Cuspidality and Hecke algebras for Langlands parameters

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Some aspects of the local Langlands program

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Goal of talk

Define all this on the Galois side, so that it matches via the local Langlands correspondence
enhanced Langlands parameters

Notations

$F$: non-archimedean local field
$G = \mathcal{G}(F)$: connected reductive group over $F$
$G^\vee = \mathcal{G}^\vee(\mathbb{C})$: complex dual group
$W_F \subset \text{Gal}(\overline{F}/F)$: Weil group of $F$

Assumption: $G$ inner twist of $F$-split group $G^*$
enhanced Langlands parameters

Notations

- $F$: non-archimedean local field
- $G = G(F)$: connected reductive group over $F$
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- Assumption: $G$ inner twist of $F$-split group $G^*$

Definition

A Langlands parameter for $G^\vee$ is an "admissible" homomorphism

$$\phi : W_F \times SL_2(\mathbb{C}) \rightarrow G^\vee$$

$G^\vee_{sc}$: simply connected cover of $G^\vee_{der}$

$S_\phi = \pi_0(Z_{G^\vee_{sc}}(\phi))$

An enhancement of $\phi$ is an irrep $\rho$ of $S_{\phi}$

$\Phi_e(G^\vee) = \{\text{enhanced } L\text{-parameters } (\phi, \rho)\} / G^\vee$-conjugation
### Local Langlands Correspondence

#### Definition

$(\phi, \rho) \in \Phi_e(G^\vee)$ is relevant for $G$ if $\rho|_{Z(G_{sc}^\vee)}$ is the Kottwitz parameter of $G$ as an inner twist of a split group $G^*$

**Notation:** $\Phi_e(G) \subset \Phi_e(G^\vee)$
### Local Langlands Correspondence

#### Definition

\((\phi, \rho) \in \Phi_e(G^\vee)\) is relevant for \(G\) if \(\rho|_{Z(G^\vee_{sc})}\) is the Kottwitz parameter of \(G\) as an inner twist of a split group \(G^*\)

**Notation:** \(\Phi_e(G) \subset \Phi_e(G^\vee)\)

#### Conjecture (Langlands, Borel, Vogan...)

There exists a bijection

\[
\irr(G) \leftrightarrow \Phi_e(G^\vee)
\]

which satisfies many nice properties, e.g.

\(\pi \in \irr(G)\) is essentially square-integrable (i.e. \(\pi|_{G_{der}}\) is square-integrable) if and only if \(\phi_\pi\) is discrete (i.e. not a \(L\)-parameter for any proper Levi subgroup of \(G\))
Cuspidality for $L$-parameters

Notations

$(\phi, \rho)$ enhanced $L$-parameter for $G^\vee$

$C_\phi = Z_{G_{sc}^\vee}(\phi(W_F))$: complex reductive group, possibly disconnected

$u_\phi = \phi(1, \frac{1}{0} \frac{1}{1})$: unipotent element of $C_\phi^\circ$

Lemma (Kazhdan–Lusztig)

Natural isomorphism $S_\phi = \pi_0(Z_{G_{sc}^\vee}(\phi(W_F \times SL_2(\mathbb{C})))) \to \pi_0(Z_{C_\phi}(u_\phi))$
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$u_\phi = \phi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$: unipotent element of $C_\phi$

Lemma (Kazhdan–Lusztig)

Natural isomorphism $S_\phi = \pi_0( Z_{G_{sc}}(\phi(W_F \times SL_2(\mathbb{C}))) ) \rightarrow \pi_0(Z_{C_\phi}(u_\phi))$

Definition

$(\phi, \rho) \in \Phi_e(G^\vee)$ is cuspidal if:

- $\phi$ is discrete;
- $(u_\phi, \rho)$ is a cuspidal pair for $C_\phi$.

i.e. $\rho \in \text{Irr}(\pi_0(Z_{C_\phi}(u_\phi)))$ and $(u_\phi, \rho)$ cannot be obtained from a proper Levi subgroup of $C_\phi$ via a certain induction procedure

Notation: $\Phi_{\text{cusp}}(G^\vee) \subset \Phi_e(G^\vee)$
Cuspidal L-parameters for $GL_n(F)$

By classification:

$(u, \rho)$ is a cuspidal pair for $GL_n(\mathbb{C})$ with $\rho|_{Z(SL_n(\mathbb{C}))} = 1$  
$\iff n = 1, u = 1$ and $\rho = 1$

Lemma

$(\phi, \rho) \in \Phi_e(GL_n(F))$ is cuspidal $\iff$

$\phi|_{SL_2(\mathbb{C})} = 1, \rho = 1$ and $\phi|_{W_F}$ is a $n$-dim irreducible rep
Cuspidal L-parameters for $GL_n(F)$

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$(\phi, \rho) \in \Phi_e(GL_n(F))$ is cuspidal $\iff$

$\phi|_{SL_2(\mathbb{C})} = 1, \rho = 1$ and $\phi|_{WF}$ is a n-dim irreducible rep

Proof

$\iff$ By the irreducibility, $\phi$ is discrete and $Z_{G_{sc}^\vee}(\phi(W_F)) = Z(G_{sc}^\vee)$.

