Square-Integrable Automorphic Forms

- $G$ a reductive algebraic group defined over a number field $F$.
- $\mathbb{A}$ is the ring of adeles of $F$.
- $X_G := G(F) \backslash G(\mathbb{A})^1$, where $G(\mathbb{A})^1 := \cap_{\chi \in X^*(G)} \ker \left| \chi \right|_\mathbb{A}$.
- $L^2(X_G)$ denotes the space of functions: $\phi : X_G \to \mathbb{C}$ such that
  $$\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} |\phi(g)|^2 dg < \infty.$$
- $\mathcal{A}_2(G)$ is the set of equivalence classes of irreducible unitary representations of $G(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\text{disc}}(X_G)$.
- $\mathcal{A}_{\text{cusp}}(G)$ is the subset of $\mathcal{A}_2(G)$ consisting of those automorphic representations of $G(\mathbb{A})$ occurring in the cuspidal spectrum $L^2_{\text{cusp}}(X_G)$. 
Theorem (Arthur, Mok, Kaletha-Minguez-Shin-White)

Let $G^*$ be an $F$-quasisplit classical group and $G$ be a pure inner form of $G^*$ over $F$. For any $\pi \in \mathcal{A}_{\text{cusp}}(G)$, there is a global Arthur parameter $\psi \in \tilde{\Psi}_2(G^*)$, which is $G$-relevant, such that

$$\pi \in \tilde{\Pi}_\psi(G)$$

where $\tilde{\Pi}_\psi(G)$ is the global Arthur packet of $G$ associated to $\psi$.

We may form the global Arthur-Vogan packet as union of the global Arthur packets $\tilde{\Pi}_\psi(G)$ over all the pure inner forms $G$ of $G^*$:

$$\tilde{\Pi}_\psi[G^*] := \bigcup_G \tilde{\Pi}_\psi(G).$$
Global Arthur Parameters $\tilde{\Psi}_2(G)$: Examples

- $G^* = \text{SO}_{2n+1}^*$, $F$-split, and $(G^*)^\vee = \text{Sp}_{2n}(\mathbb{C})$.
- Each $\psi \in \tilde{\Psi}_2(G^*)$ (global Arthur parameters) is written as a formal sum of simple Arthur parameters:

$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r$$

where $\psi_i = (\tau_i, b_i)$, with $\tau_i \in \mathcal{A}_{\text{cusp}}(\text{GL}_{a_i})$; $a_i, b_i \geq 1$; and $\sum_{i=1}^r a_i b_i = 2n$.
- If $i \neq j$, either $\tau_i \not\cong \tau_j$ or $b_i \neq b_j$, with the parity condition that $a_i \cdot b_i$ is even and $\psi_i \in \tilde{\Psi}_2(\text{SO}_{a_i b_i+1}^*)$.
- **Endoscopy Structure:** $\displaystyle 2n = \sum_{i=1}^r a_i \cdot b_i$,

$$\text{SO}_{a_1 \cdot b_1 + 1}^* \times \cdots \times \text{SO}_{a_r \cdot b_r + 1}^* \quad \Longrightarrow \quad \text{SO}_{2n+1}^*$$

$$\tilde{\Pi}_{\psi_1}(\cdot) \otimes \cdots \otimes \tilde{\Pi}_{\psi_r}(\cdot) \quad \Longrightarrow \quad \tilde{\Pi}_{\psi}(\cdot)$$
Global Arthur Parameters $\widetilde{\Psi}_2(G)$: Examples

- $G^* = \text{Sp}^*_{2n}$, $F$-split, and $(G^*)^\vee = \text{SO}_{2n+1}(\mathbb{C})$.

- Each $\psi \in \widetilde{\Psi}_2(G^*)$ is written as a formal sum of simple Arthur parameters:

$$\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r$$

where $\psi_i = (\tau_i, b_i)$, with $\tau_i \in \mathcal{A}_{\text{cusp}}(\text{GL}_{a_i})$; $a_i, b_i \geq 1$; $\sum_{i=1}^r a_i b_i = 2n + 1$; and $\prod_{i=1}^r \omega_{\tau_i}^{b_i} = 1$.

- If $i \neq j$, either $\tau_i \not\sim \tau_j$ or $b_i \neq b_j$, with the parity:
  1. If $a_i \cdot b_i$ is even, then $\psi_i \in \widetilde{\Psi}_2(\text{SO}_{a_i b_i}^*)$;
  2. If $a_i \cdot b_i$ is is odd, then $\psi_i \in \widetilde{\Psi}_2(\text{Sp}_{a_i b_i}^*)$.

