Comparison of local relative characters and the Ichino-Ikeda conjecture for unitary groups

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Ichino-Ikeda conjecture

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- $\pi = \pi_W \boxtimes \pi_V$ : cuspidal automorphic repn of $G(\mathbb{A})$;
- Period : $P_H : \pi \to \mathbb{C}$, $\varphi \mapsto \int_{[H]} \varphi(h) dh$ ($[H] := H(k) \backslash H(\mathbb{A})$).
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Conjecture (Ichino-Ikeda, N. Harris)

Assume \( \pi \) everywhere tempered. Then, for all \( \varphi = \bigotimes'_v \varphi_v \in \pi \) and \( S \) suff large we have

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|P_H(\varphi)|^2 = |S_\pi|^{-1} \mathcal{L}^S(1/2, \pi) \prod_{v \in S} P_{H_v}(\varphi_v, \varphi_v)
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L^S(s, \pi) := \prod_{i=1}^{n+1} L^S(i, \eta_{k'/k}^i) \frac{L^S(s, BC(\pi))}{L^S(s+1/2, \pi, Ad)} \quad \text{with}
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L^S(s, BC(\pi)) := L^S(s, BC(\pi_W) \times BC(\pi_V));
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- $P_{H_v}(\varphi_v, \varphi_v) := \int_{H_v} \langle \pi_v(h) \varphi_v, \varphi_v \rangle_v dh$ (where $\langle \cdot, \cdot \rangle_{\text{Pet}} = \prod_v \langle \cdot, \cdot \rangle_v$);
- $S_\pi : $ component group ass to the A-param of $\pi$ (always of the form $S_\pi \simeq (\mathbb{Z}/2\mathbb{Z})^*$).
Theorem 1 (Zhang, Xue, B.-P.)

The I-I conjecture holds provided there exists a place $v$ such that $BC(\pi_v)$ is supercuspidal (this implies $|S_\pi| = 4$).
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Remarks

- By local multiplicity one results (Aizenbud-Gourevich-Rallis-Schiffmann, Sun-Zhu), the I-I conjecture is always true up to a constant (which might however depends on $\pi$).
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- By local multiplicity one results (Aizenbud-Gourevich-Rallis-Schiffmann, Sun-Zhu), the I-I conjecture is always true up to a constant (which might however depends on \( \pi \)).
- Work in progress of Chaudouard-Zydor on the fine spectral expansions of Jacquet-Rallis trace formulas should allow to remove the above assumption.
Jacquet-Rallis trace formulas

(Simple) Relative Trace Formula for $H\backslash G/H$ : For ‘good’ test fns $f \in C_c^\infty(G(\mathbb{A}))$, we have

$$\sum_{\delta \in H(k) \backslash G_{rs}(k) / H(k)} O(\delta, f) = \sum_{\pi \in \mathcal{A}_{cusp}(G)} J_\pi(f)$$

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where $G_{rs}$ : set of regular semisimple elts for $H \times H$-action (i.e. closed orbit & trivial stab)

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O(\delta, f) := \int_{H(\mathbb{A}) \times H(\mathbb{A})} f(h_1 \delta h_2) dh_1 dh_2 \quad \text{(relative orb. int.)}
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- Reformulation of I-I in terms of relative chars: for $f = \prod_{\nu} f_\nu \in C_c^\infty(G(\mathbb{A}))$,

$$J_\pi(f) = |S_\pi|^{-1} L^S(1/2, \pi) \prod_{\nu \in S} J_{\pi_\nu}(f_\nu)$$

where $J_{\pi_\nu}(f_\nu) := \sum_{\text{ONB}(\pi_\nu)} P_{H_\nu}(\pi_\nu(f_\nu) \varphi_\nu, \varphi_\nu)$ are the local relative characters.
Set $H_1 := GL_{n,k'} \hookrightarrow G' := GL_{n,k'} \times GL_{n+1,k'} \hookrightarrow H_2 := GL_{n,k} \times GL_{n+1,k}$ and
\[ \eta : [H_2] \to \{\pm 1\} \text{ defined by } \eta(h_n, h_{n+1}) := \eta_{k'/k}(\det h_n)^{n+1} \eta_{k'/k}(\det h_{n+1})^n. \]
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(Simple) Relative Trace Formula for $H_1 \backslash G'/(H_2, \eta)$: For ‘good’ test fns $f' \in C_c(\mathcal{G}'(\mathbb{A}))$, we have

$$\sum_{\gamma \in H_1(k) \backslash \mathcal{G}'_rs(k)/H_2(k)} O(\gamma, f') = \sum_{\Pi \mapsto \mathcal{A}_{cusp}(Z(H_2) \backslash G')} l_{\Pi}(f')$$