The pair $(u = 1, \rho = 1)$ is cuspidal for $Z(G_{sc}^\vee) = Z(SL_n(\mathbb{C}))$.
Cuspidal L-parameters for $GL_n(F)$

By classification:

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Lemma

$(\phi, \rho) \in \Phi_e(GL_n(F))$ is cuspidal $\iff$

$\phi|_{SL_2(\mathbb{C})} = 1, \rho = 1 \text{ and } \phi|_{W_F}$ is a $n$-dim irreducible rep

Proof

$\iff$ By the irreducibility, $\phi$ is discrete and $Z_{G_{\text{sc}}}(\phi(W_F)) = Z(G_{\text{sc}}^{\vee})$. The pair $(u = 1, \rho = 1)$ is cuspidal for $Z(G_{\text{sc}}^{\vee}) = Z(SL_n(\mathbb{C}))$

$\Rightarrow \rho = 1$ because $G = GL_n(F)$

1. Since $\phi$ is discrete, it is a $n$-dim irreducible rep of $W_F \times SL_2(\mathbb{C})$
2. $\mathbb{C}^n = V_1 \otimes V_2$ with $V_1 \in \text{Irr}(W_F)$ and $V_2 \in \text{Irr}(SL_2(\mathbb{C}))$
3. Cuspidality forces $u_\phi = 1$, hence $V_2$ is the trivial $SL_2(\mathbb{C})$-rep
4. $\mathbb{C}^n = V_1 \in \text{Irr}(W_F)$
Theorem (Bernstein, 1984)

Let $\pi \in \text{Irr}(G) = \{\text{irreducible smooth } G\text{-reps over } \mathbb{C}\}$

- There exist a parabolic subgroup $P = L \ltimes U$ of $G$ and a $\sigma \in \text{Irr}_{\text{cusp}}(L)$ such that $\pi$ is a subquotient of the normalized parabolic induction $I^G_P(\sigma)$

- $\pi$ determines the pair $(L, \sigma)$ uniquely up to $G$-conjugation

Definition

The cuspidal support map for $G$ is

$$Sc : \text{Irr}(G) \rightarrow \bigsqcup_{\text{Levi } L} \{L\} \times \text{Irr}_{\text{cusp}}(L)/G\text{-conjugation}$$
Cuspidal support of representations

**Theorem (Bernstein, 1984)**

Let \( \pi \in \text{Irr}(G) = \{ \text{irreducible smooth } G\text{-reps over } \mathbb{C} \} \)

- There exist a parabolic subgroup \( P = L \ltimes U \) of \( G \) and a \( \sigma \in \text{Irr}_{\text{cusp}}(L) \) such that \( \pi \) is a subquotient of the normalized parabolic induction \( I^G_P(\sigma) \)
- \( \pi \) determines the pair \( (L, \sigma) \) uniquely up to \( G \)-conjugation

**Alternative presentation of the cuspidal support map**

\( \mathcal{L}e\nu(G) \): representatives for the conjugacy classes of Levi subgroups of \( G \)
\[ W(G, L) = N_G(L)/L \]

\[ Sc : \text{Irr}(G) \to \bigsqcup_{L \in \mathcal{L}e\nu(G)} \{ L \} \times (\text{Irr}_{\text{cusp}}(L)/W(G, L)) \]
Cuspidal support of enhanced L-parameters

$(\phi, \rho)$ enhanced L-parameter for $G$

**Definition**

The cuspidal support $S_{c}(\phi, \rho)$ is the $G^{\vee}$-conjugacy class of $(L^{\vee}, \psi, \epsilon)$, where:

1. $L$ is a Levi subgroup of $G$
2. $(\psi, \epsilon)$ is a cuspidal L-parameter for $L$
Cuspidal support of enhanced L-parameters

\((\phi, \rho)\) enhanced L-parameter for \(G\)

**Definition**

The cuspidal support \(\text{Sc}(\phi, \rho)\) is the \(G^\vee\)-conjugacy class of \((L^\vee, \psi, \epsilon)\), where:

