Uniform estimates for orbital integrals and their Fourier transforms

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Set-up

- \( G \) is a connected reductive group; \( \mathfrak{g} \) – its Lie algebra;
- \( F \) - a local field (often, \( F = k_v \) where \( k \) is a global field; \( k \) can be a number field or function field \textbf{of sufficiently large characteristic}.)
- \( \{ f_\lambda \} \) be a family of functions, in \( C^\infty_c(G(F)) \) or in \( C^\infty_c(g(F)) \), indexed by some parameter \( \lambda \) (very flexible, and can stand for a tuple of parameters).
- \( O_\gamma(f_\lambda), O_X(f_\lambda) \) – orbital integrals, with respect to the canonical measure.
- \( D^G(X) := \prod_{\alpha \in \Phi} |\alpha(X)| \). (an extension of Harish-Chandra’s definition; it’s no longer zero at non-regular elements).
- \( \Phi(X, f_\lambda) = D^G(X)^{1/2} O_X(f_\lambda) \) – normalized orbital integral (similarly, on the group).
For a fixed $f \in C_c^\infty(g(F))$, the function $X \mapsto \Phi(X, f)$ is bounded on $g(F)$. (Harish-Chandra: bounded on $g(F)^{rss}$, Kottwitz: all $g(F)$).

Questions:

- How does this bound depend on the field as $F = F_v$ runs through the set of finite places of a global field $F$?
- How does this bound depend on the parameters $\lambda$ in a family of test functions $f_\lambda$? (e.g. when $f_\lambda$ is the characteristic function of $K a_\lambda K$, so we can think of $\lambda$ as a tuple of integers).
There exist constants $a$, $b$ (that depend only on the root datum of $G$), such that

$$|\Phi(X, f_{\lambda})| \leq q_v^{a+b\|\lambda\|}.$$ 

Note: precise $a$ and $b$ are unknown; if geometric methods gave such $a$ and $b$ in the function field case, our result also implies that the same $a$, $b$ work in characteristic zero (for large enough $p$).
The strategy:

- Use model theory.
- Prove that $\Phi(X, f_\lambda)$ belongs to the class of the so-called constructible functions.
- Prove that if a constructible function on $S \times \mathbb{Z}^n$ is bounded, its bound has to have the form $q^{a+b\|\lambda\|}$. (Here $S$ is any definable set).
- We emphasize that we use the fact that $\Phi$ is bounded to obtain the fact that its bound has this nice form.
Theorem (Cluckers-Halupczok)

Let $H$ be a constructible (exponential) function on $X \times \mathbb{Z}^n$, such that for all $F$ with sufficiently large residue characteristic, for all $x \in X_F$, $\lim_{\|\lambda\| \to \infty} |H_F(x, \lambda)|_C = 0$. Then in fact $\exists r < 0, \alpha : X \to \mathbb{Z}$, such that

$$H_F(x, \lambda) < q_F^{\left\|\lambda\right\|}$$

for all $\lambda \in \mathbb{Z}^n$ with $\lambda > \alpha_F(x)$. 

Limits
Theorem (Cluckers-Halupczok-G.)

- *The class of constructible exponential functions is closed under taking pointwise limits, $L^p$-limits, and Fourier transforms.*

- *If a constructible exponential function $H$ is bounded for every value of a parameter $\lambda \in \mathbb{Z}^n$, then it is bounded by the expression of the form $q^{a+b\|\lambda\|}$.*
Transfer principles

The properties of **being bounded, in** $L^1$, **in** $L^2$, **identically vanishing, having a limit, converging to zero, etc.** transfer (in either direction) between fields of characteristic zero and fields of positive characteristic with isomorphic residue fields, for large enough residue characteristic.

Remark on norms: We can prove: the $L^2$-norm for a motivic exponential function $H_F$ for a field $F$ of positive characteristic is estimated by $q_F^N \| H_{F'} \|$, where $F'$ has the residue field isomorphic to that of $F$, and $N$ is a (non-effective) constant depending on the formulas defining the function $H$. There might be a better estimate.
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A list of constructible things

- Moy-Prasad filtration subgroups (and lattices $g_{x,r}$) are definable.
- Orbital integrals: $O_\gamma(f_\lambda) = H(\gamma, \lambda)$, (on a group or Lie algebra), for a definable family of test functions $\{f_\lambda\}$.
- Fourier transforms of orbital integrals on the Lie algebra: $O_X(\hat{f}) = \int_{g(F)} \hat{\mu}_X(Y)f(Y)\,dY$, where $\hat{\mu}_X(Y)$ is a locally integrable function on $g^{\text{reg}}(F)$. This function is constructible exponential; moreover, $(X, Y) \mapsto \hat{\mu}_X(Y)$ is constructible exponential.
- Shalika germs (with T.C. Hales); some extra parameters needed.
Should be on the list?

- (In progress with Tom Hales and Loren Spice): characters of supercuspidal representations obtained by J-K Yu’s construction (up to multiplicative characters)
- Not yet done: weighted orbital integrals (?)
Uniform boundedness of Fourier transforms

- (Harish-Chandra, Herb). Let $\omega$ be a compact subset of $\mathfrak{g}(F)$. Then
  \[ \sup_{\omega} |D(X)|^{1/2} |D(Y)|^{1/2} |\hat{\mu}_X(Y)| < \infty. \]

- It follows from Theorem 1: If $\omega_m$ is a family of definable compact sets, then
  \[ \sup_{\omega_m} |D(X)|^{1/2} |D(Y)|^{1/2} |\hat{\mu}_X(Y)| \leq q_F^{a+b\|m\|}. \]
Suppose $H_1, H_2$ are constructible (exponential) functions, say, on an affine space, and for every $p$-adic field $F$, $(H_1)_F = (H_2)_F$ on a neighbourhood of 0. Then this neighbourhood is of the same radius for all $F$ of sufficiently large residue characteristic, and this statement can be transferred to $F$ of positive characteristic. In particular, this applies to Shalika germ expansions.
Presburger language for $\mathbb{Z}$

The symbols are:

- '0', '1' – constants;
- '+' – a binary operation;
- $\equiv_n$, $n \geq 1$ – binary relations;
- and symbols for the variables.

