A classical example
Automorphic L-functions
Periods of motives
Adumbrating the proofs

Special values of automorphic $L$-functions

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Theorem (Shimura)

Let $\varphi \in S_k(N, \omega)_{\text{prim}}$. There exists $u^{\pm}(\varphi) \in \mathbb{C}^*$ (the periods of $\varphi$) such that for any integer $m$ with $1 \leq m \leq k - 1$, and any Dirichlet character $\chi$ we have

$$L_f(m, \varphi, \chi) \sim (2\pi i)^m \gamma(\chi) u^{\pm}(\varphi)$$

where $\chi(-1) = \pm (-1)^m$.

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$$\mathbb{Q}(\varphi, \chi) := \mathbb{Q}(a_n(\varphi), \text{values of } \chi).$$
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\textit{L-functions of modular forms}

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$$\frac{L_f(m, \varphi, \chi)}{L_f(m + 1, \varphi, \chi)} \sim (2\pi i)^{-1} \frac{u^\pm(\varphi)}{u^\mp(\varphi)}$$

Corollary (Restated...)

The quantity

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Theorem (Harder + R.)

Let $G_n = \text{GL}_n / F$, with $F$ a totally real field. Let $\pi \in \text{Coh}(G_n, \mu)$, and $\pi' \in \text{Coh}(G_{n'}, \mu')$. Assume that $n$ is even and $n'$ is odd. Put $N = n + n'$.

There exists a nonzero complex number $\Omega(\pi)$ depending only on $\pi$ such that if a combinatorial condition $C(\mu, \mu')$ holds then

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\frac{L(-N/2, \pi \times \pi')}{L(1 - N/2, \pi \times \pi')} \sim \Omega(\pi, \pi') \Omega(\pi)^{\epsilon'}
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The combinatorial lemma $C(\mu, \mu')$

The following three conditions on the weights $\mu$ and $\mu'$ are equivalent:

1. $-N/2$ and $1 - N/2$ are critical for $L(s, \pi \times \pi')$.
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3. There exists $w \in W^P$, with $l(w) = nn'/2$ and $w^{-1} \cdot (\mu + \mu')$ is dominant.

If $\pi \mapsto \pi \otimes | |^r$ then $\mu \mapsto \mu - r$ for $r \in \mathbb{Z}$. Do this as long as $C(\mu - r, \mu')$ holds. This gives us a rationality theorem for every successive pair of critical values; no more and no less!

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This theorem has a large intersection in its scope with the works of Michael Harris, Harald Grobner and Jie Lin.
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L-functions for orthogonal groups (with C. Bhagwat)

Theorem ("Theorem")

Let \( n = 2r \geq 2 \) be an even positive integer. Consider \( \text{SO}(n, n)/\mathbb{Q} \) defined so that the subgroup of all upper-triangular matrices is a Borel subgroup. Let \( \mu \) be a dominant integral weight written as \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq |\mu_n|) \), with \( \mu_j \in \mathbb{Z} \). Let \( \sigma \) be a cuspidal automorphic representation of \( \text{SO}(n, n)/\mathbb{Q} \). Assume:

1. the Arthur parameter \( \psi_{\sigma} \) is cuspidal on \( \text{GL}_{2n}/\mathbb{Q} \);
2. \( \sigma \) is globally generic;
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Let $n = 2r \geq 2$ be an even positive integer. Consider $\text{SO}(n, n)/\mathbb{Q}$ defined so that the subgroup of all upper-triangular matrices is a Borel subgroup. Let $\mu$ be a dominant integral weight written as $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq |\mu_n|)$, with $\mu_j \in \mathbb{Z}$. Let $\sigma$ be a cuspidal automorphic representation of $\text{SO}(n, n)/\mathbb{Q}$. Assume:

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Assume also that $|\mu_n| \geq 1$, and suppose $m$ and $m + 1$ are both critical, then

$$L(m, \chi \times \sigma) \approx L(m + 1, \chi \times \sigma),$$

where $\approx$ means up to an element of a number field $\mathbb{Q}(\chi, \sigma)$, and furthermore, all the successive ratios are equivariant under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. 