1. \(L\) is a Levi subgroup of \(G\)
2. \((\psi, \epsilon)\) is a cuspidal L-parameter for \(L\)
3. \(\psi = \phi\) on the inertia group \(I_F \subset \text{Gal}(\overline{F}/F)\)
4. \(\psi(\text{Frob}_F, \begin{pmatrix} q_F^{1/2} & 0 \\ 0 & q_F^{-1/2} \end{pmatrix}) = \phi(\text{Frob}_F, \begin{pmatrix} q_F^{1/2} & 0 \\ 0 & q_F^{-1/2} \end{pmatrix})\)

- (3) and (4) say that \(\text{Sc}\) preserves infinitesimal characters
Cuspidal support of enhanced L-parameters

$(\phi, \rho)$ enhanced L-parameter for $G$

Definition

The cuspidal support $S\text{c}(\phi, \rho)$ is the $G^\vee$-conjugacy class of $(L^\vee, \psi, \epsilon)$, where:

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3. $\psi = \phi$ on the inertia group $I_F \subset \text{Gal}(\overline{F}/F)$
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5. $(C_\phi \cap L^\vee_{sc}, u_\psi, \epsilon)$ is the cuspidal support of $(u_\phi, \rho)$, for the group $C_\phi = Z_{G^\vee_{sc}}(\phi(W_F))$

- (3) and (4) say that $S\text{c}$ preserves infinitesimal characters
- If $(\phi, \rho) \in \Phi_{\text{cusp}}(G)$, then by (5): $S\text{c}(\phi, \rho) = (G^\vee, \phi, \rho)$
Examples of the cuspidal support map

\[ G = GL_{5m}(F) \]

\( \phi \in \Phi(G), \rho = 1 \)

Suppose \( \phi = \phi_1 \otimes (R_2 \oplus R_2 \oplus R_1) \)

with \( \phi_1 \in \text{Irr}(W_F) \) and \( R_i = i\)-dim irrep of \( SL_2(\mathbb{C}) \)

- \( L^\vee = GL_m(\mathbb{C})^5 \)
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- \( L^\vee = GL_m(\mathbb{C})^5 \)
- for \( w \in I_F, x \in SL_2(\mathbb{C}) : \psi(w, x) = \phi_1(w) \otimes I_5 = \phi(w) \)
- \( \psi(\text{Frob}_F) = \phi(\text{Frob}_F) \otimes (q_F^{1/2}, q_F^{-1/2}, q_F^{1/2}, q_F^{-1/2}, 1) \)

Then \( Sc(\phi, \rho) = (L^\vee, \psi, \epsilon = 1) \)
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Then \( Sc(\phi, \rho) = (L^\vee, \psi, \epsilon = 1) \)

- This works also for \( GL_n(F) \)
- It fits with the Zelevinsky classification of \( \text{Irr}(GL_n(F)) \)
Examples of the cuspidal support map

\[ G = G_2(F) \]

\[ \phi|_{w_F} = 1, \ u_\phi = \text{subregular unipotent} \]
\[ \rho \in \text{Irr}(\pi_0(Z_{G_2(C)}(u_\phi))) \cong \text{Irr}(S_3) \]
- if \( \rho = \text{sign} \), then \( (\phi, \rho) \in \Phi_e(G) \) is cuspidal
Examples of the cuspidal support map

\[ G = G_2(F) \]

\[ \phi|_{w_F} = 1, \ u_\phi = \text{subregular unipotent} \]

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- if \( \rho = \text{sign} \), then \( (\phi, \rho) \in \Phi_e(G) \) is cuspidal
- if \( \rho \neq \text{sign} \), then \( Sc(\phi, \rho) = (T^\vee, \psi, 1) \),
  where \( T \) is a maximal split torus of \( G \) and

\[ \psi(\text{Frob}_F^n w, x) = \phi(1, \begin{pmatrix} q_F^{n/2} & 0 \\ 0 & q_F^{-n/2} \end{pmatrix}) \] for \( w \in I_F \)

The cuspidal support really depends on the enhancements of L-parameters.
Comparison of cuspidal support maps

For $L$ a Levi subgroup of $G$

$W(G, L) = N_G(L)/L$ is isomorphic with $N_{G^\vee}(L^\vee)/L^\vee$

Provides an action of $W(G, L)$ on $\Phi_{\text{cusp}}(L)$
Comparison of cuspidal support maps