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(Simple) Relative Trace Formula for $H_1 \backslash G' / (H_2, \eta)$: For ‘good’ test fns $f' \in C_c^\infty(G'(\mathbb{A}))$, we have
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\sum_{\gamma \in H_1(k) \backslash G'_c(k)/H_2(k)} O(\gamma, f') = \sum_{\Pi \mapsto A_{cusp}(Z(H_2) \backslash G')} I_{\Pi}(f')
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where
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O(\delta, f') := \int_{H_1(\mathbb{A}) \times H_2(\mathbb{A})} f'(h_1 \delta h_2) \eta(h_2) dh_1 dh_2 \quad \text{(relative orb. int.)}
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$$I_\Pi(f') = \sum_{ONB(\Pi)} P_{H_1}(\Pi(f') \phi) \underbrace{P_{H_2, \eta}(\phi)}_{\text{R-S period}} \underbrace{I_\Pi(f')}_{\text{F-R period}} \text{ (global relative char.)}$$
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Works of Jacquet-Piatetski-Shapiro-Shalika & Rallis, Flicker give you a factorization (for $f' = \prod_v f'_v$)

$$l_\Pi(f') = \frac{1}{4} L^S_{\text{GL}}(1/2, \Pi) \prod_{v \in S} l_{\Pi_v}(f'_v)$$

where $L^S_{\text{GL}}(s, BC(\pi)) = L^S(s, \pi)$ (and vanish at $s = 1/2$ otherwise).
Jacquet-Rallis trace formulas : Comparison

- J-R have proposed to compare the two RTF. This involves

- A matching of orbits: for all $K/k$, $H(K) \mapsto G_{rs}(K)/H(K)$;

- For every place $v$, a notion of smooth matching (or transfer):
  $C_\infty^c(G_v) \ni f_v \leftrightarrow f'_v \in C_\infty^c(G'_v)$

  if $O(\delta, f_v) = \Omega_v(\gamma)$

  transfer factor $O(\gamma, f'_v)$

  for all $G_{rs}(k_v) \ni \delta \leftrightarrow \gamma \in G'_{rs}(k_v)$.

- Z. Yun & J. Gordon have proved the fundamental lemma: for almost all places $v$ we have
  $1_{G}(O_v) \leftrightarrow 1_{G'}(O_v)$;

- W. Zhang has proved existence of smooth transfer at nonarch places: for all $f_v \in C_\infty^c(G_v)$
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- H. Xue following Zhang has proved existence of smooth transfer at archimedean places for a dense subspace of test functions.

- Output of the comparison (Zhang): if

  $f = \prod_v f_v \in C_\infty^c(G(A))$ and $f' = \prod_v f'_v \in C_\infty^c(G'(A))$ are good test functions st $f_v \leftrightarrow f'_v$ for all $v$, then

  $J_{\pi}(f) = I_{BC}(\pi)(f') = 1_{4L_S}(1/2, \pi) \prod_v I_{BC}(\pi_v)(f'_v)$

  for all $\pi \hookrightarrow \rightarrow A$ cusp ($G$) abstractly $H$-distinguished (i.e. $\text{Hom}_H(A)(\pi, C) \neq 0$).

$\Rightarrow$ to get the I-I conj for good $\pi$'s it only remains to compare $I_{BC}(\pi_v)(f'_v)$ & $J_{\pi_v}(f_v)$.
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for all $\pi \leftrightarrow A_{\text{cusp}}(G)$ abstractly $H$-distinguished (i.e. $\text{Hom}_H(\A, \C) \neq 0$).
Jacquet-Rallis trace formulas: Comparison

- J-R have proposed to compare the two RTF. This involves
  - A matching of orbits: for all $K/k$, $H(K) \backslash G_{rs}(K)/H(K) \rightarrow H_1(K) \backslash G'_{rs}(K)/H_2(K)$;
  - For every place $v$, a notion of smooth matching (or transfer):
    $C^\infty_c(G_v) \ni f_v \leftrightarrow f'_v \in C^\infty_c(G'_v)$ if
    $$O(\delta, f_v) = \Omega_v(\gamma) O(\gamma, f'_v)$$
    transfer factor
    for all $G_{rs}(k_v) \ni \delta \leftrightarrow \gamma \in G'_{rs}(k_v)$.

