LOGARITHMIC INTERPRETATION OF MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

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1. Classical theory

This short note consists of the synopsis of my talk in the Simons Symposium on Periods and L-Values of Motives. We first discuss the classical theory (characteristic zero), and then describe its positive characteristic counterpart.

1.1. Real-valued MZV’s. Let \( \mathbb{N} \) be the set of positive integers. An index is an element of \( \mathbb{N}^r \). We call an index \( s = (s_1, \ldots, s_r) \) admissible if \( s_1 \geq 2 \). Consider the following Taylor expansion:

\[
\log(1 - z) = \sum_{n=0}^{\infty} \frac{z^n}{n}.
\]

For a fixed admissible index \( s = (s_1, \ldots, s_r) \), we consider the following one-variable multiple polylogarithm:

\[
\text{Li}_s(z) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}}.
\]

The multiple zeta value (abbreviated as MZV) at the admissible index \( s \) is defined by the specialization

\[
\zeta(s) := \text{Li}_s(1) \in \mathbb{R}^\times.
\]

The weight and the depth of the presentation \( \zeta(s) \) are defined to be \( \text{wt}(s) := \sum_{i=1}^r s_i \) and \( \text{dep}(s) := r \) respectively.

For any integer \( n \geq 2 \), let \( \mathcal{Z}_n \) be the \( \mathbb{Q} \)-vector space spanned by MZV’s of weight \( n \). Then we have that \( \mathcal{Z}_n_1 \cdot \mathcal{Z}_n_2 \subseteq \mathcal{Z}_{n_1+n_2} \). In fact, there are two different ways to see the property above, which are called stuffle product and shuffle product. We refer the readers to [BGF18, Zh16]. So if we let \( \mathcal{Z} := \sum_{n \geq 2} \mathcal{Z}_n \) be the vector space spanned by all MZV’s over \( \mathbb{Q} \), then \( \mathcal{Z} \) forms a \( \mathbb{Q} \)-algebra. Goncharov conjectured that there are no non-trivial \( \mathbb{Q} \)-linear relations among different weight MZV’s.

**Conjecture 1.1.1.** (Goncharov) We have \( \mathcal{Z} = \bigoplus_{n \geq 2} \mathcal{Z}_n \).

We mention that there are many \( \mathbb{Q} \)-linear relations among the same weight MZV’s produced by regularized double shuffle relations (or called extended double shuffle relations) [IKZ06], and Ihara-Kaneko-Zagier conjectured that all the \( \mathbb{Q} \)-linear relations among the same weight MZV’s arise in this way.

**Conjecture 1.1.2.** (Ihara-Kaneko-Zagier) The regularized double shuffle relations given in [IKZ06] generate all the \( \mathbb{Q} \)-linear relations among the same weight MZV’s.

1.2. \( p \)-adic MZV’s. In what follows, we fix an admissible index \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \). Fix a prime number \( p \), and let \( \mathbb{C}_p \) be the \( p \)-adic completion of a fixed algebraic closure of the field of \( p \)-adic numbers \( \mathbb{Q}_p \). As a power series, \( \text{Li}_s(z) \) has coefficients in \( \mathbb{Q} \). We denote by \( \text{Li}_s(z)_p \) the same series as \( \text{Li}_s(z) \) but we regard it as a power series in \( \mathbb{C}_p[z] \). Fixing a branch choice \( \alpha \in \mathbb{C}_p \) of the \( p \)-adic logarithm, Furusho [F04] used Coleman’s \( p \)-adic integration theory to show that \( \text{Li}_s(z)_p \) can be \( p \)-adic analytically continued to \( \mathbb{C}_p \setminus \{1\} \), and we denote by \( \text{Li}_s^\alpha(z)_p \) the analytic continued function.
Furusho [F04] then defined the p-adic MZV by

\[(1.2.1) \quad \zeta_p(s) := \lim_{z \to 1} \left[ \text{Li}'_s(z) \right]_p,\]

where the notation \(\lim'_{z \to 1}\) is referred to the limit by taking any sequence \(\{z_n\} \subseteq \mathbb{C}_p\) for which \(\{z_n\}\) converges p-adically to 1 and the field generated by \(z_1, z_2, \ldots\) over \(\mathbb{Q}_p\) is a finitely ramified extension over \(\mathbb{Q}_p\). Furusho showed that the limit of (1.2.1) exists in \(\mathbb{Q}_p\) and is independent of the branch choices \(a\) of the p-adic logarithm. The weight and the depth of the presentation \(\zeta_p(s)\) are defined as \(\text{wt}(s)\) and \(\text{dep}(s)\) respectively.

