Some results for rank 1 orthogonal groups

Joint work with T. Kobayashi

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Outline

• Motivation
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- Irreducible representations of $G = O(n + 1, 1)$, $H = O(n, 1)$ with nontrivial $(g, K)$-cohomology and symmetry breaking
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• Distinguished representations and periods
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- Irreducible representations of $G = O(n + 1, 1)$, $H = O(n, 1)$ with nontrivial $(\mathfrak{g}, K)$-cohomology and symmetry breaking

- Distinguished representations and periods

- Nonvanishing bilinear forms on $(\mathfrak{g}; K)$-cohomologies via symmetry breaking
Motivation

Suppose that $H$ is a reductive subgroup of a reductive group $G$. We say that a smooth representation $U$ of $G$ is $H$-distinguished if there is a nontrivial continuous $H$-invariant linear functional $F^U : U \to \mathbb{C}$. 
Motivation

Suppose that $H$ is a reductive subgroup of a reductive group $G$. We say that a smooth representation $U$ of $G$ is $H$-distinguished if there is a nontrivial continuous $H$-invariant linear functional $F^U : U \to \mathbb{C}$.

If the $G$-module $U$ is $H$-distinguished, we say that $(F^U, H)$ is an $H$-period of $U$. 
If $U$ and $W$ are representations of $G$, respectively of $H$, we may consider periods

$$\text{Hom}_H(U \boxtimes W, \mathbb{C}).$$

Note:

$$\text{Hom}_H(U \boxtimes W, \mathbb{C}) = \text{Hom}_H(U, W).$$

So instead of periods we consider continuous operators in

$$\text{Hom}_H(U, W)$$

i.e. Symmetry breaking operators.
Technical Side Remark:

We use in the definition the Casselman -Wallach realization of a representation $U : G \to \text{End}(V)$ where $V$ is a nuclear Freshet space.
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Nice Properties:

- $V = V^\infty$ i.e we also have a representation of the universal enveloping algebra $U(\mathfrak{g})$

- $V$ is a module for the algebra $S(G)$ of Schwartz functions.

- $\text{Hom}_H(U \boxtimes W, \mathbb{C}) = \text{Hom}_H(U, W)$
Examples of symmetry breaking operators:

- Suppose that $G=H$ and that $U$, $W$ principal series representations induced from finite dimensional representations of the minimal parabolic subgroup. The integral intertwining operators introduced by Knapp Stein are symmetry breaking operators.
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- Suppose that $G=H$ are connected real Lie groups of real rank larger than 1 and $U$, $W$ degenerate principal series representations induced from a finite dimensional representation of a maximal parabolic subgroup. Integral intertwining (symmetry breaking) operators are constructed in special cases by Rallis Kudla, Jacquet and others:
Unusual Symmetry breaking operators which are not of "Knapp Stein type"

- $\text{O}(n+1,1)$ acts on $S^n$ and also on the differential forms $D^i(S^n)$. The operator

\[ d : D^i(S^n) \to D^{i+1}(S^n) \]

is a intertwining (symmetry breaking operator) for $G=H=\text{O}(n+1,1)$ between principal series representations of $G=H=\text{O}(n+1,1)$. It is not a Knapp Stein intertwining operator.
G = O(n+1,1), H = \text{stab}(e_n) = O(n, 1).

A spherical principal series representation \( U \) of \( G \) induced from a maximal parabolic subgroup \( P \) is realized on the Schwartz space on \( G/P = S^n \).

\( Q = H \cap P \) is a maximal parabolic subgroup of \( H \). We identify \( H/Q \) with the equator in \( G/P \).

The restriction of functions from \( G/P \) to \( H/Q \) defines a symmetry breaking operator between principal series of \( G \) and \( H \).
Irreducible representations of $G=\text{O}(n+1,1)$, $H=\text{O}(n,1)$ with nontrivial $(\mathfrak{g}, K)$-cohomology and symmetry breaking.
Irreducible representations of $G = O(n+1,1)$, $H = O(n,1)$ with nontrivial $(g, K)$-cohomology and symmetry breaking.

