Properties of rationality

$k$ a field, $X/k$ projective integral variety

We are interested in determining whether or not:

- $X$ is rational
- $X$ is stably rational
- $X$ is unirational
- $X$ is rationally connected (say $k = \mathbb{C}$).
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\( X/k \) is **universally CH\(_0\)-trivial** if \( \text{deg} : \text{CH}_0(X_L) \to \mathbb{Z} \) is an isomorphism \( \forall L/k \).

**Examples:** \( X \) smooth stably rational, does NOT necessarily hold if \( X \) is rationally connected.
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**Specialization method**: Assume \( B/\mathbb{C} \) integral of finite type,

- \( \mathfrak{X} \to B \) flat, projective map, general fiber is smooth;
- for \( b_0 \in B(\mathbb{C}) \), the fiber \( Y = \mathfrak{X}_{b_0} \) has a resolution \( f : Z \to Y \), s.t. \( f \) is a universally \( CH_0 \)-trivial map (check on fibers); \( Z \) is NOT universally \( CH_0 \)-trivial (ex. \( BrZ \neq 0 \)).

Then: \( \mathfrak{X}_b \) is not stably rational for \( b \in B(\mathbb{C}) \) very general.
Computing Brauer group birationally

- $Y/k$ integral, $H^i_{nr}(k(Y)/k)$ is
  
  $H^i_{nr}(k(Y)/k, \mathbb{Z}/2) = \bigcap_v \ker[H^i(k(Y), \mathbb{Z}/2) \xrightarrow{\partial_v} H^{i-1}(\kappa(v), \mathbb{Z}/2)];$

  if $i = 2$ and $Y/\mathbb{C}$ is smooth projective
  $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}) = Br Y[2].$

- If $Y \to \mathbb{P}^2_{\mathbb{C}}$ is a fibration in quadrics of dimension 1 or 2, generic fiber $Q/K = \mathbb{C}(\mathbb{P}^2)$ is a quadric, then:
  
  - $H^2(K, \mathbb{Z}/2) \to H^2_{nr}(K(Q)/K)$ is surjective;
  
  - to construct $\alpha \in H^2_{nr}(\mathbb{C}(Y)/\mathbb{C})$, take $\alpha_0 \in H^2(K, \mathbb{Z}/2)$ and verify that it becomes unramified on $\mathbb{C}(Y)$ (use commutative diagrams with residues).

  (method by Colliot-Thélène - Ojanguren, also explains Artin-Mumford)
$Y \to \mathbb{P}^2_{\mathbb{C}}$,
generic fiber is $Q/K = \mathbb{C}(\mathbb{P}^2)$,
$Y \subset \mathbb{P}^2 \times \mathbb{P}^3$ is given by

$$yzs^2 + xzt^2 + xyu^2 + (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0.$$  

Put $F(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. 
The hypersurface with equation

\[(u^2 + uv + ts)x^2 + (-t^2 + u^2 - v^2 - s^2)xy + (t^2 + uv + ts)y^2 + (-t^2 + u^2 - v^2 - s^2)xz + (t^2 - 16tu - u^2 + v^2 + s^2)yz + (-3uv - 3ts + s^2)z^2.\]

is smooth and satisfies the surjectivity criterion.