In [FJ07], Furusho and Jafari showed that the p-adic MZV’s satisfy the regularized double shuffle relations, and hence combining the Ihara-Kaneko-Zagier conjecture above one has the following conjecture.

**Conjecture 1.2.2.** Let \(n \geq 2\) be an integer, and let \(s_1, \ldots, s_m\) be admissible indexes with \(\text{wt}(s_i) = n\) for all \(i\). If we have a non-trivial \(\mathbb{Q}\)-linear relation \(\sum_{i=1}^{m} a_i \zeta_p(s_i) = 0\), then we also have \(\sum_{i=1}^{m} a_i \zeta_p(s_i) = 0\). In other words, if we let \(\mathcal{Z}_{n,p}\) be the \(\mathbb{Q}\)-vector space spanned by p-adic MZV’s of weight \(n\), then we have a well-defined \(\mathbb{Q}\)-linear map

\[\phi_{n,p} := (\zeta(s) \mapsto \zeta_p(s)) : \mathcal{Z}_n \to \mathcal{Z}_{n,p}.\]

The conjecture above is quite interesting as there is a route connecting the two different worlds of MZV’s. The primary result presented in this note is to describe my recent joint work with Mishiba [CM17b] verifying an analogue of Conjecture 1.2.2 in the function fields setting.

**2. Theory in positive characteristic**

Let \(A := \mathbb{F}_q[\theta]\) be the polynomial ring in the variable \(\theta\) over the finite field \(\mathbb{F}_q\) with characteristic \(p\), and \(k\) be its quotient field. Let \(|\cdot|_\infty\) be the non-archimedean absolute value with respect to the infinite place of \(k\) for which \(|\theta|_\infty = q\). We let \(k_\infty\) be the completion of \(k\) with respect to \(|\cdot|_\infty\), and let \(\mathbb{C}_\infty\) be the \(\infty\)-adic completion of a fixed algebraic closure of \(k_\infty\).

**2.1. \(\infty\)-adic MZV’s.** In [T04], Thakur defined the \(\infty\)-adic MZV’s: for any index \(s = (s_1, \ldots, s_r) \in \mathbb{N}^r\),

\[\zeta_A(s) := \sum a_i^{s_1i} \cdots a_r^{s_r} \in k_\infty,\]

where \(a_1, \ldots, a_r\) run over all monic polynomials in \(A\) for which \(|a_1|_\infty > \cdots > |a_r|_\infty\). Since \(|\cdot|_\infty\) is non-archimedean, we see that \(\zeta_A(s)\) converges in \(k_\infty\) for any index \(s\). As same as the terminology in the classical theory, the weight and the depth of the presentation \(\zeta_A(s)\) are defined as \(\text{wt}(s)\) and \(\text{dep}(s)\) respectively. Note that \(\infty\)-adic MZV’s of depth one are called Carlitz zeta values initiated by Carlitz [Ca35].

We mention that there are several monic polynomials with the same absolute value, so it is not clear to obtain the non-vanishing of \(\zeta_A(s)\). However, Thakur [T04] showed that it is indeed the case that every \(\infty\)-adic MZV is non-vanishing. These special values have arithmetic-geometric interpretation in the sense that they occur as periods of certain mixed Carlitz-Tate t-motives by the work of Anderson-Thakur [AT09].

Given a positive integer \(n\), we let \(\mathcal{Z}_n\) be the \(k\)-vector space spanned by \(\infty\)-adic MZV’s of weight \(n\). In [T09], Thakur showed that \(\mathcal{Z}_{n_1} \cdot \mathcal{Z}_{n_2} \subseteq \mathcal{Z}_{n_1+n_2}\), and therefore the \(k\)-vector space \(\mathcal{Z} := \sum_{n \geq 1} \mathcal{Z}_n\) spanned by all \(\infty\)-adic MZV’s form a \(k\)-algebra. In
this positive characteristic world, we have the following result as an analogue of Gon
charov’s conjecture as well as Baker-Wüstholz philosophy for \( \infty \)-adic MZV’s established in [C14].

**Theorem 2.1.1.** The \( k \)-algebra \( \mathcal{X} \) is a graded algebra in the sense that \( \mathcal{X} = \bigoplus_{n \geq 1} \mathcal{X}_n \).
Moreover, if the \( \infty \)-adic MZV’s \( \lambda_1, \ldots, \lambda_m \) are linear independent over \( k \), then \( 1, \lambda_1, \ldots, \lambda_m \)
are linearly independent over \( \bar{k} \), where \( \bar{k} \) is the algebraic closure of \( k \) inside \( C_\infty \).

As an important consequence, we obtain that each \( \infty \)-adic MZV is transcendental
over \( k \). We mention that these results generalize the work of Yu [Yu91, Yu97] for
Carlitz zeta values.