- **Endoscopy Structure:** $2n + 1 = \sum_{i=1}^r a_i \cdot b_i$,

$$\prod_{a_i b_i = 2l_i} \text{SO}_{2l_i}^* \times \prod_{a_j b_j = 2l_j + 1} \text{Sp}_{2l_j}^* \implies \text{Sp}^*_{2n}$$

$$\otimes_{a_i b_i = 2l_i} \widetilde{\Pi}_{\psi_i}(\cdot) \otimes \otimes_{a_j b_j = 2l_j + 1} \widetilde{\Pi}_{\psi_j}(\cdot) \implies \widetilde{\Pi}_{\psi}(\cdot)$$
Global Arthur Parameters $\tilde{\Psi}_2(G)$: Examples

- A parameter $\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \in \tilde{\Psi}_2(G^*)$ is **generic** if $b_1 = \cdots = b_r = 1$.

- **Generic global Arthur parameters** $\phi \in \tilde{\Phi}_2(G^*)$ are:

$$
\phi = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_r, 1)
$$

with $\tau_i \in A_{\text{cusp}}(GL_{a_i})$ that $\tau_i \not\sim \tau_j$ if $i \neq j$. They are of either **symplectic** or **orthogonal** type, depending on $G^*$.

- The pure inner forms of $G^* = SO^*_m$ are $G = SO_m(V, q)$ for non-deg. quad. spaces $(V, q)$ over $F$ with the same dimension and discriminant.

- If $G$ is a pure inner form of $G^*$, then $L_G = L_{G^*}$.

- For $\phi \in \tilde{\Phi}_2(G^*)$, the endoscopic classification may define the global Arthur packet $\tilde{\Pi}_\phi(G^*)$ and also define the global Arthur packet $\tilde{\Pi}_\phi(G)$, which is non-empty if $\phi$ is $G$-relevant.
Endoscopic Classification and Langlands Functoriality

\[ \mathcal{A}(GL_{NG}) \]

\[ \pi_\psi \]

\[ \uparrow \]

\[ \tilde{\Psi}_2(G^*)_G \]

\[ \psi \]

\[ \downarrow \quad \downarrow \]

\[ \mathcal{A}_2(G) \cap \tilde{\Pi}_\psi(G) \quad \iff \quad \tilde{\Pi}_\psi(G^*) \cap \mathcal{A}_2(G^*) \]
A Simple Question: \( \tilde{\Pi}_\psi(G) \cap A_{\text{cusp}}(G) = \emptyset \)?

If \( \tilde{\Pi}_\psi(G) \cap A_{\text{cusp}}(G) \neq \emptyset \), call \( \psi \) cuspidal.

What can one say about the cuspidal \( \psi \)?

Write \( \psi = (\tau_1, b_1) \boxplus \cdots \boxplus (\tau_r, b_r) \). How to bound these integers \( b_1, \cdots, b_r \) if \( \psi \) is cuspidal?

This leads to a Ramanujan type bound for \( A_{\text{cusp}}(G) \).

For \( \pi \in A_{\text{cusp}}(G) \), how to determine which \((\tau, b)\) occurs in the global Arthur parameter \( \psi \) of \( \pi \)?

This leads to the \((\tau, b)\)-theory that characterizes the \((\tau, b)\) factor of \( \pi \) in terms of basic invariants of \( \pi \).

If \( \psi \) is cuspidal, how to construct explicit modules for the members in \( \tilde{\Pi}_\psi(G) \cap A_{\text{cusp}}(G) \)?

This leads to the theory of twisted automorphic descents and endoscopy correspondences via integral transforms.
Remarks

- If $G^*$ is $F$-quasisplit, the automorphic descent of Ginzburg-Rallis-Soudry constructs a **generic** member in $\widetilde{\Pi}_\phi(G^*) \cap A_{\text{cusp}}(G^*)$ for each generic global Arthur parameter $\psi = \phi$.

- If $G$ is a pure inner form of $G^*$, **all members** in the set $\widetilde{\Pi}_\phi(G) \cap A_{\text{cusp}}(G)$ can be constructed by using the **twisted automorphic descent** developed in my work with Lei Zhang, assuming the extended Arthur-Burger-Sarnak principle.

- A different approach is taken up also joint with Baiying Liu on this issue.