- Z. Yun & J. Gordon have proved the fundamental lemma: for almost all places $v$ we have
  $1_{G(O_v)} \leftrightarrow 1_{G'(O_v)}$;
- W. Zhang has proved existence of smooth transfer at nonarch places: for all
  $f_v \in C^\infty_c(G_v)$ there exists $f'_v \in C^\infty_c(G'_v)$ such that $f_v \leftrightarrow f'_v$ and conversely;
- H. Xue following Zhang has proved existence of smooth transfer at archimedean places for a dense subspace of test functions.

Output of the comparison (Zhang): if $f = \prod_v f_v \in C^\infty_c(G(\mathbb{A}))$ and
$f' = \prod_v f'_v \in C^\infty_c(G'(\mathbb{A}))$ are good test functions st $f_v \leftrightarrow f'_v$ for all $v$, then

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for all $\pi \rightarrow \mathcal{A}_{cusp}(G)$ abstractly $H$-distinguished (i.e. $\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$).

$\Rightarrow$ to get the I-I conj for good $\pi$'s it only remains to compare $I_{BC(\pi_v)}(f'_v)$ & $J_{\pi_v}(f_v)$. 
Spectral transfer: Zhang’s conjecture

Zhang has defined explicit constants $(\kappa_v)_v$ satisfying $\prod_v \kappa_v = 1$ and made the following conjecture:

Conjecture (Zhang)

Let $v$ be a place of $k$. Then, for all $\pi_v \in \text{Temp}_{H_v}(G_v)$ (set of $H_v$-distinguished tempered representations of $G_v$) and all $C_\infty_c(G_v) \ni f_v \leftrightarrow f'_v \in C_\infty_c(G'_v)$, we have

$$J_{\pi_v}(f_v) = \kappa_v I_{BC}(\pi_v)(f'_v)$$

Theorem 2

Zhang's conjecture holds for all places $v$.

Remarks

Zhang has verified his conjecture for unramified and supercuspidal repns at non-Archimedean places $v$; the conjecture is easy to check at places $v$ which split in $k'$. 

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**Remarks**

- Zhang has verified his conjecture for unramified and supercuspidal repns at non-Achimedean places $v$;
- The conjecture is easy to check at places $v$ which split in $k'$. 
Hiraga-Ichino-Ikeda conjecture on formal degrees

Along the way we also prove

**Theorem 3**

Let $v$ be a nonsplit place of $k$. Then, up to a constant depending on choices of measures, for every square-integrable representation $\pi_v$ of $G_v$ we have

$$d(\pi_v) = \frac{|\gamma(0, \pi_v, \text{Ad}, \psi'_v)|}{|S_{\pi_v}|}$$

where $d(\pi_v)$ denotes the formal degree of $\pi_v$ and $S_{\pi_v}$ the centralizer of the $L$-parameter of $\pi_v$. 
Local Plancherel formula for $GL_n(F) \backslash GL_n(E)$

- We move to a local setting: $E/F$ quad extn of local fields;
Local Plancherel formula for $GL_n(F) \backslash GL_n(E)$

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- An important ingredient for the proof of Theorems 2 & 3 is an ‘explicit’ Plancherel formula for $GL_n(F) \backslash GL_n(E)$;

Let $\varphi \in C^\infty_c(GL_n(F) \backslash GL_n(E))$ and choose $f \in C^\infty_c(GL_n(E))$ s.t.

$$\varphi(x) = \int_{GL_n(F)} f(hx) \, dh$$

For all $\Pi \in \text{Temp}(GL_n(E))$ set

$$f_{\Pi}(x) = \sum_{W \in W(\Pi, \psi)} \beta(\Pi(f)W) \beta(\Pi(x)W)$$

where $W(\Pi, \psi)$ is the Whittaker model of $\Pi$ relative to a char $\psi: E/F \to S^1$, the sum is over an ONB for the scalar product $\langle W, W' \rangle := \int_{N_n(E) \backslash P_n(E)} W W' \, N_n$, and $\beta(W) := \int_{N_n(F) \backslash P_n(F)} W R$.}

Raphaël Beuzart-Plessis

Comparison of local relative characters and the Ichino-Ikeda conjecture for unitary groups

