\tau(\mathcal{K}_X) := \int_{X(F) \subset X(\mathbb{A}_F)} \omega_{\mathcal{K}_X}.$$

This constant appears in the constant $c = c(-\mathcal{K}_X)$ in Manin’s conjecture above.
Let $X$ be a smooth projective variety over a local field $F$, $D$ an effective divisor on $X$, $f_D$ the canonical section of $\mathcal{O}_X(D)$, and $U = X \setminus |D|$. 
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A form $\omega \in \Omega^d(U)$ defines a measure $|\omega|$ as before.
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A metrization of the canonical line bundle $K_X$ gives a global measure on $X(F)$

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A metrization of $K_X(D)$ defines a measure on $U(F)$

$$\tau(X,D) = |\omega|/||\omega f_D||.$$
When $X$ is an equivariant compactification of an algebraic group $G$ and $\omega$ a left-invariant differential form on $G$, we have $\text{div}(\omega) = -D$, so that $K_X(D)$ is a trivial line bundle, equipped with a canonical metrization. We may assume that its section $\omega f_D$ has norm 1. Then

$$\tau(X,D) = |\omega|/\|\omega f_D\| = |\omega|$$

is a Haar measure on $G(F)$. 
Let $L$ be an effective divisor with support $|D| = X \setminus U$, equipped with a metrization. Then

$$\{ u \in U(F) \mid \| f_L(u) \| \geq 1/B \}$$

is a height ball, i.e., it is compact of finite measure $\text{vol}(B)$.
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To compute the volume, for $B \to \infty$, we use the Mellin transform

$$Z(s) := \int_0^\infty t^{-s} \text{dvol}(t)$$
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To compute the volume, for $B \to \infty$, we use the Mellin transform

$$Z(s) := \int_0^\infty t^{-s}d\text{vol}(t) = \int_{U(F)} \| f_L \|^s \tau(X,D),$$

combined with a Tauberian theorem.
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where $D_{\alpha}$ are geometrically irreducible, smooth, and intersecting transversally.
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By the transversality assumption, $D_A \subset X$ is smooth, of codimension $\#A$ (or empty).
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By the transversality assumption, $D_A \subset X$ is smooth, of codimension $\#A$ (or empty). Write

$$D = \sum \rho_\alpha D_{\alpha}, \quad L = \sum \lambda_\alpha D_{\alpha}.$$
The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^\circ(F)$ it takes the form

$$\int \prod_\alpha \|f_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha} d\tau x(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_\alpha dx_\alpha dy.$$
The Mellin transform $Z(s)$ can be computed in charts, via partition of unity. In a neighborhood of $x \in D_A^0(F)$ it takes the form

$$\int \prod_{\alpha} \|f_{D_\alpha}\|(x)^{\lambda_\alpha s - \rho_\alpha} d\tau x(x) = \int \prod_{\alpha \in A} |x_\alpha|^{\lambda_\alpha s - \rho_\alpha} \phi(x; y; s) \prod_{\alpha} dx_\alpha dy.$$ 

Essentially, this is a product of integrals of the form

$$\int_{|x| \leq 1} |x|^{s-1} dx.$$ 

Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification $(D_A)$.
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**Abscissa of convergence** $= \max_{D_\alpha(F) \neq \emptyset} \frac{\rho_\alpha - 1}{\lambda_\alpha}$, where $\lambda_\alpha > 0$.

**Order of pole** $= \text{number of } \alpha \text{ that achieve equality};$
Analytic properties of $Z(s)$ are encoded in the combinatorics of the stratification ($D_A$).

**Abscissa of convergence** = $\max_{D_\alpha(F) \neq \emptyset} \frac{\rho_\alpha - 1}{\lambda_\alpha}$;

**Order of pole** = number of $\alpha$ that achieve equality;

**Leading coefficient** = sum of integrals over all $D_A$ of minimal dimension where $A$ consists only of such $\alpha$s.
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Global theory
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Igusa integrals and volume asymptotics
Let $X$ be a smooth projective variety over a number field $F$, $D$ an effective divisor on $X$, $U = X \setminus |D|$. Fix an adelic metric on $K_X(D)$; this defines measures $\tau_{(X,D),\nu}$ on $U(F_\nu)$ for all $\nu$. Assume that

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$ 

Let

$$\text{EP}(U) = \Gamma(U_\bar{\mathbb{F}}, \mathcal{O}_X^*)/\bar{\mathbb{F}}^* - \text{Pic}(U_\bar{\mathbb{F}})/\text{torsion}$$

be the virtual Galois module.
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Let

$$\text{EP}(U) = \Gamma(U_{\overline{F}}, \mathcal{O}_X^*)/\overline{F}^* - \text{Pic}(U_{\overline{F}})/\text{torsion}$$

be the virtual Galois module. Put

$$\lambda_\nu = L_\nu(1, \text{EP}(U)), \quad \nu \nmid \infty, \quad \lambda_\nu = 1, \quad \nu \mid \infty.$$ 

