Fourier expansion of modular forms on exceptional groups

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Fourier expand “modular form”

**Question:** What does F.E. of “modular form” look like on exceptional group.

**Example:** $GSp_{2n}$. Holomorphic Siegel modular forms:

$$GSp_{2n} = \{ g \in GL(2n) : ^t g \begin{pmatrix} 1_n & 0_{n \times n} \\ 0_{n \times n} & -1_n \end{pmatrix} = \nu(g) \begin{pmatrix} 1_n & 0_{n \times n} \\ 0_{n \times n} & -1_n \end{pmatrix} \}$$

the similitude.

$$S_n := n \times n$$ symmetric matrices

Then

$$\mathcal{H}_n = \{ Z = X + iY : X, Y \in S_n(\mathbb{R}), Y > 0 \} = Sp_{2n}(\mathbb{R})/U(n)$$

Siegel upper half-space, and $GSp_{2n}(\mathbb{R})$ acts on

$$GSp_{2n}(\mathbb{R})/ (\mathbb{R} \times U(n)) = \mathcal{H}_n^{\pm}$$

via $g \cdot Z = (aZ + b)(cZ + d)^{-1}, \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 
Siegel modular form: $f : \mathcal{H}_n^\pm \to \mathbb{C}$ holomorphic,

$$f(Z) = \sum_{T \in S_n(Z)^\vee, T \geq 0} a(T) e^{2\pi i \text{tr}(TZ)}.$$ 

OR: $f \to \varphi_f : \text{GSp}_{2n}(\mathbb{A}) \to \mathbb{C}$ with

$$\varphi_f \left( \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y^{1/2} & \\ & Y^{-1/2} \end{pmatrix} \right) = \varphi_f(n(X)m)$$

$$= \sum_{T \in S_n(\mathbb{Q})} a(T) e^{2\pi i \text{tr}(TX)} e^{-2\pi \text{tr}(TY)}.$$

where $iY = m \cdot i$ in $\mathcal{H}_n$. 
Example: $G_2$.

- No holomorphic structure
- no $P = MN \subseteq G_2$ with $N$ abelian.

Gross-Wallach, Gan-Gross-Savin:

- Notion of modular form on $G_2(\mathbb{A})$ (particular discrete series $\pi_n$ at infinity)
- Special $P$: Heisenberg parabolic.

$P = MN$ with $M = \text{GL}_2$, $N \supseteq N_0 = [N, N] \supseteq 0,$

$$N/N_0 = \text{Sym}^3(V_2) \otimes \det(V_2)^{-1} := W, N_0 = \det(V_2).$$
Gross-Wallach, Wallach, Gan-Gross-Savin

1. \( \varphi \) a modular form on \( G_2 \)
2. \( \varphi_0(g) := \int_{[N_0]} \varphi(ng) \, dn \)
3. Fourier expand \( \varphi_0 \) along \( N/N_0 = W \):

\[
\varphi_0(xm) = \sum_{\ell \in W(\mathbb{Z})^\vee} a(\ell) e^{2\pi i \langle \ell, x \rangle} F_\ell(m)
\]

for some special functions \( F_\ell(m) \).

Remark:

1. Wallach: \( \dim Hom_{N(R)}(\pi_n, \chi) \leq 1 \Rightarrow \) can normalize \( F_\ell \)'s relative to one another
2. Gan-Gross-Savin: Study \( a(\ell) \)'s for \( \Theta \) functions, Eisenstein series (via work of Jiang-Rallis)
Min $K = (SU(2) \times SU(2))/\mu_2$-type of $\pi_n$ is $Sym^{2n}V_2 \boxtimes 1$. If $\varphi \leftrightarrow x^{n+v}y^{n-v} \in Sym^{2n}V_2$ then:

**Theorem**

$\ell \leftrightarrow p_\ell$, cubic polynomial. Then $F_\ell(m)$ is

$$
\det(m)^n |\det(m)| \left( \frac{|p_\ell(z)j(m, i)|}{p_\ell(z)j(m, i)} \right)^v K_V(|p_\ell(z)j(m, i)|)
$$

where $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, $z = m \cdot i$, $j(m, i) = (ci + d)^3 \det(m)^{-1}$, and $a(\ell) = 0$ unless $p_\ell$ has all real roots.
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Modular form on $G_2$

What is a modular form on $G_2$?