For $L$ a Levi subgroup of $G$

$$W(G, L) = N_G(L)/L \text{ is isomorphic with } N_{G^\vee}(L^\vee)/L^\vee$$

Provides an action of $W(G, L)$ on $\Phi_{\text{cusp}}(L)$

---

**Conjecture**

*Let $G$ be a connected reductive $p$-adic group. The local Langlands correspondence makes following diagram commute*

\[
\begin{array}{ccc}
\Phi_e(G) & \xleftarrow{\text{LLC}} & \text{Irr}(G) \\
\text{Sc} & & \text{Sc} \\
\bigsqcup_{L \in \text{Lev}(G)} \Phi_{\text{cusp}}(L)/W(G, L) & \xleftarrow{\text{LLC}} & \bigsqcup_{L \in \text{Lev}(G)} \text{Irr}_{\text{cusp}}(L)/W(G, L)
\end{array}
\]
Bernstein components for representations

\( X_{nr}(G) \): group of unramified characters \( G \to \mathbb{C}^\times \)

**Definition**

Let \( \sigma \in \text{Irr}_{\text{cusp}}(L) \)

\( s_L = (L, X_{nr}(L) \otimes \sigma) \subset \{ L \} \times \text{Irr}_{\text{cusp}}(L) \)

\( s = [L, \sigma]_G = G\text{-}conjugacy\ class\ of\ s_L \)

\( s \) is an inertial equivalence class for \( G \)

\( \Omega(G) \): set of such classes

\( \text{Irr}(G)^s = Sc^{-1}(s_L) \), a Bernstein component of \( \text{Irr}(G) \)
 Bernstein components for representations

$X_{nr}(G)$: group of unramified characters $G \rightarrow \mathbb{C}^\times$

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$s$ is an inertial equivalence class for $G$

$\Omega(G)$: set of such classes

$\text{Irr}(G)^s = Sc^{-1}(s_L)$, a Bernstein component of $\text{Irr}(G)$

**Theorem (Bernstein, 1984)**

$\text{Irr}(G) = \bigsqcup_{s \in \Omega(G)} \text{Irr}(G)^s$

$\text{Rep}(G) = \prod_{s \in \Omega(G)} \text{Rep}(G)^s$
Bernstein components for L-parameters

$G$ inner twist, so $\chi_{nr}(G) \cong Z(G^\vee)^\circ$

**Definition**

Let $(\phi, \rho) \in \Phi_{\text{cusp}}(L^\vee)$

$s^\vee_L = (L^\vee, Z(L^\vee)^\circ \phi, \rho) \subset \{L^\vee\} \times \Phi_{\text{cusp}}(L^\vee)$

$s^\vee = [L^\vee, \phi, \rho]^G = G^\vee$-conjugacy class of $s^\vee_L$

$s^\vee$ is an inertial equivalence class for $\Phi_e(G^\vee)$
Bernstein components for L-parameters

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$s^\vee$ is an inertial equivalence class for $\Phi_e(G^\vee)$

$\Omega(G^\vee)$: set of such classes

$\Phi_e(G^\vee)^{s^\vee} = Sc^{-1}(s^\vee_L)$, a Bernstein component of $\Phi_e(G^\vee)$

**Lemma**

Any Bernstein component of $\Phi_e(G^\vee)$ is relevant for a unique inner twist of $G^*$

$\Phi_e(G^\vee) = \bigcup \bigcup \Phi_e(G^\vee)^{s^\vee}$

inner twists $G$ of $G^*$ $G$-relevant $s^\vee \in \Omega(G^\vee)$
Example

Suppose:

\( L^\vee = \) maximal torus of \( G^\vee \)
\( \phi_1(I_F \times SL_2(\mathbb{C})) = 1, \phi_1(Frob_F) \in L^\vee \)
\( s_L^\vee = (L^\vee, \phi_1, \text{triv}_{S_{\phi_1}}) \)
Example

Suppose:

\[ L^\vee = \text{maximal torus of } G^\vee \]
\[ \phi_1(I_F \times SL_2(\mathbb{C})) = 1, \phi_1(\text{Frob}_F) \in L^\vee \]
\[ s_L^\vee = (L^\vee, \phi_1, \text{triv}_{s\phi_1}) \]

Then:

\[ s^\vee \] is relevant for the split form of \( G \)
\[ \Phi_e(G^\vee)^{s^\vee} = \{ (\phi, \rho) \in \Phi_e(G^\vee) : \phi(I_F) = 1, \rho \text{ appears in } H_*(B^\phi) \} \]
\[ B^\phi = \text{variety of Borel subgroups of } G^\vee \text{ which contain the image of } \phi \]
Conjecture

Let $G$ be a connected reductive $p$-adic group.