- The idea is to consider Fourier coefficients of automorphic representations associated to nilpotent orbits, which leads to the information on the **automorphic wave-front set**.
Fourier Coefficients and Nilpotent Adjoint Orbits

- $G^*$ is an $F$-quasi-split classical group and $\mathfrak{g}^*$ is the Lie algebra.
- Let $N_{G^*}$ be the dimension for the defining embedding $G^* \to \text{GL}(N_{G^*})$.
- Over algebraic closure $\overline{F}$ of $F$, all the nilpotent elements in $\mathfrak{g}^*(\overline{F})$ form a conic algebraic variety, called the nilcone $\mathcal{N}(\mathfrak{g}^*)$.
- Under the adjoint action of $G^*$, $\mathcal{N}(\mathfrak{g}^*)$ decomposes into finitely many adjoint $G^*$-orbits $O$, which are parameterized by the corresponding partitions of $N = N_{G^*}$ of type $G^*$.
- Over $F$, each $\overline{F}$-orbit reduces to an $F$-stable adjoint $G^*(F)$-orbits $O^{st}$, and hence the $F$-stable adjoint orbits in the nilcone $\mathcal{N}(\mathfrak{g}^*)$ are also parameterized by the corresponding partitions of an integer $N = N_{G^*}$ of type $G^*$. 
Fourier Coefficients and Nilpotent Adjoint Orbits

- For $X \in \mathcal{N}(\mathfrak{g}^*)$, use $\mathfrak{sl}_2$-triple (over $F$) to define a unipotent subgroup $V_X$ and a character $\psi_X$.
- Let $\{X, H, Y\}$ be an $\mathfrak{sl}_2$-triple (over $F$). Under the adjoint action of $\text{ad}(H)$,

$$
\mathfrak{g}^* = \mathfrak{g}_{-r} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r.
$$

- $\text{Ad}(G^*)(Y) \cap \mathfrak{g}_{-2}$ and $\text{Ad}(G^*)(X) \cap \mathfrak{g}_2$ are Zariski dense in $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$, respectively.
- Take $V_X$ to be the unipotent subgroup of $G^*$ such that the Lie algebra of $V_X$ is equal to $\bigoplus_{i \geq 2} \mathfrak{g}_i$.
- Let $\psi_F$ be a non-trivial additive character of $F \backslash \mathbb{A}$. The character $\psi_X$ of $V_X(F)$ or $V_X(\mathbb{A})$ is defined by

$$
\psi_X(v) = \psi_F(\text{tr}(Y \log(v))).
$$
The Fourier coefficient of $\varphi \in \pi \in A_2(G^*)$ is defined by

$$\mathcal{F}^{\psi X}(\varphi)(g) := \int_{V_X(F) \backslash V_X(\mathbb{A})} \varphi(vg)\psi_X(v)^{-1}dv.$$ 

Since $\varphi$ is automorphic, the nonvanishing of $\mathcal{F}^{\psi X}(\varphi)$ depends only on the $G^*(F)$-adjoint orbit $O_X$ of $X$.

The set $n(\varphi) := \{X \in N(g) \mid \mathcal{F}^{\psi X}(\varphi) \neq 0\}$ is stable under the $G^*(F)$-adjoint action.

Denoted by $p(\varphi)$ the set of partitions $p$ of $N_{G^*}$ of type $G^*$ corresponding to the $F$-stable orbits $O_{p}^{\text{st}}$ that have non-empty intersection with $n(\varphi)$.

$p^m(\varphi)$ is the set of all maximal partitions in $p(\varphi)$, according to the partial ordering of partitions.
Maximal Fourier Coefficients of Automorphic Forms

- For $\pi \in A_2(G)$, denote by $p^m(\pi)$ the set of maximal members among $p^m(\varphi)$ for all $\varphi \in \pi$.
- We would like to know:
  - *How to determine $p^m(\pi)$ in terms of other invariants of $\pi$?*
  - *What can one say about $\pi$ based on the structure of $p^m(\pi)$?*
- **Folklore Conjecture:** For any irreducible automorphic representation $\pi$ of $G$, the set $p^m(\pi)$ is singleton.
- Write $\underline{p} = [p_1 p_2 \cdots p_r] \in p^m(\pi)$ with $p_1 \geq p_2 \geq \cdots \geq p_r$.
- *What can we say about the largest part $p_1$ if $\pi$ is cuspidal?*
- This problem can be formulated to have close relation with the extended Arthur-Burger-Sarnak principle, which is important to the theory of twisted automorphic descent.
Examples: $G = \text{GL}_n$, the $G(F)$-stable orbits in $\mathcal{N}(g)$ are parameterized by partitions of $n$.

Theorem (Piatetski-Shapiro; Shalika): If $\pi \in \mathcal{A}_2(\text{GL}_n)$ is cuspidal, $p^m(\pi) = \{ [n] \}$. This says that any irreducible cuspidal automorphic representation has a nonzero Whittaker-Fourier coefficient.

What happens if $\pi \in \mathcal{A}_2(\text{GL}_n)$ is not cuspidal?

