SKELETON AND DUAL COMPLEX

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This is a note to the talk I gave in Simons Symposium on Non-Archimedean and Tropical Geometry, held on February 2-6, organized by Matthew Baker and Sam Payne. I want to thank them for the invitation!

Here we give a map on the recent joint works [dFKX12], [NX13] and [KX15], which is our attempt to study the dual complex and establish its relationship with the topology of the analytification of a variety defined over $K = \mathbb{C}[[t]]$. We also ask some questions which are worthy to be further studied.

1. Essential skeleton

Let $X$ be a normal variety, and $D$ a Weil divisor on $X$. If $D = \sum D_i$ has the following two properties

(1) An irreducible component $W$ of intersections $\bigcap_{i=1,\ldots,k} D_i$ are all normal.

(2) $W$ is of codimension $k$ in $X$.

Then we can construct a dual complex $\mathcal{D}(D)$ by associate $W$ a $k$-dimensional cell attaching on the vertices $v_1,\ldots,v_k$.

A very useful category of pairs satisfying the above assumption comes from a dlt pair $(X,\Delta)$, where we choose $D = \Delta = 1$.

Definition 1.1. A log pair $(X,\Delta)$ is called divisorial log terminal (dlt), if there is an open set $U \subset X$ such that $U$ is smooth and $\Delta|_U$ is a reduced simple normal crossing divisor, and for any divisorial valuation $E$ with center on $X \setminus U$, we have $a(E,X,\Delta) > -1$.

Then we know that $D = \Delta = 1$ satisfies our assumptions (1) and (2), so we can define $\mathcal{D}(D)$.

Definition 1.2. For any log pair $(Y,E)$, if $K_Y + E$ is $\mathbb{Q}$-Cartier, then we can define a partial resolution, called dlt modification $f : (X,\Delta) \to Y$ satisfying

(1) Let $\Delta$ be the sum of the birational transform of $E$ and the reduced exceptional divisor, then $(X,\Delta)$ is dlt.

(2) $K_X + \Delta$ is $f$-nef.

In [dFKX12], we investigate how the dual complex changes under the minimal model program. As a corollary, we show in various cases, the modification is indeed a collapse (which is a special kind of deformation retract). In particular, we have the following which gives interesting results in the Berkovich space setting.

Theorem 1.3 ([dFKX12]). Let $X_K$ be a smooth projective variety over $K = k((t))$. Let $X_i$ ($i=1,2$) be two projective models over an algebraic curve whose base changes give $X_K$. Assume there is a morphism $f : X_1 \to X_2$ and $(X_i,(X_i)_{\text{red}})$ are dlt, where $X_i$ are the special fibers.

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Then $\mathcal{D}(X_1)$ collapses to $\mathcal{D}(X_2)$.

When $(X_i, (X_i)_{\text{red}})$ are snc, this kind of results can be obtained by weak factorization theorem. But for the general case, we need to invoke the minimal model program.

Now if $X$ is a smooth model which is a base change of a relatively projective algebraic model $\mathcal{X}$, such that $(\mathcal{X}, X_{\text{red}})$ is dlt. Assume $K_{\mathcal{X}, K}$ is semi ample, i.e., $|mK_{\mathcal{X}, K}|$ is base point free for some $m > 0$. Then we can run a relative minimal model program to obtain $\mathcal{X}_{\text{min}}$, which satisfies that $(\mathcal{X}_{\text{min}}, X_{\text{red}}^{\text{min}})$ is dlt and $K_{\mathcal{X}_{\text{min}}} + X_{\text{red}}^{\text{min}}$ is relatively semi ample. This model $\mathcal{X}_{\text{min}}$ is important, since we have the following

**Theorem 1.4** ([NX13]). The Kontsevich-Soibelman essential skeleton of $X_{K}^{\text{an}}$ is naturally isomorphic to $\mathcal{D}(X_{\text{red}}^{\text{min}})$.

Recall that the Kontsevich-Soibelman essential skeleton defined in [MN12] is a natural subspace embedded in $X^{\text{an}}$, which does not depend on the choice of the models.

**Corollary 1.5.** The analytic space $X_{K}^{\text{an}}$ admits a deformation retract to the essential skeleton.

Here the deformation retract we choose depends on the MMP process, which is in general not unique. So it is natural to ask

**Question 1.6.** Does there exist a more canonically defined way to yield the deformation retract?

2. **Topology of the dual complex**

In [dFKX12], we show the following

**Theorem 2.1.** If $X_K$ is a rationally connected variety, then $\mathcal{D}(X_{\text{red}})$ is always contractible.

Later this result is also used in a relative setting in [BF14]. One interesting question is that the process of minimal model program indeed yields a special component, called the Kollár component. It depends on the MMP process, so in general it is not unique. However, it has been shown that it carries interesting geometric properties. A natural question is

**Question 2.2.** Study Kollár component from the Berkovich viewpoint.

When $X_K$ is of general type, we do not know many non-trivial restrictions on $\mathcal{D}(X_{\text{red}})$. So the most interesting case seems to be on the border line when $X_K$ is a Calabi-Yau, which also natural appears in many other questions.

A probably naive question is the following,

**Question 2.3.** If $X_K$ is a simply connected Calabi-Yau manifold, such that $H^i(X_K, \mathcal{O}) = 0$ for any $0 < i < \dim X$.

Let $\mathcal{X}$ be a semistable model with maximal degeneration and $K_{\mathcal{X}} \sim 0$, then $\mathcal{D}(X)$ is isomorphic to the sphere $S^{\dim X}$. 

This question is far from known to be true. In a more general setting, given any $K_X$ with $K_X \sim_\mathbb{Q} 0$, we can study the dual complex of $\mathcal{D}(X_{\text{min}})$. For any $E \subset X_{\text{min}}$, the link of each vertex $v_E$ in this complex is given by $\mathcal{D}(D_E)$. Here we define 

$$(K_{X_{\text{min}}} + X_{\text{red}})|_E = K_E + \Delta_E,$$

and $D_E = \Delta_E^{-1}$. Thus $(E, \Delta_E)$ is a dlt log Calabi-Yau pair.

Then it is natural to study $\mathcal{D}(D)$ for any dlt log Calabi-Yau pair $(X, \Delta)$. Using MMP, we obtain the following results

**Theorem 2.4 ([KX15]).** Let $(X, \Delta)$ be a dlt log Calabi-Yau pair and $D = \Delta^{-1}$. Then

1. $H^i(\mathcal{D}(D), \mathbb{Q}) = 0$ for $0 < i < \dim \mathcal{D}(D)$.
2. If $\dim \mathcal{D}(D) > 1$, then $\pi_1(X_{\text{sm}})$ admits a surjection to $\pi_1(\mathcal{D}(D))$. In particular, the pro-finite completion $\hat{\pi}_1(\mathcal{D}(D))$ is finite.

**Corollary 2.5.** Question 2.3 has an affirmative answer when $\dim(X_K) \leq 4$.

The main idea of proving Theorem 2.4 is using MMP to construct a new birational log Calabi-Yau lc model $(X', \Delta')$, such that $D'$ supports an ample divisor. Then after doing some standard birational modification, we can conclude.

Since after Theorem 2.4, we have a good understanding of the rational homology and fundamental group, the remaining important question is

**Question 2.6.** How to compute $H^i(\mathcal{D}(D), \mathbb{Z})$? Or similarly, how to compute $H^i(\mathcal{D}(X_{\text{min}}), \mathbb{Z})$? Is $H^i(\mathcal{D}(X), \mathbb{Z}) = 0$ in the setting of Question 2.3 for $0 < i < \dim(X_K)$?

**References**


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