The local Langlands correspondence makes following diagram commute

\[
\begin{array}{ccc}
\Phi_e(G) & \stackrel{LLC}{\leftrightarrow} & \text{Irr}(G) \\
\downarrow \text{Sc} & & \downarrow \text{Sc} \\
\bigsqcup_{L \in \mathcal{L}ev(G)} \Phi_{\text{cusp}}(L)/W(G, L) & \stackrel{\text{LLC}_{\text{cusp}}}{\leftrightarrow} & \bigsqcup_{L \in \mathcal{L}ev(G)} \text{Irr}_{\text{cusp}}(L)/W(G, L)
\end{array}
\]

If this holds and $\text{LLC}_{\text{cusp}}$ is compatible with unramified twists, then LLC induces a bijection between Bernstein components.

Known for:
- inner twists of $\text{GL}_n(F)$ and $\text{SL}_n(F)$ (ABPS)
- $\text{Sp}_{2n}(F)$ and $\text{SO}_n(F)$ (Moussaoui)
- principal series representations of split groups (ABPS)
- unipotent representations of adjoint groups (Lusztig)
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If this holds and $LLC_{\text{cusp}}$ is compatible with unramified twists, then LLC induces a bijection between Bernstein components.

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- $Sp_{2n}(F)$ and $SO_n(F)$ (Moussaoui)
- principal series representations of split groups (ABPS)
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Hecke algebras for Bernstein blocks of representations

Example

If $G$ is $F$-split and $I$ is an Iwahori subgroup, then

- $\mathcal{H}(G, I) = C_c^\infty(I \backslash G/I)$ is the affine Hecke algebra associated to the root datum of $G^\vee$ and the parameter $q_F$.
- $\text{Mod}(\mathcal{H}(G, I))$ is Morita equivalent to $\text{Rep}(G)[T,1]^G$.

Conjecture

For every inertial equivalence class $s$ for $G$ there exists a slight generalization $H_s$ of an affine Hecke algebra, such that $\text{Rep}(G)[T,1]^G \cong \text{Mod}(H_s)$.

There is a bijection $\text{Irr}(G)[T,1]^G \leftrightarrow \text{Irr}(H_s)$.
Hecke algebras for Bernstein blocks of representations

Example

If $G$ is $F$-split and $I$ is an Iwahori subgroup, then

- $\mathcal{H}(G, I) = C_c^\infty(I \backslash G/I)$ is the affine Hecke algebra associated to the root datum of $G^\vee$ and the parameter $q_F$
- $\text{Mod}(\mathcal{H}(G, I))$ is Morita equivalent to $\text{Rep}(G)^{[T, 1]}_G$

Conjecture

For every inertial equivalence class $s$ for $G$ there exists a slight generalization $\mathcal{H}^s$ of an affine Hecke algebra, such that

$\text{Rep}(G)^s \cong \text{Mod}(\mathcal{H}^s)$.

There is a bijection $\text{Irr}(G)^s \longleftrightarrow \text{Irr}(\mathcal{H}^s)$.
Hecke algebras for Bernstein components of L-parameters

Data from a Bernstein component $\Phi_e(G^\vee)^{\mathfrak{s}^\vee}$

- $\mathfrak{s}_L^\vee = (L^\vee, Z(L^\vee) \circ \phi, \rho)$
- torus $T_{\mathfrak{s}^\vee} := (Z(L^\vee) \circ \phi, \rho) \subset \Phi_{\text{cusp}}(L)$
- complex reductive group $Z_{G_{sc}^\vee}(\phi(1))$
- finite group $\mathcal{W}_{\mathfrak{s}^\vee} = \text{Stab}_{\mathcal{W}(G^\vee, L^\vee)}(\mathfrak{s}_L^\vee)$
Data from a Bernstein component $\Phi_e(G^\vee)^{\mathfrak{s}^\vee}$

- $\mathfrak{s}_L^\vee = (L^\vee, Z(L^\vee)^\phi, \rho)$
- torus $T_{\mathfrak{s}^\vee} := (Z(L^\vee)^\phi, \rho) \subset \Phi_{\text{cusp}}(L)$
- complex reductive group $Z_{G_{\mathfrak{s}^c}^\vee}((\phi(\mathbf{1}_F)))$
- finite group $W_{\mathfrak{s}^\vee} = \text{Stab}_{W(G^\vee, L^\vee)}(\mathfrak{s}_L^\vee)$

Example

$L^\vee = G^\vee, \mathfrak{s}^\vee = \mathfrak{s}_L^\vee$ is cuspidal
algebra: $\mathcal{O}(T_{\mathfrak{s}^\vee})$

Example

$L^\vee = \text{maximal torus}, u_\phi = 1, W_{\mathfrak{s}^\vee} = W(G^\vee, L^\vee)$
algebra: $\mathcal{H}(T_{\mathfrak{s}^\vee}, W_{\mathfrak{s}^\vee}, \mathfrak{v})$, the affine Hecke algebra for the root datum of $(G^\vee, L^\vee)$ and the single parameter $q_F = \mathfrak{v}^2$
Theorem