Moeglin-Waldspurger Theorem: Any $\pi \in \mathcal{A}_{\text{disc}}(\text{GL}_n)$ has form $\Delta(\tau, b)$ (Speh residue with cuspidal support $\tau \otimes b$), where $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_a)$ and $n = ab$.

If $\pi = \Delta(\tau, b)$, then $p^m(\pi) = \{ [a^b] \}$ (Ginzburg (2006), J.-Liu (2013) gives a complete global proof).

In particular, the Folklore Conjecture is verified for all $\pi \in \mathcal{A}_2(\text{GL}_n)$!
Maximal Fourier Coefficients and Arthur Parameters

- **How to understand this in terms of Arthur parametrization?**
- \( \Delta(\tau, b) \) has the Arthur parameter \( \psi = (\tau, b) \).
- The partition attached to \( \psi \) is \( p_{\psi} := [b^a] \) and \( p^m(\Delta(\tau, b)) = \{[a^b]\} \).
- \( \eta([b^a]) = [a^b] \) is given by the Barbasch-Vogan duality \( \eta \) from \( GL_n^\vee \) to \( GL_n \). In this case, it is just the transpose.
- Take an Arthur parameter for \( GL_n \): for \( \tau_i \in A_{\text{cusp}}(GL_{a_i}) \),
  \[
  \psi = (\tau_1, b_1) \boxplus (\tau_2, b_2) \boxplus \cdots \boxplus (\tau_r, b_r).
  \]
- The partition attached to \( \psi \) is \( p_{\psi} = [b_1^{a_1} b_2^{a_2} \cdots b_r^{a_r}] \).
- The Arthur representation is an isobaric sum
  \[
  \pi_{\psi} = \Delta(\tau_1, b_1) \boxplus \Delta(\tau_2, b_2) \boxplus \cdots \boxplus \Delta(\tau_r, b_r).
  \]
- **Conjecture:** \( p^m(\pi_{\psi}) = \{\eta_{GL_n^\vee, GL_n}(p_{\psi})\} \).
Maximal Fourier Coefficients and Arthur Parameters

For $\psi = \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r \in \tilde{\Psi}_2(G^\ast)$, where $\psi_i = (\tau_i, b_i)$ with $\tau_i \in A_{\text{cusp}}(GL_{a_i})$ and $b_i \geq 1$, $p_{\psi} = [b_1^{a_1} \cdots b_r^{a_r}]$ is the partition of $N_{G^*}^\vee$ attached to $(\psi, (G^*)^\vee)$ and $\eta(p_{\psi})$ is the Barbasch-Vogan duality of $p_{\psi}$ from $(G^*)^\vee$ to $G^*$. 

Conjecture (J.-2014):

(1) For every $\pi \in \tilde{\Pi}_{\psi}(G^*) \cap A_2(G^*)$, any partition $p \in p^m(\pi)$ has the property that $p \leq \eta(p_{\psi})$.

(2) There exists at least one member $\pi \in \tilde{\Pi}_{\psi}(G) \cap A_2(G)$ for some pure inner form $G$ of $G^*$ that have the property: $\eta(p_{\psi}) \in p^m(\pi)$.

Remark: For a pure inner form $G$ of $G^*$, assume that the global Arthur parameter $\psi$ is $G$-relevant and the Barbasch-Vogan duality $\eta(p_{\psi})$ is a $G$-relevant partition of $N_G = N_{G^*}$ of type $G^*$. The definition of Fourier coefficients also work.
Examples of the Barbasch-Vogan duality

- $G = \text{SO}_{2n+1}$ and $2n = ab$; Take $\psi = (\tau, b)$ for $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_a)$, and

$$b = \begin{cases} 
2\ell, & \text{if } \tau \text{ is orthogonal}, \\
2\ell + 1, & \text{if } \tau \text{ is symplectic}.
\end{cases}$$

- $p_\psi = [b^a]$ is the partition of $2n$ of type $(\psi, \text{Sp}_{2n}(\mathbb{C}))$.

- The Barbasch-Vogan duality is given as follows:

$$\eta(p_\psi) = \begin{cases} 
[(a + 1)a^{b-2}(a - 1)1] & \text{if } b = 2\ell \text{ and } a \text{ is even}; \\
[a^b1] & \text{if } b = 2\ell \text{ and } a \text{ is odd}; \\
[(a + 1)a^{b-1}] & \text{if } b = 2\ell + 1.
\end{cases}$$
Examples of the Barbasch-Vogan duality