### 2.2. Carlitz multiple polylogarithms.

Let \( t \) be a new variable. The Carlitz \( \mathbb{F}_q[t] \)-
module \( C \) plays the role of the multiplicative group \( G_m \) in the function field case, see
[To04]. It’s logarithm is defined by the power series

\[
\log_{C}(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{L_i},
\]

where \( L_0 := 1 \) and \( L_i := (\theta - \theta q) \cdots (\theta - \theta q^i) \). Without confusion, for any index \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \) we still denote by \( L_i(s) := (z_1, \ldots, z_r) \) the \( r \)-variable \( s \)th Carlitz multiple
polylogarithm defined in [C14], and further define its star version by:

\[
(2.2.1) \quad \mathcal{L}_s^*(z_1, \ldots, z_r) := \sum_{i_1 \geq \cdots \geq i_r \geq 0} \frac{z_1^{q_{i_1}} \cdots z_r^{q_{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in k[z_1, \ldots, z_r].
\]

It was shown in [AT90] for depth one case and in [C14] for arbitrary case that \( \zeta_A(s) \)
can be expressed as an explicit \( k \)-linear combination of \( \mathcal{L}_s \) at some integral points,
and so we have the following explicit formula ([CM17b, Thm. 5.2.5]): for any index
\( s \in \mathbb{N}^r \), there exist some explicit indexes \( s_\ell \) with \( \text{wt}(s_\ell) = \text{wt}(s) \) and \( \text{dep}(s_\ell) \leq \text{dep}(s) \),
coefficients \( \alpha_\ell \in k^\times \), and integral points \( u_\ell \in A^{\text{dep}(s_\ell)} \) so
that

\[
(2.2.2) \quad \zeta_A(s) = \sum_\ell \alpha_\ell \mathcal{L}_{s_\ell}(u_\ell).
\]

### 2.3. \( v \)-adic MZV’s.

Fix a field \( k \subseteq L \subseteq \bar{k} \). A \( d \)-dimensional \( t \)-module over \( L \) is a pair
\( G = (G_{a/L}, \rho) \), where \( G_{a/L} \) is a \( d \)-dimensional additive group defined over \( L \), and \( \rho \) is
an \( \mathbb{F}_q \)-linear ring homomorphism

\[
\rho : \mathbb{F}_q[t] \to \text{End}_{\mathbb{F}_q} \left( G_{a/L} \right)
\]

for which \( \partial \rho_t - \theta \cdot I_d \) is a nilpotent matrix. Here \( \partial \rho_t \) is referred to the induced morphism
of \( \rho_t \) at identity on the Lie algebra of \( G_{a/L} \). By Anderson’s theory [A86], there is an
exponential function (associated to \( G \))

\[
\exp_G(z) = \sum_{i=0}^{\infty} A_i z^{(i)},
\]

where \( A_0 = I_d \) and \( A_i \in \text{Mat}_d(L) \) for all \( i \), and \( z := (z_1, \ldots, z_d)^{tr} \) and \( z^{(i)} := (z_1^{q^i}, \ldots, z_d^{q^i})^{tr} \).
Over \( C_\infty \), the exponential \( \exp_G : \text{Lie } G(C_\infty) \to G(C_\infty) \) is entire and \( \mathbb{F}_q[t] \)-linear. Here,
we note that the \( \mathbb{F}_q[t] \)-module structure on \( \text{Lie } G(\mathbb{C}_\infty) \) is via \( \partial_p \) for \( \alpha \in \mathbb{F}_q[t] \). As vector-valued power series, the formal inverse of \( \exp_G \) is called the logarithm of \( G \) and is denoted by \( \log_G \), i.e., \( \log_G \circ \exp_G(z) = \mathbf{z} = \exp_G \circ \log_G(z) \).

Fix a finite place \( \mathfrak{v} \) of \( k \). Let \( k_\mathfrak{v} \) be the completion of \( k \) at \( \mathfrak{v} \) and \( C_\mathfrak{v} \) be the \( \mathfrak{v} \)-adic completion of a fixed algebraic closure of \( k_\mathfrak{v} \). For any index \( s \in \mathbb{N}^r \), we denote by \( \text{Li}^*_s(z_1, \ldots, z_r)_\mathfrak{v} \) the same power series as \( \text{Li}^*_s(z_1, \ldots, z_r) \) but we regard it in \( C_\mathfrak{v}[z_1, \ldots, z_r] \).