Canonical associated to $s^\vee$ is an algebra $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)$ such that:

- $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)$ is an extension of an affine Hecke algebra by a finite dimensional algebra
- For every $v \in \mathbb{R}_{>0}$ there is a canonical bijection

$$\Phi_e(G^\vee)^{s^\vee} \longleftrightarrow \text{Irr}(\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)/(v - v))$$
Theorem

Canonically associated to $s^\vee$ is an algebra $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)$ such that:

- $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)$ is an extension of an affine Hecke algebra by a finite dimensional algebra
- For every $\nu \in \mathbb{R}_{>0}$ there is a canonical bijection

$$\Phi_e(G^{\vee})^{s^\vee} \longleftrightarrow \text{Irr}(\mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)/(\nu - v))$$

- $\text{Mod } \mathcal{H}(T_{s^\vee}, W_{s^\vee}, v)$ provides a categorification of $\Phi_e(G^{\vee})^{s^\vee}$
- The module category depends only on $Z_{G_{sc}}(\phi(I_F))$ and some cuspidal data. This explains many equivalences between different Bernstein blocks
- The Galois side of the LLC can be phrased entirely in terms of cuspidal L-parameters and the groups $W_{s^\vee}$
This helps to reduce the proof of the LLC to the cuspidal case.
Specialization at $\mathbf{v} = q_F^{1/2}$

\[
\begin{align*}
\text{Rep}(G)^s & \xrightarrow{J^G_L} \text{Rep}(L)^{s_L} \\
\text{Mod}\left(\mathcal{H}(T_{s^\vee}, W_{s^\vee}, \mathbf{v})/(\mathbf{v} - q_F^{1/2})\right) & \xrightarrow{\text{Res}_{\mathcal{O}(T_{s^\vee})}} \text{Mod}\left(\mathcal{O}(T_{s^\vee})\right)
\end{align*}
\]
Specialization at $\mathbf{v} = q_F^{1/2}$

$$\begin{array}{c}
\text{Rep}(G)^s \xrightarrow{J_L^G} \text{Rep}(L)^{s_L} \\
\text{Mod}(\mathcal{H}(T_5^\vee, W_5^\vee, \mathbf{v})/(\mathbf{v} - q_F^{1/2})) \xrightarrow{\text{Res}_{\mathcal{O}(T_5^\vee)}} \text{Mod}(\mathcal{O}(T_5^\vee))
\end{array}$$

Specialization at $\mathbf{v} = 1$

$$\begin{array}{c}
\text{Irr}(G)^s \xrightarrow{\text{Sc}} \text{Irr}_{\text{cusp}}(L)^{s_L}/W_5 \\
\Phi_e(G)^{s^\vee} \xrightarrow{\text{Sc}} \Phi_{\text{cusp}}(L)^{s_L^\vee}/W_5^{\vee} \\
\text{Irr}(\mathcal{O}(T_5^\vee) \rtimes \mathbb{C}[W_5^\vee, t_5^\vee]) \rightarrow T_5^\vee/W_5^{\vee}
\end{array}$$
Generalizations of the Springer correspondence

\[ \text{Unip}_e(\mathcal{H}) = \{(u, \rho) : u \in \mathcal{H} \text{ unipotent}, \rho \in \text{Irr}(\pi_0(Z_{\mathcal{H}}(u)))\}/\mathcal{H}\text{-conjugacy} \]

\[ \text{Unip}_{\text{cusp}}(\mathcal{H}) = \text{cuspidal part of } \text{Unip}_e(\mathcal{H}) \]
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**Theorem (Lusztig, 1984)**

1. There exists a unique cuspidal support map

   \[
   \text{Sc}_{H^\circ} : \text{Unip}_e(\mathcal{H}^\circ) \to \bigsqcup_{\text{Levi } \mathcal{L}} \{\mathcal{L}\} \times \text{Unip}_{\text{cusp}}(\mathcal{L}) / \mathcal{H}^\circ\text{-conjugacy}
   \]

   such that \((u, \rho)\) appears in some induction of \(\text{Sc}_{H^\circ}(u, \rho)\).
Generalizations of the Springer correspondence

\[ \text{Unip}_e(\mathcal{H}) = \{(u, \rho) : u \in \mathcal{H} \text{ unipotent}, \rho \in \text{Irr}(\pi_0(Z_{\mathcal{H}}(u)))\}/\mathcal{H}\text{-conjugacy} \]

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**Theorem (Lusztig, 1984)**

1. There exists a unique cuspidal support map

   \[ Sc_{\mathcal{H}^\circ} : \text{Unip}_e(\mathcal{H}^\circ) \to \bigcup_{\text{Levi } \mathcal{L}} \{\mathcal{L}\} \times \text{Unip}_{\text{cusp}}(\mathcal{L})/\mathcal{H}^\circ\text{-conjugacy} \]

   such that \((u, \rho)\) appears in some induction of \(Sc_{\mathcal{H}^\circ}(u, \rho)\).