Simons Symposium April 2018
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(W, W') := \int_{N_n(E) \backslash P_n(E)} W \overline{W'}
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$N_n : \text{std maxl unipotent sbgp} \subset P_n : \text{mirabolic sbgp}$,
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$$\beta(W) := \int_{N_n(F) \backslash P_n(F)} W$$
Let $\text{Temp}(U(n))/\text{stab}$ denote the set of tempered $L$-packets of a (quasi-split) unitary gp of rank $n$;
Let \( \text{Temp}(U(n))/\text{stab} \) denote the set of tempered \( L \)-packets of a (quasi-split) unitary gp of rank \( n \);

We have a base-change map (Mok, Kaletha-Minguez-Shin-White)
\( \text{BC} : \text{Temp}(U(n))/\text{stab} \rightarrow \text{Temp}(GL_n(E)) \) (to be precise we consider \textit{unstable} bc if \( n \) even);
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**Theorem 4**

For all $\pi \in \text{Temp}(U(n))/\text{stab}$, $f_{BC}(\pi)$ descents to $\varphi_{BC}(\pi) \in C^\infty(\text{GL}_n(F) \backslash \text{GL}_n(E))$ and moreover we have a spectral decomposition

$$\varphi = \int_{\text{Temp}(U(n))/\text{stab}} \varphi_{BC}(\pi) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi$$

where $\gamma^*(0, \pi, \text{Ad}, \psi') = (\zeta_F(s)^{n\pi} \gamma(s, \pi, \text{Ad}, \psi'))_{s=0}$ and $d\pi$ is locally given by a Haar measure (conn. components of $\text{Temp}(U(n))/\text{stab}$ are quotients of tori by finite groups).
This induces an $L^2$-decomposition

$$L^2(GL_n(F) \setminus GL_n(E)) = \int_{\text{Temp}(U(n))/\text{stab}} BC(\pi) d\mu_{\text{Planch}}(\pi)$$

where $d\mu_{\text{Planch}}(\pi) = \frac{|\gamma^*(0,\pi,\text{Ad},\psi^')|}{|S_\pi|} d\pi$.
Remarks

- *This induces an $L^2$-decomposition*

\[
L^2(\text{GL}_n(F) \backslash \text{GL}_n(E)) = \int_{\text{Temp}(\text{U}(n)) / \text{stab}} BC(\pi) d\mu_{\text{Planch}}(\pi)
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where \( d\mu_{\text{Planch}}(\pi) = \frac{\gamma^*(0,\pi,\text{Ad},\psi')}{|S_{\pi}|} d\pi \);

- *This is in accordance with a very general conjecture of Sakellaridis-Venkatesh on the spectral decomposition of spherical varieties (the ‘L-group’ of \( X = \text{GL}_n(F) \backslash \text{GL}_n(E) \) is \( L\text{U}(n) \)).*
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The proof follows closely the work of Flicker and Rallis on $\text{GL}_n(F)$-periods for global cuspidal autom repns: by some ‘local unfolding’ we express $\varphi(x)$ as the residue at $s = 0$ of a certain local Zeta integral $Z(s, W_f, \phi_0)$ where $\phi_0 \in C_\infty^c(F^n)$;
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The proof follows closely the work of Flicker and Rallis on $GL_n(F)$-periods for global cuspidal autom repns: by some ‘local unfolding’ we express $\phi(x)$ as the residue at $s = 0$ of a certain local Zeta integral $Z(s, W_f, \phi_0)$ where $\phi_0 \in C^\infty_c(F^n)$;

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Local functional equations for this kind of Zeta integrals (Kable) allows to isolate the representations contributing to the residue. Roughly these are the tempered repns $\Pi$ for which $L(s, \Pi, As)$ has a pole of maxl order at $s = 0$. 
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- **This induces an** $L^2$-**decomposition**

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The proof follows closely the work of Flicker and Rallis on $\text{GL}_n(F)$-periods for global cuspidal autom repns: by some ‘local unfolding’ we express $\phi(x)$ as the residue at $s = 0$ of a certain local Zeta integral $Z(s, W_f, \phi_0)$ where $\phi_0 \in C^\infty_c(F^n)$;

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Local functional equations for this kind of Zeta integrals (Kable) allows to isolate the representations contributing to the residue. Roughly these are the tempered repns $\Pi$ for which $L(s, \Pi, As)$ has a pole of maxl order at $s = 0$.

The set of such tempered repns is precisely the image of base-change.
On the proof of Theorems 2 & 3

Let $W \subset V$ be herm spaces over $E$ of dim $n$, $n+1$ and set

$$H = U(W) \hookrightarrow G = U(W) \times U(V)$$

$$H_1 = GL_n(E) \hookrightarrow G' = GL_n(E) \times GL_{n+1}(E) \hookleftarrow H_2 = GL_n(F) \times GL_{n+1}(F)$$

with $\eta : H_2 \rightarrow \{\pm 1\}$ given as before.