Recall:

- (Harish-Chandra) $G(\mathbb{R})$ has discrete series $\leftrightarrow \text{rank}G = \text{rank}K$;
- $(\pi, V_\pi)$ disc series; $V \subseteq V_\pi$ minimal $K$-type. Occurs with multiplicity 1 in $V_\pi$.
- Roughly: $\varphi \in \mathcal{A}(G_2(\mathbb{A}))$ with $\varphi = \varphi_f \otimes \varphi_\infty$ with $\varphi_\infty$ in minimal $K$-type of particular discrete series $\pi_n$ on $G_2(\mathbb{R})$.
- Literally: For $n \geq 2$, there is discrete series $\pi_n$ on $G_2(\mathbb{R})$. Modular form on weight $n \geq 2$ on $G_2$: $\varphi \in \text{Hom}_{G_2(\mathbb{R})}(\pi_n, \mathcal{A}(G_2(\mathbb{R}))).$
Holomorphic case

$Sp_{2n}(\mathbb{R}) \supseteq U(n)$

- $U(n)$ has 1-dimensional representation $\text{det} \otimes^k$
- $\pi_k$ discrete series with 1-dimensional min $K$-type: scalar weight modular form.

Compare with $G_2$:

- $K = (SU(2) \times SU(2))/(\Delta \mu_2) \Rightarrow$ no nontrivial 1-dimensional representations
- $\pi_n$: $V_n = S^{2n}(V_2) \boxtimes 1$ min $K$-type.
- vector valued forms: $\varphi \in Hom_{G_2(\mathbb{R})}(\pi_n, \mathcal{A}(G_2(A)))$, get

$$F_\varphi : G(A) \to V_n^\vee,$$
$$F_\varphi(g) = \varphi(v)(g), \quad F_\varphi(gk) = k^{-1}F_\varphi(g).$$
\( \pi_n \): “quaternionic” discrete series:

\[ T_1(G_2(\mathbb{R})/K) = \mathbb{H} \oplus \mathbb{H} \]

as representation of

\[ \mathbb{H}^{n=1} = \text{SU}(2) \rightarrow K. \]

So, invariant “quaternionic” structure.
Suppose

- $G/R$ reductive group, $G(R)$ connected
- $\pi$ discrete series for $G$
- $P = MN \subseteq G$ parabolic
- $\chi : N(R) \to \mathbb{C}^\times$ character
- $L : V_\pi \to \mathbb{C}$, $L(nv) = \chi(n)L(v)$ for all $n \in N(R)$, $v \in V_\pi$.
- Define $\mathcal{W}_\nu(m) = L(mv)$, generalized Whittaker function

What is $\mathcal{W}_\nu(m)$ as function of $m \in M(R)$, $\nu \in V_{\min}$ $K$-type?

In other words: Prove an Archimedean Casselman-Shalika formula
Examples

- $G$ Hermitian tube domain, $\pi$ holomorphic discrete series, $P = (\ast \ast)\text{ Siegel parabolic, } e^{-2\pi \text{tr}(TY)}. \text{ Classical.}$
- Koseki-Oda $\text{SU}(2, 1)$ large discrete series, Whittaker model
- Moriyama, $\text{Sp}_4(\mathbb{R})$, large discrete series, Whittaker model
- Miyazaki $\text{Sp}_4(\mathbb{R})$, large discrete series, Bessel model
- Yamashita, P-Shah $\text{SU}(2, 2)$, (quaternionic) discrete series, Heisenberg model
- Taniguchi, $\text{SU}(n, 1)$, $\text{Spin}(n, 2)$, quasilarge discrete series, Whittaker model
Schmid’s equations: $G \supseteq K =$ maximal compact.

1. $\varphi \in \text{Hom}_{G(\mathbb{R})}(\pi, \mathcal{A}(G(\mathbb{A})))$
2. $V \subseteq \pi_{\text{min}} K$-type
3. $F_{\varphi} : G(\mathbb{A}) \to V^\vee$, $F_{\varphi}(g)(v) = \varphi(v)(g)$.

Schmid $F_{\varphi}$ satisfies certain differential equations.
More precisely...