2. Let \(t = (\mathcal{L}, \nu, \epsilon)\) be a cuspidal support for \(\mathcal{H}^\circ\).

   There exists a canonical bijection

   \[ \Sigma_t : Sc_{\mathcal{H}^\circ}^{-1}(t) \to \text{Irr}(W(\mathcal{H}^\circ, \mathcal{L})), \]

   realized in the cohomology of a certain sheaf.
Generalizations of the Springer correspondence

Desired: Lusztig–Springer correspondence for disconnected complex reductive groups \( \mathcal{H} \)

**Modifications**

1. Cuspidal support \((\mathcal{L}, \nu, \epsilon)\) is replaced by a cuspidal quasi-support \(q\tau = (q\mathcal{L}, \nu, q\epsilon)\), where:
   - \(q\mathcal{L} \subset \mathcal{H}\) is quasi-Levi: \(q\mathcal{L} = Z_{\mathcal{H}}(Z(\mathcal{L})^\circ)\) for a Levi subgroup \(\mathcal{L} \subset \mathcal{H}^\circ\)
   - \(\nu \in q\mathcal{L}^\circ\) is unipotent
   - \(q\epsilon \in \text{Irr}(\pi_0(Z_{\mathcal{H}}(\nu)))\) such that \(q\epsilon|_{\pi_0(Z_{\mathcal{H}^\circ}(\nu))}\) is a sum of cuspidal reps
Generalizations of the Springer correspondence

Desired: Lusztig–Springer correspondence for disconnected complex reductive groups $\mathcal{H}$

Modifications

1. Cuspidal support $(\mathcal{L}, \nu, \epsilon)$ is replaced by a cuspidal quasi-support $q\mathcal{t} = (q\mathcal{L}, \nu, q\epsilon)$, where:
   - $q\mathcal{L} \subset \mathcal{H}$ is quasi-Levi: $q\mathcal{L} = Z_{\mathcal{H}}(Z(\mathcal{L})^\circ)$ for a Levi subgroup $\mathcal{L} \subset \mathcal{H}^\circ$
   - $\nu \in q\mathcal{L}^\circ$ is unipotent
   - $q\epsilon \in \text{Irr}(\pi_0(Z_{\mathcal{H}}(\nu)))$ such that $q\epsilon|_{\pi_0(Z_{\mathcal{H}}^\circ(\nu))}$ is a sum of cuspidal reps

2. $W(\mathcal{H}^\circ, \mathcal{L})$ is extended to $W(\mathcal{H}, q\mathcal{L}, q\epsilon) = N_{\mathcal{H}}(q\mathcal{L}, q\epsilon)/q\mathcal{L}$

3. $\text{Irr}(W(\mathcal{H}, \mathcal{L}))$ is replaced by $\text{Irr}(\mathbb{C}[W(\mathcal{H}, q\mathcal{L}, q\epsilon), \sharp_q\epsilon])$, where
   $\sharp_q\epsilon : (W(\mathcal{H}, q\mathcal{L}, q\epsilon)/W(\mathcal{H}^\circ, \mathcal{L}))^2 \to \mathbb{C}^\times$ is a 2-cocycle
Theorem (Generalization of the Lusztig–Springer correspondence)

Let $\mathcal{H}$ be a complex reductive group, possibly disconnected.

1. There exists a canonical cuspidal support map

$$S_{c\mathcal{H}} : \text{Unip}_e(\mathcal{H}) \to \bigsqcup_{\text{quasi-Levi } \mathcal{L}} \{\mathcal{L}\} \times \text{Unip}_{\text{cusp}}(\mathcal{L})/\mathcal{H}\text{-conjugacy}$$

- $S_{c\mathcal{H}}$ is used in the cuspidal support map for enhanced L-parameters
Theorem (Generalization of the Lusztig–Springer correspondence)

Let $\mathcal{H}$ be a complex reductive group, possibly disconnected.