(Raphaël Beuzart-Plessis (Aix-Marseille Université))

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We want: for all $\pi \in \text{Temp}_H(G)$ (= set of $H$-dist tempered repn of $G$) and $C_c^\infty(G) \ni f \leftrightarrow f' \in C_c^\infty(G')$,

$$J_\pi(f) = \kappa I_{BC}(\pi)(f')$$

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- For this, we will compare two pairs of distributions on $G$ and $G'$;
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First, by a global-to-local argument (Ichino-Lapid-Mao) and multiplicity one results, we have a weak comparison : for all $\pi \in \text{Temp}_H(G)$, there exists $\kappa(\pi) \in \mathbb{C}$ st

$$J_\pi(f) = \kappa(\pi) I_{BC(\pi)}(f')$$

for all $C_c^\infty(G) \ni f \leftrightarrow f' \in C_c^\infty(G')$. 
Local JR trace formulas

- Local RTF for $H \backslash G / H$: for all $f_1, f_2 \in C^\infty_c(G)$,

$$
\int_{H \backslash G / H} O(\delta, f_1) \overline{O(\delta, f_2)} d\delta = \int_{\text{Temp}_H(G)} J_\pi(f_1) \overline{J_\pi(f_2)} \mu_G(\pi) d\pi
$$

where $\mu_G(\pi)$ is Harish-Chandra $\mu$-function (so that $\mu_G(\pi) d\pi$ is the Plancherel measure for $G$).

$\rightarrow$ no analytic difficulties as generic stabilizers are trivial and $H$ is strongly tempered (i.e. tempered coeff are integrable over $H$).
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- Local RTF for $H_1 \backslash G' / (H_2, \eta)$ for all $f'_1, f'_2 \in C_c^\infty(G')$,

$$\int_{H_1 \backslash G' / H_2} O(\gamma, f'_1) \overline{O(\gamma, f'_2)} d\gamma = \int_{\text{Temp}(G) / \text{stab}} l_{BC}(\pi)(f'_1) \overline{l_{BC}(\pi)(f'_2)} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi$$
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where $\mu_G(\pi)$ is Harish-Chandra $\mu$-function (so that $\mu_G(\pi) d\pi$ is the Plancherel measure for $G$).

→ no analytic difficulties as generic stabilizers are trivial and $H$ is strongly tempered (i.e. tempered coeff are integrable over $H$).

- Local RTF for $H_1 \backslash G' / (H_2, \eta)$ for all $f'_1, f'_2 \in C_c^\infty(G')$,

$$\int_{H_1 \backslash G' / H_2} O(\gamma, f'_1) \overline{O(\gamma, f'_2)} d\gamma = \int_{\text{Temp}(G) / \text{stab}} l_{BC}(\pi)(f'_1) \overline{l_{BC}(\pi)(f'_2)} \left| \frac{\gamma^*(0, \pi, \text{Ad}, \psi')}{|S_{\pi}|} \right| d\pi$$

→ need to know the Plancherel formula for $G' / H_2$. Apart from this, no analytical difficulties as generic stabilizers are trivial and $H_1$ is strongly tempered.
When $f_i \leftrightarrow f'_i$, $i = 1, 2$, comparing the geometric sides we get

\[
\int_{\text{Temp}(G)/\text{stab}} l_{BC(\pi)}(f'_1) l_{BC(\pi)}(f'_2) |\gamma^*(0, \pi, \text{Ad}, \psi')| \frac{|S_{\pi}|}{|S_{\pi}|} \, d\pi =
\int_{\text{Temp}_{H}(G)} J_{\pi}(f_1) J_{\pi}(f_2) \mu_G(\pi) \, d\pi
\]
When $f_i \leftrightarrow f'_i$, $i = 1, 2$, comparing the geometric sides we get

$$\int_{\text{Temp}(G)/\text{stab}} l_{BC}(\pi)(f'_1) l_{BC}(\pi)(f'_2) \frac{\gamma^*(0, \pi, \text{Ad}, \psi')}{|S_\pi|} d\pi =$$