Schmid’s equations:

- Cartan involution $\Rightarrow g = p + k$.
- $X_i$ basis of $p$, $X_i^*$ dual basis of $p^*$.
- Define $\tilde{DF} : G(\mathbb{R}) \rightarrow V^\vee \otimes p^*$ via

$$\tilde{DF}(g) = \sum_i (X_iF)(g) \otimes X_i^*.$$

- Representation theory: $pr_- : V^\vee \otimes p^* \rightarrow V_-$; $K$-equiv surjection, image is reps that cannot occur in $\pi$
- $D := pr_- \circ \tilde{D}$, $DF \in C^\infty(G(\mathbb{R}), V_-)$.

Schmid: $DF = 0$. 
E.g., $HDS \rightarrow \bar{\partial}F_\varphi = 0$.

$W_\chi(m)$ satisfies same equations

Use differential equations plus $N(\mathbb{R})$-equivariance to determine $W_\chi(m)$.

If $\dim Hom_{N(\mathbb{R})}(\pi, \chi) \leq 1$, you can hope to succeed

To get formula for $W$: Carry out the above explicitly in useful coordinates
Above: $G_2$. $G$ “quaternionic” (Gross-Wallach). This is when $G/K$ has invariant “quaternionic” structure:

**Examples** with Heisenberg parabolic:

1. Classical: $\text{SU}(n, 2), \text{SO}(N, 4)$
2. Exceptional: $G_2$ (split) $F_4, E_6, E_7, E_8$ (rank four over $\mathbb{R}$).
More info on $G$ quaternionic

$$K = (SU(2) \times L)/(\Delta \mu_2)$$

- e.g., $SU(n,2)$: $K \approx SU(2) \times SU(n)$
- e.g., $SO(N,4)$:
  $$K \approx SO(4) \times SO(N) = SU(2) \times (SU(2) \times SO(N)).$$

**Heisenberg parabolic**: $P = MN$

- e.g., $SU(n,2)$: $M \approx \text{Res}_{\mathbb{C}/\mathbb{R}}(GL_1) \times SU(n-1,1)$.
- e.g., $SO(N,4)$: $M = GL_2 \times SO(N-2,2)$.

In all cases: $M$ has hermitian tube structure,

$N \supset N_0 = [N,N] \supseteq 0$, $N_0$ one dimensional center of $N$,

$N/N_0 =: W$ abelian.
Result of Gross-Wallach

\[ \Theta \text{ Cartan involution: } g = \mathfrak{k} \oplus \mathfrak{p}. \]

To give you some taste for QDS, ala Gross-Wallach:

**Theorem (Gross-Wallach)**

*Suppose* \( G \) *has QDS, and is of adjoint type. Then* \((\mathfrak{p}_0)_c = \mathfrak{p} = V_2 \boxtimes W\) *as representation of* \( K \). *For* \( 2n \geq \dim(W) \), *there is QDS* \( \pi_n \) *for which*

\[
\pi_n|_K = \bigoplus_{k \geq 0} S^{2n+k} V_2 \boxtimes S^k(W) = S^{2n}(V_2) \boxtimes 1 + \cdots
\]

*In particular, minimal* \( K \)-*type is* \( S^{2n} V_2 \boxtimes 1 \).
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Multiplicity one result of Wallach

The above $G$ have $P = MN$ maximal "Heisenberg" parabolic.

Definition

Say $\chi : N(\mathbb{R}) \to \mathbb{C}^\times$ is generic if $\chi(n) = \psi_\infty((v, n))$ and $v$ in open orbit of $M(\mathbb{R})$ on $(N^{ab})^\vee$.

Theorem (Wallach)

Suppose $\chi : N(\mathbb{R}) \to \mathbb{C}^\times$ is generic. Then $\dim \text{Hom}_{N(\mathbb{R})}(\pi_n, \chi) \leq 1$ (and condition on when $\dim = 1$). More generally, multiplicity for all $\pi$. 
The group $F_4$

$F_4$ “build up” from $\text{GSp}_6$ and classical stuff:

Set

1. $W_6$ the defining representation of $\text{GSp}_6$,
2. $W = \wedge^3_0(W_6) \otimes \nu^{-1}$, where
   
   $\wedge^3_0(W_6) = \ker\{\wedge^3 W_6 \to W_6 \otimes \nu\}$.
   (given by contracting along the invariant symplectic form.)
3. $W$ is 14-dimensional
4. Restriction to $\text{SL}_3 \subseteq \text{Sp}_6$:
   