- Take $G = \text{Sp}_{2n}$ and $\psi = (\tau, 2b + 1) \boxtimes \boxplus_{i=2}^r (\tau_i, 1) \in \widetilde{\Psi}_2(G)$.
- $p_{\psi} = [(2b + 1)^a (1)^{2m+1-a}]$ with $2m + 1 = (2n + 1) - 2ab$.
- When $a \leq 2m$ and $a$ is even,

$$\eta(p_{\psi}) = \eta([(2b + 1)^a (1)^{2m+1-a}]) = [(2b + 1)^a (1)^{2m-a}]^t = [(a)^{2b+1}] + [(2m - a)] = [(2m)(a)^{2b}]$$

- When $a \leq 2m$ and $a$ is odd,

$$\eta(p_{\psi}) = \eta([(2b + 1)^a (1)^{2m+1-a}]) = \eta([(2b + 1)^a (1)^{2m-a}]_{\text{Sp}_{2n}}^t = [(2b + 1)^{a-1} (2b)(2)(1)^{2m-a-1}]^t = [(a - 1)^{2b+1}] + [(1)^{2b}] + [(1)^2] + [(2m - 1 - a)] = [(2m)(a + 1)(a)^{2b-2}(a - 1)].$$
Remarks on the Conjecture

- It is true when $G = \text{GL}_n$ and $\psi$ is an Arthur parameter for the discrete spectrum.

- If $\phi \in \tilde{\Phi}_2(G^*)$ is generic, i.e. $b_1 = \cdots = b_r = 1$, the partition $\underline{p}_\phi = [1\, N(G^*)^\vee]$. 

- The Barbasch-Vogan duality of $\underline{p}_\phi$ is $\eta([1\, N(G^*)^\vee]) = [N_{G^*}]_{G^*}$.

- It is clear that the partition $\eta([1\, N(G^*)^\vee])$ is $G$-relevant only if $G = G^*$ is quasi-split. In this case, it is the regular partition.

- The conjecture claims that any generic global Arthur packet contains a generic member for quasi-split $G^*$, and hence implies the global **Shahidi conjecture** on genericity of tempered packets.

- This special case can be proved by the Arthur-Langlands transfer from $G$ to $\text{GL}_{NG}$ and the Ginzburg-Rallis-Soudry descent.
Remarks on the Conjecture

- The conjecture is known for various cases of $\text{Sp}_{2n}$ (J.-Liu).
- The conjecture provides an upper bound partition for $\pi \in \tilde{\Pi}_\psi(G) \cap A_2(G)$, with a given Arthur parameter $\psi$.
- We are to obtain a lower bound partition, with a given $\psi$, for $\pi \in \tilde{\Pi}_\psi(G) \cap A_2(G)$.
- It is very interesting, but harder problem to determine $p^m(\pi)$, with a given $\psi$, for general members $\pi \in \tilde{\Pi}_\psi(G) \cap A_2(G)$.
- The theory of singular automorphic forms of Howe and Li provides a lower bound partition for $\pi \in A_{\text{cusp}}(G)$.
- It is not hard to find that the lower bound partition provided by the theory of Howe and Li may not be the best for all $\pi \in A_{\text{cusp}}(G)$.
Singular Partitions and Rank in the Sense of Howe

- $G = \text{Sp}_{2n}$. Take $P_n = M_nU_n$ to be the Siegel parabolic of $\text{Sp}_{2n}$, with $M_n \cong \text{GL}_n$ and the elements of $U_n$ are of form

$$u(X) = \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}.$$

- By the Pontryagin duality, one has

$$\hat{U}_n(F) \backslash U_n(\mathbb{A}) \cong \text{Sym}^2(F^n).$$

- For a fixed nontrivial additive character $\psi_F$ of $F\backslash \mathbb{A}$, a $T \in \text{Sym}^2(F^n)$ corresponds to the character $\psi_T$ by

$$\psi_T(u(X)) := \psi_F(\text{tr}(TwX)),$$

with $w$ anti-diagonal, and the action of $\text{GL}_n$ on $\text{Sym}^2(F^n)$ is induced from its adjoint action on $U_n$. 
Singular Partitions and Rank in the Sense of Howe

For an automorphic form $\varphi$ on $\text{Sp}_{2n}(\mathbb{A})$, the $T$- or $\psi_T$-Fourier coefficient is defined by

$$\mathcal{F}^{\psi_T}(\varphi)(g) := \int_{U_m(F) \backslash U_n(\mathbb{A})} \varphi(u(X)g)\psi_T^{-1}(u(X))du(X).$$

An automorphic form $\varphi$ on $\text{Sp}_{2n}(\mathbb{A})$ is called non-singular if $\varphi$ has a nonzero $\psi_T$-Fourier coefficient with a non-singular $T$.

$\varphi$ is called singular if $\varphi$ has the property that if a nonzero $\psi_T$-Fourier coefficient $\mathcal{F}^{\psi_T}(\varphi)$ is nonzero, then $\det(T) = 0$.