A. Raghuram
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Adumbrating the proofs

**L-functions for orthogonal groups (with C. Bhagwat)**

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A. Raghuram
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Let $E/F$ be a quadratic extension of totally real fields. Suppose $\pi \in \text{Coh}(\text{GL}_n/E, \mu)$. Let $\chi$ be a finite order character over $F$. Then the critical set for degree $n^2$ twisted Asai $L$-function $L(s, \pi, \text{As}^\pm \otimes \chi)$ is an explicit contiguous string of half-integers determined by $\mu$. Suppose $m$ and $m + 1$ are both critical for $L(s, \pi, \text{As}^\pm \otimes \chi)$ then we have:

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Asai $L$-functions (with Muthu Krishnamurthy)

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Deligne’s Conjecture

Let $M$ be a pure motive over $\mathbb{Q}$ with coefficients in a number field $E$. We have the three realizations:

1. Betti realization $H_B(M)$ with Hodge decomposition
   
   $H_B(M) \otimes_E \mathbb{C} = \bigoplus H^{p,q}$.

2. de Rham realization $H_{dR}(M)$ with a Hodge filtration.

3. $\ell$-adic realization $H_\ell(M)$ with a Galois action.

The comparison isomorphism between

$$H_B(M) \otimes_E \mathbb{C} \rightarrow H_{dR}(M) \otimes_E \mathbb{C}$$

gives two periods $c^{\pm}(M) \in (E \otimes \mathbb{C})^\times / E^\times$.

The Artin $L$-function attached to the Galois representation on $H_\ell(M)$ is the motivic $L$-function $L(s, M)$.

$$L_f(m, M) \sim (2\pi i)^{md^{\pm}(M)} c^{\pm}(M), \quad \pm 1 = (-1)^m.$$
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Let $M$ be a pure motive over $\mathbb{Q}$ with coefficients in a field $E$. Suppose $\mathcal{S}$ some multilinear algebraic structure on $M$. We let $G$ be the structure group of $(M, \mathcal{S})$ defined as:

$$G := \{ g \in \text{GL}(M) : g\mathcal{S} = \mathcal{S}\}^\circ,$$

Suppose we are given an algebraic representation of $(\sigma, V)$ of $G$ defined over $E$. To this data $\{M, \mathcal{S}, G, (\rho, V)\}$ we can attach a motive $M_V$. Assume that $M_V$ has no middle Hodge type. Assume also that the real Frobenius $\iota$ of $M$ is an element of $G$. Decompose $V = V^+ \oplus V^-$ into the eigenspaces for the action of $\rho(\iota)$, i.e., $V^\pm := \{v \in V : \rho(\iota)v = \pm v\}$. 

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Let $M$ be a pure motive over $\mathbb{Q}$ with coefficients in a field $E$. Suppose $\mathcal{G}$ some multilinear algebraic structure on $M$. We let $G$ be the structure group of $(M, \mathcal{G})$ defined as:

$$G := \{ g \in \text{GL}(M) : g\mathcal{G} = \mathcal{G}\}^\circ,$$

Suppose we are given an algebraic representation of $(\sigma, V)$ of $G$ defined over $E$. To this data $\{M, \mathcal{G}, G, (\rho, V)\}$ we can attach a motive $M_V$. Assume that $M_V$ has no middle Hodge type. Assume also that the real Frobenius $\iota$ of $M$ is an element of $G$. Decompose $V = V^+ \oplus V^-$ into the eigenspaces for the action of $\rho(\iota)$, i.e., $V^\pm := \{ v \in V : \rho(\iota)v = \pm v\}$. 