   $W = 1 \oplus S_3 \oplus S_3^* \oplus 1$

5. Typical element $(a, b, c, d)$ with $a, d \in \mathbb{Q}$, $b, c \in S_3(\mathbb{Q})$
   ($S_3 = 3 \times 3$ symmetric matrices)
Facts about $W$

Note

- $W$ has invariant symplectic form:

$$\wedge^2 W \to \wedge^6 W_6 \otimes \nu^{-2} \cong \nu$$

$(\text{det} = \nu^3 \text{ on } \text{GSp}_6)$, i.e. $\nu_1, \nu_2 \in W$,

$$\langle \nu_1, \nu_2 \rangle = \nu_1 \wedge \nu_2 \in 1 \otimes \nu.$$

- **FACT**: $W$ also has invariant quartic form: $\text{Sym}^4(W) \to \nu^2$. 
The Lie algebra of $F_4$

The Lie algebra: 5-step $\mathbb{Z}$-grading:

$$f_4 = g = \nu^{-1} \oplus (W \otimes \nu^{-1}) \oplus \mathfrak{sp}_6 \oplus W \oplus \nu.$$  

in degrees $-2, -1, 0, 1, 2$.

- The map $\text{Sym}^2(W) \to \mathfrak{sp}_6$ involves the invariant quartic form on $W$, and the symplectic form.
- This satisfies Jacobi identity. Define $G = \text{Aut}(g)^0$; this is $F_4$. 

The Heisenberg parabolic

- $P$: Heisenberg parabolic; subgroup of $G$ corresponding to stuff in degree $\geq 0$;
- Equivalently, stabilizing the line spanned by degree 2: $P = \{ p \in G : p \cdot (g)_2 = (g)_2 \}$.
- $P = MN$, $M = \text{GSp}_6$, $N \supseteq N^0 = [N, N] = (g)_2$ with

$$N^{ab} = N/([N, N]) = N/(N_0) = W.$$
Theorem on $F_4$: Setup

- Given $w \in W$, define

$$\chi_w : N(\mathbb{R}) \to \mathbb{C}^\times, \quad \chi_w(n) = \psi(\langle w, \bar{n} \rangle) = \exp(i \langle w, \bar{n} \rangle)$$

where $\bar{n} \in N^{ab} \simeq W$.

- Suppose $\pi = \pi_n$ a QDS, so that min $K$-type of $\pi$ is $\text{Sym}^2 V_2 \boxtimes 1$ as representation of $K = (\text{SU}(2) \times L)/\mu(2)$. ($L$ is compact form of $\text{Sp}_6$.)

- Suppose $\ell \in \text{Hom}_{N(\mathbb{R})}(\pi_n, \chi_w)$, get generalized Whittaker function $\mathcal{W}^\ell : G(\mathbb{R}) \to V^\vee$ via $\mathcal{W}^\ell(g)(v) = \ell(gv)$, $g \in G(\mathbb{R})$ and $v \in V$, min $K$-type.
Theorem on $F_4$: More setup

- Write
  \[ \mathcal{W}^\ell(g) = \sum_{-n \leq j \leq n} \mathcal{W}_j(g) \frac{x^{n+j} y^{n-j}}{(n+j)!(n-j)!}. \]
- Suppose $w = (a, b, c, d)$, where $a, d \in \mathbb{R}$ and $b, c \in \text{Sym}_2$, and $Z \in \mathcal{H}_3^\pm$ the Siegel upper half-space of degree three.
- Define
  \[ q_w(Z) = a \det(Z) + \text{tr}(bZ^\#) + \text{tr}(cZ) + d \]
  \[ (Z^\# = \det(Z)Z^{-1} \text{ the adjoint matrix}). \]
- Furthermore, for $g \in \text{GSp}_6(\mathbb{R})$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, define
  \[ j(g, Z) = \nu(g)^{-1} \det(\gamma Z + \delta). \]
Theorem on $F_4$: Statement

**Theorem**

Let the notations be as above. Then:

1. If $\exists Z_0 \in H_3^\pm$ s.t. $q_w(Z_0) = 0$, $\text{Hom}_N(\mathbb{R})(\pi_n, \chi_w) = 0$.
2. If $q_w(Z)$ never $0$ on $H_3^\pm$, then (up to scalar multiple) for $g \in \text{GSp}_6(\mathbb{R})$,

$$\mathcal{W}_\nu(g) = \left( \frac{|q_w(Z)j(g, i)|}{q_w(Z)j(g, i)} \right)^\nu \nu(g)^n \nu(g) |K_v (|q_w(Z)j(g, i)|)$$

where $Z = g \cdot i \in H_3^\pm$. 