Howe shows in 1981 that if an automorphic form $\varphi$ on $\text{Sp}_{2n}(\mathbb{A})$ is singular, then $\varphi$ can be expressed as a linear combination of certain theta functions.

Jianshu Li shows in 1989 that any cuspidal automorphic form of $\text{Sp}_{2n}(\mathbb{A})$ is non-singular.
For a split $\text{SO}_m$ defined by a non-deg. quad. space $(V, q)$ over $F$ of dim $m$ with the Witt index $\lfloor \frac{m}{2} \rfloor$, let $X^+$ be a maximal totally isotropic subspace of $V$ with dim $\lfloor \frac{m}{2} \rfloor$ and let $X^-$ be the dual to $X^+$:

$$V = X^- + V_0 + X^+$$

with $V_0$ the orth. complement of $X^- + X^+$ (dim $V_0 \leq 1$).

The generalized flag $\{0\} \subset X^+ \subset V$ defines a maximal parabolic subgroup $P_{X^+}$, whose Levi part $M_{X^+}$ is isomorphic to $\text{GL}[\frac{m}{2}]$ and whose unipotent radical $N_{X^+}$ is abelian if $m$ is even; and is a two-step unipotent subgroup with its center $Z_{X^+}$ if $m$ is odd. When $m$ is even, we set $Z_{X^+} = N_{X^+}$. 
By the Pontryagin duality, \( Z_{X+}(\overline{F}) \backslash Z_{X+}(\mathbb{A}) \cong \wedge^2(F^{[m/2]}) \).

For any \( T \in \wedge^2(F^{[m/2]}) \),

\[
\psi_T(z(X)) := \psi_F(\text{tr}(TwX)).
\]

For an automorphic form \( \varphi \) on \( G(\mathbb{A}) \), the \( T \) or \( \psi_T \) Fourier coefficient is defined by

\[
\mathcal{F}^{\psi_T}(\varphi)(g) := \int_{Z_{X+}(\overline{F}) \backslash Z_{X+}(\mathbb{A})} \varphi(z(X)g)\psi_T^{-1}(z(X))dz(X).
\]

An automorphic form \( \varphi \) on \( G(\mathbb{A}) \) is called non-singular if \( \varphi \) has a non-zero \( \psi_T \)-Fourier coefficient with \( T \in \wedge^2(F^{[m/2]}) \) of maximal rank.
Denote by $p_{\text{ns}}$ the partition corresponding to the non-singular Fourier coefficients.

For $\text{Sp}_{2n}$, $p_{\text{ns}} = [2^n]$, which is a special partition for $\text{Sp}_{2n}$.

For $\text{SO}_{2n+1}$, one has

$$p_{\text{ns}} = \begin{cases} [2^{2e} 1] & \text{if } n = 2e; \\ [2^{2e} 1^3] & \text{if } n = 2e + 1. \end{cases}$$

This is not a special partition of $\text{SO}_{2n+1}$.

For $\text{SO}_{2n}$, one has

$$p_{\text{ns}} = \begin{cases} [2^{2e}] & \text{if } n = 2e; \\ [2^{2e} 1^2] & \text{if } n = 2e + 1. \end{cases}$$

This is a special partition of $\text{SO}_{2n}$. 
J.-Liu-Savin show 2015 that for any automorphic representation $\pi$, the set $p_m(\pi)$ contains only special partitions.

For $SO_{2n+1}$, $p_{ns} \notin p_m(\pi)$ for any $\pi \in A_{\text{cusp}}(SO_{2n+1})$.

Any $p \in p_m(\pi)$ as $\pi$ runs in $A_{\text{cusp}}(SO_{2n+1})$ must be bigger than or equal to the following partition

$$p_{SO_{2n+1}} = \begin{cases} [32^{2e-2}1^2] & \text{if } n = 2e; \\ [32^{2e-2}1^4] & \text{if } n = 2e + 1. \end{cases}$$

$p_{G_n}^{G_n}$ denotes the $G_n$-expansion of the partition $p_{ns}$, i.e., the smallest special partition which is bigger than or equal to $p_{ns}$.

**Proposition:** For a split classical group $G_n$, $p_{ns}^{G_n}$ is a lower bound for partitions in the set $p_m(\pi)$ as $\pi$ runs in $A_{\text{cusp}}(G_n)$. 
It is natural to ask whether the lower bound $p_{ns}^{G_n}$ is sharp?

For $n = 2e$ even, and $F$ to be totally real, the Ikeda lifting gives $\pi \in A_{cusp}(Sp_{4e})$ with the global Arthur parameter $(\tau, 2e) \boxplus (1, 1)$, where $\tau \in A_{cusp}(GL_2)$ is self-dual and has the trivial central character.