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Motivic periods (with Pierre Deligne)

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Let $Z_G(\iota) := \{ g \in G : g\iota = \iota g \}$ be the centralizer of $\iota$ in $G$. Then $Z_G(\iota)$ stabilizes $V^\pm$ under the representation $\rho$. Define algebraic characters $\chi^\pm : Z_G(\iota) \to E^\times$ by

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\begin{align*}
Z_G(\iota) &\quad \longrightarrow \quad GL(V^\pm) \\
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If $\chi^+ = \chi^-$ then $c^+(M_V) \sim c^-(M_V)$.
Let $M$ be a pure motive over $\mathbb{Q}$ of rank $2n$ and purity weight $w$, and suppose $\beta : M \otimes M \to \mathbb{Q}(\chi)(-w)$ is a symmetric nondegenerate morphism of motives. (Such a $\beta$ gives an orthogonal structure on $M$.) Assume

1. $M$ has no middle Hodge type, (hence $d^+ = d^- = n$),  
2. $n$ is even, and  
3. $\varepsilon_\beta = 1$, i.e., $\chi(-1) = (-1)^w$, 

then $c^+(M) \sim c^-(M)$.

This result, together with Deligne’s conjecture and the Langlands program, implies the results on special values of degree-$2n$ $L$-functions for $\text{SO}(n, n)$.
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Example: Asai motives

Let $F/\mathbb{Q}$ be a real quadratic extension. Let $M$ be a pure motive of rank $n$ over $F$ with coefficients in $E$. Then we have the Asai motives $\text{As}^{\pm}(M)$ both of which are rank $n^2$-motives over $\mathbb{Q}$ with coefficients in $E$. Assume $n$ is even, and that $\text{As}^{\pm}(M)$ have no middle Hodge type. Then $c^+(\text{As}^{\pm}(M)) \sim c^-(\text{As}^{\pm}(M))$.

This result, together with Deligne’s conjecture and the Langlands program, implies the results on special values of degree-$n^2$ Asai $L$-functions for $\text{GL}(n)/F$. 
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Let $G$ be a connected reductive group over $\mathbb{Q}$. Let $K_\infty$ be the maximal compact subgroup of $G_\infty = G(\mathbb{R})$ thickened by the maximal central split torus. Define

$$S^G_{K_f} := G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_f.$$ 

Let $E_\lambda$ be a finite-dimensional irreducible representation of $G$ with highest weight $\lambda$. It is defined over $\mathbb{Q}$. Let $\mathcal{E}_\lambda$ be the corresponding local system on $S^G_{K_f}$. We are interested in the arithmetic information contained in the $G(\mathbb{A}_f) \times \pi_0(G_\infty)$-modules

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Eine kleine Einführung: Eisenstein Kohomologie

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$$H^\bullet(S^G, \mathcal{E}_\lambda) := \lim_{\to} H^\bullet(S_{K_f}^G, \mathcal{E}_\lambda).$$
Inner cohomology is defined as:

\[ H_i^\bullet(S^G, \mathcal{E}_\lambda) := \text{Image}(H_c^\bullet(S^G, \mathcal{E}_\lambda) \to H^\bullet(S^G, \mathcal{E}_\lambda)) \]

Inside inner cohomology is a transcendentally defined subspace called cuspidal cohomology.

\[ H_{\text{cusp}}^\bullet(\ldots) \subset H_i^\bullet(\ldots) \subset H^\bullet(\ldots) \]
Inner cohomology

Inner cohomology is defined as:

\[ H^i_c(S^G, \mathcal{E}_\lambda) := \text{Image}(H^i_c(S^G, \mathcal{E}_\lambda) \to H^i(S^G, \mathcal{E}_\lambda)) \]

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Cuspidal cohomology

Cuspidal cohomology is defined by the diagram:

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\begin{array}{c}
H^\bullet(S^G, \mathcal{E}_\lambda) \rightarrow H^\bullet(g_\infty, K^0_\infty; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E_\lambda) \\
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H^\bullet_{\text{cusp}}(S^G, \mathcal{E}_\lambda) \rightarrow H^\bullet(g_\infty, K^0_\infty; C^\infty_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E_\lambda)
\end{array}
\]