We have a global measure on $U(\mathbb{A}_F)$ given by

$$\tau_{(X,D)} = L^*(1, \text{EP}(U))^{-1} \cdot \prod_\nu \lambda_\nu \tau_{(X,D),\nu}$$

Igusa integrals and volume asymptotics
Height on the adelic space $U(\mathbb{A}_F)$

Let $\mathcal{L} = (L, (\| \cdot \|_v))$ be an adelicly metrized effective divisor supported on $|D|$. This defines a height function on $U(\mathbb{A}_F)$

$$H_{\mathcal{L}}((x_v)) = \prod_v \| f_L(x_v) \|_v^{-1}.$$
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To compute the volume of the height ball

$$\text{vol}(B) := \{ x \in U(\mathbb{A}_F) \mid H_{\mathcal{L}}(x) \leq B \},$$

for $\mathcal{L}$ and $\tau(X,D)$, we use the adelic Mellin transform:

$$Z(s) = \int_0^\infty t^{-s} d\text{vol}(t) = \int_{U(\mathbb{A}_F)} H_{\mathcal{L}}(x)^{-s} d\tau(X,D)(x) = \prod_v \int_{U(F_v)} \cdots.$$
Denef’s formula

Recall that

\[ D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha. \]

Choosing adelic metrics on \( \mathcal{O}_X(D_\alpha) \) one has:

\[ Z(v(s)) = \int_X (F_v) \prod_\alpha \| f \|_{D_\alpha}^s \lambda_\alpha - \rho_\alpha v \, \mathrm{d} \tau_X, \]

By the local analysis, this converges absolutely for \( \Re(s) > \max((\rho_\alpha - 1)/\lambda_\alpha) \).

For almost all \( v \) and \( \Re(s) > (\rho_\alpha - 1)/\lambda_\alpha \), one has

\[ Z(v(s)) = \sum A(D^\circ A)(F_q) q^{\dim X} \prod_{\alpha \in A} q^{-1} s \lambda_\alpha - \rho_\alpha + 1 - 1. \]
Recall that

\[ D = \sum \rho_\alpha D_\alpha, \quad L = \sum \lambda_\alpha D_\alpha. \]

Choosing adelic metrics on \( \mathcal{O}_X(D_\alpha) \) one has:

\[ Z_v(s) = \int_{X(F_v)} \prod_\alpha \| f_{D_\alpha} \|_v^{s\lambda_\alpha - \rho_\alpha} d\tau_{X,v}(x). \]
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Choosing adelic metrics on \( O_X(D_\alpha) \) one has:

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For almost all \( v \) and \( \Re(s) > (\rho_\alpha - 1)/\lambda_\alpha \), one has

\[ Z_v(s) = \sum_A \frac{\#D_A^0(\mathbb{F}_q)}{q^{\dim X}} \prod_{\alpha \in A} \frac{q - 1}{q^{s \lambda_\alpha - \rho_\alpha + 1} - 1}. \]
Analyzing the Euler product

Let $a := \max(\rho_\alpha / \lambda_\alpha)$ and let $A(L, D)$ be the set of $\alpha$ where equality is achieved; put $b = \#A(L, D)$. 
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\lim_{s \to a} Z(s)(s - a)^b \prod_{\alpha \in A(L, D)} \lambda_\alpha = \int_{X(\mathbb{A}_F)} H_E(x)^{-1} \, d\tau_X(x).
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A Tauberian theorem implies the volume asymptotics with respect to \( L \) and \( \tau(X, D) \), for \( B \to \infty \), of the form

\[
B^a \log(B)^{b-1} \left( a(b - 1)! \prod_{\alpha \in A(L, D)} \lambda_\alpha \right)^{-1} \int_{X(\mathbb{A}_F)} H_E(x)^{-1} \, d\tau_X(x).
\]
Integral points

- $F$ number field, $\mathcal{O}_F$ ring of integers
- $S$ finite set of places of $F$, $S \supset S_\infty$
- $X$ smooth projective variety over $F$, $D \subset X$ subvariety
- $\mathcal{D} \subset \mathcal{X}$ models over $\text{Spec}(\mathcal{O}_F)$

A rational point $x \in X(F)$ gives rise to a section

$$\sigma_x : \text{Spec}(\mathcal{O}_F) \to \mathcal{X}.$$ 

A $(\mathcal{D}, S)$-integral point on $X$ is a rational point $x \in X(F)$ such that $\sigma_{x,v} \notin \mathcal{D}_v$ for all $v \notin S$. 