1. There exists a canonical cuspidal support map

$$Sc_{\mathcal{H}} : \text{Unip}_e(\mathcal{H}) \to \bigsqcup_{\text{quasi-Levi } \mathcal{L}} \{\mathcal{L}\} \times \text{Unip}_{\text{cusp}}(\mathcal{L})/\mathcal{H}\text{-conjugacy}$$

2. Let $q_t = (q_\mathcal{L}, v, q_\epsilon)$ be a cuspidal quasi-support for $\mathcal{H}$. There exists a (almost canonical) bijection

$$\Sigma_{q_t} : Sc_{\mathcal{H}}^{-1}(q_t) \to \text{Irr}(\mathbb{C}[\mathcal{W}(\mathcal{H}, q_\mathcal{L}, q_\epsilon), \sharp q_\epsilon])$$

realized in the cohomology of a certain sheaf.

- $Sc_{C_\phi}$ is used in the cuspidal support map for enhanced L-parameters
- Sometimes the 2-cocycle $\sharp q_\epsilon$ is nontrivial. Used in $\mathcal{H}(T_s, W_s, v)$
Cuspidal support of enhanced L-parameters

\((\phi, \rho)\) enhanced L-parameter for \(G\)

**Definition**

The cuspidal support \(S_{c}(\phi, \rho)\) is the \(G^{\vee}\)-conjugacy class of \((L^{\vee}, \psi, \epsilon)\), where:

1. \(L\) is a Levi subgroup of \(G\)
2. \((\psi, \epsilon)\) is a cuspidal L-parameter for \(L\)
3. \(\psi = \phi\) on the inertia group \(I_{F} \subset \text{Gal}(\overline{F}/F)\)
4. \(\psi(\text{Frob}_{F}, \begin{pmatrix} q_{F}^{1/2} & 0 \\ 0 & q_{F}^{-1/2} \end{pmatrix}) = \phi(\text{Frob}_{F}, \begin{pmatrix} q_{F}^{1/2} & 0 \\ 0 & q_{F}^{-1/2} \end{pmatrix})\)
5. \((C_{\phi} \cap L_{sc}^{\vee}, u_{\psi}, \epsilon)\) is the cuspidal support of \((u_{\phi}, \rho)\), for the group \(C_{\phi} = Z_{G_{sc}^{\vee}}(\phi(W_{F}))\)
A Bernstein component \( \Phi_e(G^\vee)s^\vee \) consists of the enhanced L-parameters with the same cuspidal support, up to unramified twists

\[
\Phi_e(G^\vee) = \bigsqcup \bigsqcup \Phi_e(G^\vee)s^\vee
\]

inner twists \( G \) of \( G^* \) G-relevant \( s^\vee \in \Omega(G^\vee) \)

Can the local Langlands correspondence be categorified to \( \text{Rep}(G) \cong \bigsqcup \bigsqcup \text{Irr}(H(Ts^\vee, Ws^\vee, v))/ (v - v) \)?

Is there some sheaf with endomorphism algebra \( H(Ts^\vee, Ws^\vee, v) ? \)
Summary

- A Bernstein component \( \Phi_e(G^\vee) s^\vee \) consists of the enhanced L-parameters with the same cuspidal support, up to unramified twists

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inner twists \( G \) of \( G^* \) G-relevant \( s^\vee \in \Omega(G^\vee) \)

- To every \( s^\vee \) we can attach \( \mathcal{H}(T_s^\vee, W_s^\vee, v) \), which is almost an affine Hecke algebra.

- For every \( v \in \mathbb{R}_{>0} \)

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\Phi_e(G^\vee) = \bigsqcup \bigsqcup \text{Irr}(\mathcal{H}(T_s^\vee, W_s^\vee, v)/(v-v))
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inner twists \( G \) of \( G^* \) G-relevant \( s^\vee \in \Omega(G^\vee) \)
Summary

- A Bernstein component $\Phi_e(G^\vee)^{s^\vee}$ consists of the enhanced $L$-parameters with the same cuspidal support, up to unramified twists
  
  \[
  \Phi_e(G^\vee) = \bigsqcup \bigsqcup \Phi_e(G^\vee)^{s^\vee}
  \]
  inner twists $G$ of $G^*$ $G$-relevant $s^\vee \in \Omega(G^\vee)$

- To every $s^\vee$ we can attach $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, \nu)$, which is almost an affine Hecke algebra.

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- Can the local Langlands correspondence be categorified to
  
  \[
  \text{Rep}(G) \sim \bigsqcup \text{Irr}(\mathcal{H}(T_{s^\vee}, W_{s^\vee}, \nu))
  \]
  $G$-relevant $s^\vee \in \Omega(G^\vee)$

- Is there some sheaf with endomorphism algebra $\mathcal{H}(T_{s^\vee}, W_{s^\vee}, \nu)$?