(Note: no multiplication!)

A set is called definable if it can be defined by '$\phi(x)$ is true', where $\phi$ is a formula and $x$ – a tuple of variables.

A function is called definable if its graph is a definable set. Definable functions are linear combinations of piecewise-linear and periodic functions.
Language of rings

The language of rings has:
- $0, 1$ – symbols for constants;
- $+, \times$ – symbols for binary operations;
- countably many symbols for variables.

The formulas are built from these symbols, the standard logical operations, and quantifiers. Any ring is a structure for this language.

Example
A formula: $\exists y, f(y, x_1, \ldots, x_n) = 0'$, where $f \in \mathbb{Z}[x_0, \ldots, x_n]$.
If we choose a structure that happens to be an algebraically closed field, then definable sets are the constructible sets in the sense of algebraic geometry (thanks to quantifier elimination for algebraically closed fields).
Denef-Pas Language (for the valued field)

Formulas are allowed to have variables of three sorts:

- valued field sort, \((+ , \times , '0', '1', \text{ac}(\cdot), \text{ord}(\cdot))\)
- value sort \((\mathbb{Z})\), \((+ , '0', '1', \equiv_n, n \geq 1)\)
- residue field sort, (language of rings: \(+, \times, '0', '1\))

Formulas are built from arithmetic operations, quantifiers, and symbols \(\text{ord}(\cdot)\) and \(\text{ac}(\cdot)\). **Example:**

\[ \phi(y) = '\exists x , y = x^2', \text{ or, equivalently, } \]

\[ \phi(y) = '\text{ord}(y) \equiv 0 \mod 2 \land \exists x : \text{ac}(y) = x^2'. \]
Constructible Motivic Functions

Generally, we’ll be able to consider functions of the form $f(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_r)$, where $x_1, \ldots, x_m$ are allowed to range over the valued field, $y_1, \ldots, y_n$ over the residue field, and $z_1, \ldots, z_r$ – over $\mathbb{Z}$ (i.e., residue field and value group parameters are allowed).

The ring of constructible functions is built from definable functions, and functions of the form $\mathbb{L}^f$, where $f$ is a definable function and $\mathbb{L}$ is a formal symbol. Constructible motivic functions take values in

$$\mathbb{Z} \left[ \mathbb{L}, \mathbb{L}^{-1}, (1 - \mathbb{L}^{-i})^{-1}, i \geq 1 \right].$$
Specialization

$(F, \omega)$ gives a structure for Denef-Pas language: we allow the variables of the corresponding sorts to range over $F$ and the residue field of $F$, respectively.

- The function $\text{ord}(\cdot)$ specializes to the usual valuation;
- the function $\text{ac}(\cdot)$ specializes to the first nonzero coefficient of the $\omega$-adic expansion.
- The symbol $\mathbb{L}$ specializes to $q = \# k$.
- Any formula in the language of rings with $n$ free variables ranging over the residue field specializes to the number of the points in $k^n$ that satisfy it.
Introduction

An old result (2012)

New results
Model theory results
Harmonic analysis
Uniform results for orbital integrals

How it works
Languages
Constructible motivic functions and specialization

Integration

(due to R. Cluckers and F. Loeser)
When we integrate \( f(x_1, \ldots, x_m, \bar{y}, \bar{z}) \), say, with respect to \( x_1 \), the answer is the same type of function \( f(x_2, \ldots, x_m, \bar{y}, \bar{z}) \), but the formulas defining it might have \textit{more} bound residue field variables.

\[
\int_{K^m} f(x_1, \ldots, x_m, \bar{y}, \bar{z}) \, dx_1 \ldots \, dx_m = F(\bar{y}, \bar{z})
\]

where \( F \) is built from formulas in the language of rings over the residue field. If all these formulas were quantifier-free, it would have been a constructible set over the residue field.
Constructible functions

A constructible function on a definable subset of $F^n$ has the form:

$$f_F(x) = \sum_{i=1}^{N} q_F^{\alpha_i F}(x) \#(p_i^{-1}(x)) \left( \prod_{j=1}^{N'} \beta_{ij} F(x) \right) \left( \prod_{\ell=1}^{N''} \frac{1}{1 - q_F^{a_{i\ell}}} \right),$$

where:

- $a_{i\ell}$ are negative integers;
- $\alpha_i : X \rightarrow \mathbb{Z}$ are $\mathbb{Z}$-valued definable functions;
- $Y_i$ are definable sets such that $Y_{iF} \subset k_F^{r_i} \times X_F$ for some $r_i \in \mathbb{Z}$, and $p_i : Y_i \rightarrow X$ is the coordinate projection.
Constructible exponential functions

Roughly, these are of the form

\[ H(x) = \sum_{i=1}^{N} a_i(x)\psi(h_i(x)), \]

where \( a_i \) are constructible, and \( h_i \) are definable functions on \( F^n \), and \( \psi : F \to \mathbb{C}^* \) is an additive character of level 0. We are interested in questions such as:

- when is \( H \) in \( L^1(F^n) \)?
- what can we say about its bounds?

It turns out that the answer entirely depends on \( a_i(x) \) (and integrability cannot happen due to delicate cancellations).