Let $G = \text{GL}_n/\mathbb{Q}$ and $b_n = [n^2/4]$. As a $G(\mathbb{A}_f) \times \pi_0(G_\infty)$-module, we have a multiplicity free decomposition:

\[
H^b_{\text{cusp}}(S^G, \mathcal{E}_\lambda) = \bigoplus H^b_{\text{cusp}}(S^G, \mathcal{E}_\lambda)[\Pi_f \times \epsilon]
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\]
For brevity, let $M = S_{K_f}^G$, and let $\bar{M}$ be the Borel-Serre compactification. Here are a few details:

- $\bar{M} = M \cup \partial \bar{M}$, where the ‘boundary’ $\partial \bar{M}$ is stratified as $\partial \bar{M} = \bigcup_P \partial_P M$, the union running over conjugacy classes of parabolic $\mathbb{Q}$-subgroups $P$ of $G$.
- $\bar{M}$ is a manifold with corners.
- $M$ is the interior of $\bar{M}$ and the inclusion $M \hookrightarrow \bar{M}$ is a homotopy equivalence, hence, $H^\bullet(M, \mathcal{E}) = H^\bullet(\bar{M}, \mathcal{E})$. 
For brevity, let $M = S^G_K$, and let $\tilde{M}$ be the Borel-Serre compactification. Here are a few details:

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A long exact sequence

A fundamental long exact sequence associated to the pair $(\widetilde{M}, \partial \widetilde{M})$ is

$$
\cdots \longrightarrow H_c^i(M, \mathcal{E}) \overset{\iota^*}{\longrightarrow} H^i(\widetilde{M}, \mathcal{E}) \overset{r^*}{\longrightarrow} H^i(\partial \widetilde{M}, \mathcal{E}) \longrightarrow H_{c}^{i+1}(M, \mathcal{E}) \longrightarrow \cdots
$$
What is Eisenstein Cohomology?

- Eisenstein cohomology gets you back into the manifold $S^K_f$ from the boundary $\partial S^K_f$:

$$
\cdots \to H_c^i(M, \mathcal{E}) \to H^i(\bar{M}, \mathcal{E}) \xrightarrow{r^*} H^i(\partial \bar{M}, \mathcal{E}) \xrightarrow{\text{Eis}^*} \cdots
$$

- Eisenstein cohomology is the image of global cohomology in the cohomology of the boundary

$$
H^i_{\text{Eis}}(S^K, \mathcal{E}) = \text{Image}(H^i(S^K, \mathcal{E}) \to H^i(\partial S^K, \mathcal{E})).
$$

- Eisenstein cohomology consists of cohomology classes represented by cocycles built out of Eisenstein series.
What is Eisenstein Cohomology?

- Eisenstein cohomology gets you back into the manifold $S^G_{K_f}$ from the boundary $\partial S^G_{K_f}$:

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Cohomology of the (maximal) boundary strata

Let $P = M_P U_P$ be a maximal proper parabolic subgroup.

$$H^\bullet(\partial_P S^G, \mathcal{E}_\lambda) = a \text{Ind}^{G(\mathbb{A}_f) \times \pi_0(G_\infty)}_{P(\mathbb{A}_f) \times \pi_0(P_\infty)}(H^\bullet(S^{M_P}, \mathcal{H}^\bullet(u_P, \mathcal{E}_\lambda)))$$

$$= \bigoplus_{w \in W_P} a \text{Ind}^{G(\mathbb{A}_f) \times \pi_0(G_\infty)}_{P(\mathbb{A}_f) \times \pi_0(P_\infty)}(H^\bullet - l(w)(S^{M_P}, \mathcal{E}_{w^\lambda})).$$
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\]

\[
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So far: Representations induced from cohomological cuspidal representations appear in boundary cohomology.

We will see: The Eisenstein classes attached to sections of these induced representations carry arithmetic information about the associated Langlands $L$-functions.
A classical example

Automorphic L-functions

Periods of motives

Adumbrating the proofs

Gist of it...

- So far: Representations induced from cohomological cuspidal representations appear in boundary cohomology.
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