First, for any $p \in p^m(\pi)$, one has $p \leq [2^{2e}] = p_{ns}^{Sp_{4e}}$.

Then, one also has $p_{ns}^{Sp_{4e}} = [2^{2e}] \leq p$.

**Proposition:** If $F$ is totally real, the Ikeda construction $\pi$ of $Sp_{4e}$ has $p^m(\pi) = \{p_{ns}^{Sp_{4e}}\} = \{[2^{2e}]\}$, and hence the non-singular partition $p_{ns}^{Sp_{4e}} = p_{ns} = [2^{2e}]$ is the sharp lower bound for $A_{cusp}(Sp_{4e})$.

Note that the construction of Ikeda is not known when $F$ is not totally real or $n = 2e + 1$ is odd.

What happens in such situations?
Singular Partitions and Small Cuspidal Representations

- J.-Liu (2015): If $F$ is totally imaginary and $n \geq 5$, the global Arthur packet $\tilde{\Pi}(\tau, n) \boxtimes (\epsilon, 1)(\text{Sp}_{2n})$, with $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_2)$ self-dual, has no cuspidal members, where $\epsilon = 1$ if $n = 2e$; and $\epsilon = \omega_\tau$ if $n = 2e + 1$.

- In this case, the construction of the Ikeda lifting is **impossible**!

- **Theorem (J.-Liu 2015):**
  1. For any $\pi \in \mathcal{A}_{\text{cusp}}(\text{Sp}_{2n})$, $p^m(\pi) = [2^n]$ if and only if $\pi$ is hypercuspidal in the sense of I. Piatetski-Shapiro.
  2. For $\psi = (\tau, 2e) \boxtimes (1, 1) \in \tilde{\Psi}_2(\text{Sp}_{4e})$ with $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_2)$, any cuspidal $\pi \in \tilde{\Pi}_\psi(\text{Sp}_{4e})$ has that $p^m(\pi) = [2^{2e}]$.
  3. For $\psi = (\tau, 2e + 1) \boxtimes (\omega_\tau, 1) \in \tilde{\Psi}_2(\text{Sp}_{4e+2})$ with $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_2)$, any cuspidal $\pi \in \tilde{\Pi}_\psi(\text{Sp}_{4e+2})$ has that $p^m(\pi) = [2^{2e+1}]$.
  4. For $\psi = (\tau, 2e + 1) \in \tilde{\Psi}_2(\text{Sp}_{6e+2})$ with $\tau \in \mathcal{A}_{\text{cusp}}(\text{GL}_3)$, any cuspidal $\pi \in \tilde{\Pi}_\psi(\text{Sp}_{6e+2})$ has that $p^m(\pi) = [2^{3e+1}]$. 
Theorem (J.-Liu 2015)

Assume that $F$ is totally imaginary, and Part (1) of the Conjecture holds. For $\psi = \bigoplus_{i=1}^{r}(\tau_i, b_i)$ in $\tilde{\Psi}_2(\text{Sp}_{2n})$ with $\tau_i \in \mathcal{A}_{\text{cusp}}(\text{GL}_{a_i})$ for $i = 1, 2, \cdots, r$, $2n + 1 = \sum_{i=1}^{r} a_i b_i$, and $\prod_{i=1}^{r} \omega_{\tau_i}^{b_i} = 1$, there exist constants $N_{a} \geq N_{a,b}^{(1)} \geq N_{a,b}^{(2)}$, depending on $(a_1, \cdots, a_r)$ and $(b_1, \cdots, b_r)$, such that if $2n > N_0$ for $N_0$ to be one of the $N_{a}, N_{a,b}^{(1)}, N_{a,b}^{(2)}$, the set $\tilde{\Pi}_{\psi}(\text{Sp}_{2n}) \cap \mathcal{A}_{\text{cusp}}(\text{Sp}_{2n})$ is empty.

Remarks:

1. $N_{a} = \begin{cases} (\sum_{i=1}^{r} a_i)^2 + 2(\sum_{i=1}^{r} a_i) & \text{if } \sum_{i=1}^{r} a_i \text{ is even;} \\ (\sum_{i=1}^{r} a_i)^2 - 1 & \text{otherwise.} \end{cases}$

2. $N_{a,b}^{(1)}, N_{a,b}^{(2)}$ are sharper bound, but not easy to be defined.