It is shown in [CM17] that \( \text{Li}^*_s(z_1, \ldots, z_r)_\mathfrak{v} \) converges on the open unit ball of \( C_\mathfrak{v}^r \). By relating \( \text{Li}^*_s \) to certain coordinate of the logarithm of certain \( t \)-module and using suitable twists, one can extend its convergence domain to the closed unit ball of \( C_\mathfrak{v}^r \) [CM17, Sec. 4.1], and hence \( \text{Li}^*_s(u)_\mathfrak{v} \) is defined for any \( u \in A^r \). Using (2.2.2) we define the \( \mathfrak{v} \)-adic MZV \( \zeta_A(s)_\mathfrak{v} \) by

\[
\zeta_A(s)_\mathfrak{v} := \sum_{\ell} \alpha_\ell \cdot \text{Li}^*_\ell(u_\ell)_\mathfrak{v} \in k_\mathfrak{v}.
\]

Such as before, the weight and the depth of the presentation \( \zeta_A(s)_\mathfrak{v} \) are defined to be \( \text{wt}(s) \) and \( \text{dep}(s) \) respectively. Note that in [Go79] Goss defined the \( \mathfrak{v} \)-adic zeta values at positive integers, which are identical to our values above modulo rational multiples in \( k \) in the depth one case. The main result of [CM17b] is to give a positive answer of the analogue of Conjecture 1.2.2.

**Theorem 2.3.1.** Fix a finite place \( \mathfrak{v} \) of \( k \). Let \( n \) be a positive integer, and let \( \mathcal{X}_{n,\mathfrak{v}} \) be the \( k \)-vector space spanned by \( \mathfrak{v} \)-adic MZV's of weight \( n \). Then we have the following well-defined \( k \)-linear map

\[
\psi_{n,\mathfrak{v}} := (\zeta_A(s) \mapsto \zeta_A(s)_\mathfrak{v}) : \mathcal{X}_{n} \to \mathcal{X}_{n,\mathfrak{v}}.
\]

3. **Key ingredients of the proof**

3.1. **Strategy of the proof.** There are two key ingredients in the proof of Theorem 2.3.1.

(1) Give logarithmic interpretation for \( \infty \)-adic and \( \mathfrak{v} \)-adic MZV's.

(2) Apply Yu's sub-\( t \)-module theorem [Yu97].

Note that in the depth one case, (1) originated from Anderson-Thakur's work [AT90], which is an important bridge for Yu [Yu91] to show the transcendence of Carlitz zeta values and Goss \( \mathfrak{v} \)-adic zeta values at "odd" positive integers. In the function fields setting, Yu's sub-\( t \)-module theorem plays the analogue of Wüstholz's analytic subgroup theorem stated below.

**Theorem 3.1.1.** (Wüstholz [W89]) Let \( G \) be a commutative algebraic group defined over \( \overline{\mathbb{Q}} \), and let \( \exp_G : \text{Lie } G(\mathbb{C}) \to G(\mathbb{C}) \) be the exponential map when regarding \( G(\mathbb{C}) \) as a complex Lie group. Suppose that we have a vector \( Z \in \text{Lie } G(\mathbb{C}) \) for which \( \exp_G(Z) \in G(\overline{\mathbb{Q}}) \). Let \( V_Z \subset \text{Lie } G(\mathbb{C}) \) be the smallest linear subspace that is defined over \( \overline{\mathbb{Q}} \) and contains \( Z \). Then \( V_Z = \text{Lie } H(\mathbb{C}) \) for some algebraic subgroup \( H \) of \( G \) defined over \( \overline{\mathbb{Q}} \).

The spirit of Wüstholz's analytic subgroup theorem is that the \( \overline{\mathbb{Q}} \)-linear relations among the coordinates of \( Z \) are explained by the defining equations of \( \text{Lie } H \) over \( \overline{\mathbb{Q}} \).

We go back to (1) above. It is to show that given an index \( s \) with \( n := \text{wt}(s) \), one explicitly constructs a \( t \)-module \( G_s \) defined over \( k \), and a rational point \( v_s \in G(k) \), and shows that there exists a vector \( Z_s \in \text{Lie } G_s(\mathbb{C}_\infty) \) for which

- \( \exp_{G_s}(Z_s) = v_s \),
- the \( n \)-th coordinate of \( Z_s \) is \( c_s \zeta_A(s) \) for some explicit constant \( c_s \in k^\times \),
• the logarithm $\log_{G_s}$ converges $v$-adically at $v_s$,
• the $n$th coordinate of $\log_{G_s}(v_s)_v$ is $c_s \zeta_A(s)_v$.

3.2. A remark. The transcendence theory we used in the proof of Theorem 2.3.1 is Yu’s sub-t-module theorem which is parallel to Wüstholz’s analytic subgroup theorem in the classical transcendence theory. A similar question is to ask whether one is able to give a logarithmic interpretation for classical MZV’s fitting into Wüstholz’s theorem. However, this is not known yet.

References


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