Integral points of bounded height
Let $X$ be a projective equivariant compatification of $G = \mathbb{G}_a^n$, and

$$\bigcup_{\alpha \in \mathcal{A}} D_\alpha = X \setminus G$$

the boundary divisor, whose irreducible components $D_\alpha$ are smooth and intersect transversally. Choose a subset $\mathcal{A}_D \subseteq \mathcal{A}$ and put

$$U = X \setminus \bigcup_{\alpha \in \mathcal{A}_D} D_\alpha.$$
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$$X \setminus U = D = \sum_{\alpha \in A_D} D_\alpha.$$

Let $\mathcal{L}$ be an adelically metrized line bundle on $X$.

**Problem**

Establish an asymptotic formula for

$$N(B) := \#\{\gamma \in G(F) \cap U(\mathbb{O}_F, S) | H_\mathcal{L}(\gamma) \leq B\}.$$
Techniques

Height pairing

\[ G(\mathbb{A}_F) \times \bigoplus_{\alpha} \mathbb{C} D_{\alpha} \rightarrow \mathbb{C} \]
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Height pairing

\[ G(\mathbb{A}_F) \times \bigoplus_{\alpha} \mathbb{C}D_{\alpha} \to \mathbb{C} \]

Height zeta function

\[ Z(g, s) = \sum_{\gamma \in G(F) \cap U(\mathcal{O}_F, s)} H(\gamma g, s)^{-1}, \]

is holomorphic for \( \Re(s) \gg 0 \) and all \( g \).
“Fourier” expansion - “Poisson formula”

\[ Z(g, s) = \sum_{\psi} \hat{H}(s, \psi), \]

a sum over all (automorphic) characters of \( G(\mathbb{A}_F)/G(F) \).
**Techniques**

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**Main term = trivial character**

\[ \int_{G(\mathbb{A}_F) \cap U(\mathcal{O}_F, S)} H(g, s)^{-1}, \]
Techniques

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Main term = trivial character

\[ \int_{G(\mathbb{A}_F) \cap U(\mathcal{O}_F, S)} H(g, s)^{-1}, \]

a volume integral computed above.

Integral points of bounded height
For $L = -(K_X + D)$ we obtain


\[ N(B) \sim c \cdot B \log(B)^{b^{-1}}, \]

\[ b := \text{rk}(\text{Pic}(U)) + \sum_{v \in S} (1 + \dim C_{\text{an}}^{\text{max}}(D)), \]

the analytic Clemens complex of the stratification of $D$, and
Asymptotics

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$$N(B) \sim c \cdot B \log(B)^{b-1},$$

$$b := \text{rk}(\text{Pic}(U)) + \sum_{v \in S} (1 + \dim C^\text{an}_{F_v}(D)),$$

the analytic Clemens complex of the stratification of $D$, and

$$c = \alpha \beta \tau,$$

- $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$;
- $\tau = \tau^S_{(X,D)}(U(\mathcal{O}_S)) \cdot \prod_{v \in S} \left( \sum_{\sigma \in C^\text{an}_{\max,F_v}(D_v)} \tau_v(\sigma) \right)$
- $\tau_v(\sigma)$ Tamagawa volume of $\sigma$, (adjunction!).

Integral points of bounded height
Contributions from nontrivial characters

\[ \hat{H}(s, \psi) = \int_{G(\mathbb{A}_F)} H(g, s)^{-1} \psi(g) \, dg. \]
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For \( D = \emptyset \), i.e., rational points on \( X \), the nontrivial characters contribute a pole of smaller order, coming from the Euler product.
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Only unramified \( \psi \) appear. Uniform bounds needed for summation over the lattice of these \( \psi \) are (relatively) easy to obtain.
Fourier transforms at $\nu \in S$ have poles interacting with the main term.
Complications for integral points

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$$\int_\sigma \prod_{\alpha} |x_\alpha|^{s_\alpha} \psi(u(x)x^\lambda)\phi(x, s, \psi)dx,$$

where $\lambda = (\lambda_\alpha)$ and $\sigma$ is a certain cone in $F^d_v$. 
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We proved uniform bounds on (meromorphic continuations) of these integrals, in all parameters (2009).
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We proved uniform bounds on (meromorphic continuations) of these integrals, in all parameters (2009). Similar integrals appeared in the work of Cluckers (2010) on Analytic van der Corput Lemma....
Summary

- Geometric Igusa integrals (Mellin transforms) allow to compute volume asymptotics of all balls arising in analytic and adelic geometry, in particular, height balls.
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The spectral method to establish asymptotics for the number of integral points of bounded height leads to interesting $\nu$-adic oscillatory integrals. This should allow to establish asymptotics for $\mathcal{O}_F, \mathcal{S}$-integral points on general quasi-projective embeddings of algebraic groups.
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The \textit{spectral} method to establish asymptotics for the number of integral points of bounded height leads to interesting $\nu$-adic oscillatory integrals. This should allow to establish asymptotics for $\mathcal{O}_{F,S}$-integral points on general quasi-projective embeddings of algebraic groups.

A framework to generalize Manin’s conjectures to \textit{integral} points.