3. Conjecture: $N_{a,b}^{(2)}$ is sharp in the sense that if $2n = N_{a,b}^{(2)}$, then the set $\tilde{\Pi}_{\psi}(\text{Sp}_{2n}) \cap \mathcal{A}_{\text{cusp}}(\text{Sp}_{2n})$ is not empty
Example (1): Take $\psi = (\tau_1, 1) \boxplus (\tau_2, 8)$ with $\tau_1 \in A_{\text{cusp}}(GL_5)$ and $\tau_2 \in A_{\text{cusp}}(GL_2)$. We have

$$N_a = 48 > N_{a,b}^{(1)} = 24 > N_{a,b}^{(2)} = 16.$$ 

Example (2): Take $\psi = (1, b_1) \boxplus (\tau, b_2)$ with $b_1 \geq 1$ odd, $\tau \in A_{\text{cusp}}(GL_2)$ of symplectic type and $b_2$ even. Then

$$N_a = N_{a,b}^{(1)} = N_{a,b}^{(2)} = 8$$

and the global Arthur packet $\tilde{\Pi}_\psi(Sp^*)$ contains no cuspidal members except that

$$(b_1, b_2) = (1, 2), (1, 4), (3, 2), \text{ or } (5, 2).$$

Mœglin (2008, 2011) also provide criterion for the cuspidality of global Arthur packets, which is in a different nature from what we developed here.
(τ, b)-theory and the Ramanujan Type Bound

- This is to bound b for (τ, b) to occur in π ∈ A_{cusp}(Sp_{2n}).
- For π ∈ A_2(Sp_{2n}), if (τ, b) occurs in π, then b ≤ 2n (sharp!).

- **Theorem (Kudla-Rallis (1994))**
  For τ = χ with χ^2 = 1, and π ∈ A_{cusp}(Sp_{2n}), if (χ, b) occurs in π, then b ≤ 2[n^2] + 1.
  - This bound can also be deduced from my Conjecture.
  - If (τ, b) occurs in π ∈ A_{cusp}(Sp_{2n}), then b ≤ 2[n^2] + 1.
  - **Question:** Is 2[n^2] + 1 sharp upper bound for b?
  - When n is **even**, Piatetski-Shapiro and Rallis (1987) constructed a π_{(χ,n+1)} ∈ A_{cusp}(Sp_{2n}) with a global Arthur parameter (χ, n + 1), and hence this bound 2[n^2] + 1 is sharp!
  - **Question:** When n is **odd**, how to construct a π_{(χ,n)} in A_{cusp}(Sp_{2n}) with a global Arthur parameter (χ, n)?
(τ, b)-theory and the Ramanujan Type Bound

- For \( \pi \in A_{\text{cusp}}(\text{Sp}_{2n}) \), one reads the Satake parameters of \( \pi \) at a unramified local place \( v \) from
  \[
  \text{Ind}_{B(F_v)}^{\text{Sp}_{2n}(F_v)} \chi_1|\cdot|^{\alpha_1} \otimes \chi_2|\cdot|^{\alpha_2} \otimes \cdots \otimes \chi_n|\cdot|^{\alpha_n},
  \]
  where \( B \) is the standard Borel of \( \text{Sp}_{2n} \), and \( \chi_i \)s are unitary.

- For \( \theta \in \mathbb{R}_{\geq 0} \), we say that \( \pi \) has \( R(\theta) \) if at each unramified local place \( v \), one has \( 0 \leq \alpha_i \leq \theta \) for all \( i \).

- When \( n \) is even, \( \pi(\chi,n+1) \) has \( R\left(\frac{n}{2}\right) \).

- **Theorem**
  For any number field \( F \), if \( n \) is even, every \( \pi \in A_{\text{cusp}}(\text{Sp}_{2n}) \) has \( R\left(\frac{n}{2}\right) \), which is sharp.
  
  - The proof uses the Ramanujan bound for \( \tau \in A_{\text{cusp}}(\text{GL}_2) \) given by Kim-Sarnak (2003) and by Blomer-Brumley (2011).
(τ, b)-theory and the Ramanujan Type Bound

Theorem
For any number field $F$, if $n$ is odd, every $\pi \in A_{\text{cusp}}(\text{Sp}_{2n})$ has $R\left(\frac{7}{64} + \frac{n-1}{2}\right)$.

- The proof is the same. In this case, it is a hard problem to find a sharp bound even assuming the Ramanujan conjecture.

Theorem (J.-Liu (2015))
If $F$ is totally imaginary, and if $n \geq 5$ is odd, then every $\pi$ in $A_{\text{cusp}}(\text{Sp}_{2n})$ has $R\left(\frac{n-1}{2}\right)$.

- It is not known if the bound is sharp.
- One expects that the above discussion should be extended to other classical groups.
- My work in progress joint with Lei Zhang and Baiying Liu is to figure out the small cuspidal members in the global Arthur packets $\widetilde{\Pi}_{\psi}(G)$ for general global Arthur parameters $\psi$. 