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Corollary on Fourier expansion

Denote $n : W \simeq N^{ab} = N/[N, N]$, and for $\varphi$ on $G(A)$, set

$$\varphi_0(g) = \int_{N_0(Q) \backslash N_0(A)} \varphi(ng) \ dn$$

where $N_0 = [N, N]$.

Corollary

*Suppose* $\varphi \in \text{Hom}_{G(R)}(\pi_n, A^0(G(A)))$. *There are* $a(w) \in C$ *for* $w \in 2\pi W(Q)$ *so that if* $x \in W(R)$, $m \in \text{GSp}_6(R)$ *then*

$$\varphi_0(x^{n+\nu}y^{n-\nu})(n(x)m) = \left( \frac{|q_w(Z)j(g,i)|}{q_w(Z)j(g,i)} \right)^{\nu} \nu(g)^n \times$$

$$\sum_{w \in 2\pi W(Q), w \gg 0} a(w) e^{i\langle w, x \rangle} |\nu(g)| K_v (|q_w(Z)j(g,i)|).$$
On $E_8$...

Theorem is true and proved uniformly on $G_2, F_4, E_6, E_7, E_8$, by explicit uniform calculation.

How to build $E_8$? Out of $GE_7$, like we built $F_4$ out of $GSp_6$:

Denote by $\Theta$ the definite octonions.

Define

$$J_{\Theta} = \left\{ \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{R}, x_1, x_2, x_3 \in \Theta \right\}.$$  

Then $\dim J_{\Theta} = 27$. (This is the smallest nontrivial representation of $E_6$.)

$$\Theta \overset{\sim}{\to} \mathbb{R} \Rightarrow J_{\Theta} \overset{\sim}{\to} S_3(\mathbb{Q}),$$

as in $F_4$-case.
Similar $W$ and $GE_7$

Define

$$W_{\Theta} = \mathbb{R} \oplus J_{\Theta} \oplus J_{\Theta}^\vee \oplus \mathbb{R}$$

- 56-dimensional space
- with symplectic $\langle \ , \ \rangle$ form
- and quartic form $q : \text{Sym}^4(W_{\Theta}) \to \mathbb{R}$ (Freudenthal), just like the 14-dimensional $W$ above.

$$GE_7 = \{ g \in \text{GSp}(W_{\Theta}, \langle \ , \ \rangle) : g \cdot q = \nu(g)^2 q \}.$$
$GE_7$ has tube domain

This form of $GE_7$ is the one that has the tube domain:

$$\mathcal{H}_\Theta = \{ X + iY : X, Y \in J_\Theta, Y > 0 \}.$$ 

Then $GE_7$ acts on $\mathcal{H}_\Theta^\pm$.

Set

$$g = e_8 = \nu^{-1} \oplus (W \otimes \nu^{-1}) \oplus ge_7 \oplus W \oplus \nu 1.$$ 

Again, $\mathbb{Z}$-graded, in degrees $-2, -1, 0, 1, 2$.

- Endow with Lie bracket;
- interesting part $Sym^2(W) \rightarrow ge_7$ in terms of quartic form on $W$ and symplectic form on $W$.
- Satisfies Jacobi identity, $G = E_8 = Aut(g)^0$.

Theorem will be identical to $F_4 \supseteq GSp_6$ case.
Theorem on $E_8$

Suppose $\mathcal{W} = \sum_{-n \leq v \leq n} \mathcal{W}_v \frac{x^{n+v}y^{n-v}}{(n+v)!(n-v)!}$ is generalized Whittaker function for $\pi_n$, on $E_8$ (or, in general) with (recall) minimal $K$-type $\text{Sym}^2 V_2 \boxtimes 1$ of $K = (\text{SU}(2) \times L)/\mu_2$. Then

$$\mathcal{W}_v(g) = \left( \frac{|q_w(Z)j(g, i)|}{q_w(Z)j(g, i)} \right)^v \nu(g)^n|\nu(g)|K_v (|q_w(Z)j(g, i)|)$$

with $g \cdot i = Z$ in $\mathcal{H}^\perp_\Theta$. 
Thank you for your attention!