$$\int_{\text{Temp}_H(G)} J_\pi(f_1) J_\pi(f_2) \mu_G(\pi) d\pi = \int_{\text{Temp}_H(G)} |\kappa(\pi)|^2 l_{BC}(\pi)(f'_1) l_{BC}(\pi)(f'_2) \mu_G(\pi) d\pi$$
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Local Gan-Gross-Prasad conj : $\text{Temp}_H(G) \rightarrow \text{Temp}(G)/\text{stab}$ is a bijection (actually need to take Pure Inner Forms into account...);
When $f_i \leftrightarrow f'_i$, $i = 1, 2$, comparing the geometric sides we get

$$\int_{\text{Temp}(G)/\text{stab}} I_{BC}(\pi)(f'_1) I_{BC}(\pi)(f'_2) \frac{|\gamma^*(0, \pi, \text{Ad, } \psi')|}{|S_\pi|} d\pi =$$

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Thus we get

$$\int_{\text{Temp}(G)/\text{stab}} I_{BC}(\pi)(f'_1) I_{BC}(\pi)(f'_2) \left( \frac{|\gamma^*(0, \pi, \text{Ad, } \psi')|}{|S_\pi|} - |\kappa(\pi)|^2 \mu_G(\pi) \right) d\pi = 0$$

for all $f'_1, f'_2 \in C^\infty_c(G')$;
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Separating each spectral contribution, we obtain

$$
|\kappa(\pi)|^2 = \mu_G(\pi)^{-1} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}, \quad \pi \in \text{Temp}_H(G)
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Next, we compare two distributions supported on the ‘nilpotent cone’.
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For $H \backslash G / H$, the ‘trivial’ relative orb integral $O(1, f) = \int_H f(h)$ admits the following spectral expansion:

$$O(1, f) = \int_{\text{Temp}(G)} J_{\pi}(f) \mu_G(\pi) d\pi$$

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(1) $O(1, f) = \int_{\text{Temp}_{H}(G)} J_{\pi}(f) \mu_G(\pi) d\pi$

→ again this is easy using the strong temperedness of $H$. 

For $H \backslash G / H$, there is no invariant distribution supported on the ‘trivial’ double coset (b/c of the character $\eta$). However, there are two regular unipotent orbits $H_1 \xi_+ + H_2$, $H_1 \xi_- H_2$ where $\xi_+ / \xi_- \text{ is upper/lower triangular}$ and they support $(\text{regularized}) \text{ orbital int} f' \mapsto O(\xi_{\pm}, f')$;

Using the Plancherel formula for $G' / H_2$, we can derive the following spectral expansion:

(2) $O(\xi_+, f') = \int_{\text{Temp}(G') / \text{stab}(I_{BC})(\pi)} |\gamma^*(0, \pi, \text{Ad}, \psi')| |S_{\pi}| d\pi$

Moreover, for $f \leftrightarrow f'$, we have $O(1, f) = \kappa O(\xi_+, f')$: this follows from analogs of (1) & (2) at the Lie algebra level (i.e. expansions in terms of FT of orbital integrals) + Compatibility of transfer with FT (Zhang).

Comparing (1) & (2), we obtain $\kappa(\pi) = \kappa \mu_G(\pi) - 1 |\gamma^*(0, \pi, \text{Ad}, \psi')| |S_{\pi}|$, for all $\pi \in \text{Temp}_{H}(G)$. 
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Moreover, for $f \leftrightarrow f'$, we have $O(1, f) = \kappa O(\xi_+, f')$ : this follows from analogs of (1) & (2) at the Lie algebra level (i.e. expansions in terms of FT of orbital integrals). Compatibility of transfer with FT (Zhang).
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Moreover, for $f \leftrightarrow f'$, we have $O(1, f) = \kappa O(\xi_+, f')$: this follows from analogs of (1) & (2) at the Lie algebra level (i.e. expansions in terms of FT of orbital integrals)+ Compatibility of transfer with FT (Zhang).

Comparing (1) & (2), we obtain $\kappa(\pi) = \kappa \mu_G(\pi)^{-1} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}$, for all $\pi \in \text{Temp}_H(G)$. 
End of the proof

All in all, we have obtained

$$|\kappa(\pi)|^2 = \mu_G(\pi)^{-1} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} \quad \text{and} \quad \kappa(\pi) = \kappa \mu_G(\pi)^{-1} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}$$
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This easily implies

$$\kappa(\pi) = \kappa \quad \text{and} \quad \mu_G(\pi) = \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}$$
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All in all, we have obtained

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In the particular case where \( \pi \) is square-integrable this gives

\[ d(\pi) = \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}. \]
Thank you!