Technical Warning: $O(n,1)$ has a maximal compact subgroup $K = O(n) \times \mathbb{Z}_2$ and has 4 connected components. The character group

\[ \hat{O}(n,1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \]

The characters are denoted by $\chi_{\epsilon,\delta}, \epsilon, \delta \in \mathbb{Z}_2$.

If you are not familiar with representations of the disconnected group ignore the subscripts. For precise details about the parametrization of irreducible representations of this disconnected group please ask after class or see "Symmetry breaking for representations of the rank one orthogonal group." on the arxiv.
Irreducible representations of $G = O(n+1,1)$, $H = O(n,1)$ with nontrivial $(g,K)$-cohomology and symmetry breaking.

**Theorem 1.** 1. Irreducible admissible representations of $G$ with infinitesimal character $\rho$ can parametrized by

$$\text{Irr}(G)_\rho = \{ \Pi_{\ell,\delta} : 0 \leq \ell \leq n+1, \delta = \pm \}.$$  

2. Every $\Pi_{\ell,\delta}$ ($0 \leq \ell \leq n+1, \delta = \pm$) is unitarizable.
Examples:

- There are four one-dimensional representations of $G$, they are given by characters

  \[ \{ \Pi_{0,+} \simeq 1, \quad \Pi_{0,-} \simeq \chi_{+-}, \quad \Pi_{n+1,-} \simeq \chi_{-+}, \quad \Pi_{n+1,-} \simeq \chi_{--}(=\det) \} \]

  and

  \[ \Pi_{i,+} \otimes \chi_{+-} = \Pi_{i,-} \]

  \[ \Pi_{i,+} \otimes \chi_{-+} = \Pi_{n+1-i,+} \]
• If $n$ is odd, $G$ has discrete series representations. In particular \( \Pi_{n,\delta} \) are discrete series representations.

• If $n$ is even $G$ has no discrete series representations and \( \Pi_{n-1,\delta} \) and \( \Pi_{n,\delta} \) are tempered induced representations.
For $\Pi \in \text{Irr}(G)_\rho$ and $\pi \in \text{Irr}(H)_\rho$ we define the multiplicity

$$m(\Pi, \pi) = \dim_\mathbb{C} \text{Hom}_H(\Pi|_H, \pi)$$
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$$m(\Pi, \pi) \leq 1$$

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We determined the multiplicities precisely
Theorem 2.

1. Suppose $0 \leq i \leq n + 1$, $0 \leq j \leq n$ and $\delta, \varepsilon \in \{\pm\}$. If $j = i - 1$ or $i$ and if $\delta \varepsilon = +$, then

$$\dim \mathbb{C} \text{Hom}_{G'}(\Pi_{i,\delta} \mid_{G'}, \pi_{j,\varepsilon}) = 1.$$
Theorem 2.

1. Suppose $0 \leq i \leq n + 1$, $0 \leq j \leq n$ and $\delta, \varepsilon \in \{\pm\}$. If $j = i - 1$ or $i$ and if $\delta \varepsilon = +$, then

$$\dim_{\mathbb{C}} \text{Hom}_{G'}(\Pi_{i,\delta|G'},\pi_{j,\varepsilon}) = 1.$$ 

2. Suppose $0 \leq i \leq n + 1$, $0 \leq j \leq n$, $\delta, \varepsilon \in \{\pm\}$.

   (1) If $j \neq i, i - 1$ then $\text{Hom}_{G'}(\Pi_{i,\delta|G'},\pi_{j,\varepsilon}) = \{0\}$.

   (2) If $\delta \varepsilon = -$, then $\text{Hom}_{G'}(\Pi_{i,\delta|G'},\pi_{j,\varepsilon}) = \{0\}$. 
Remark:

The theorem confirms the Gross Prasad conjecture for the tempered representations with infinitesimal character $\rho$ of the special orthogonal groups of rank one.
Distinguished representations and periods

A period can be obtained by the composition of the symmetry breaking operators with respect to the chain of subgroups:

\[ G = O(n + 1, 1) \supset O(n, 1) \supset O(n - 1, 1) \supset \cdots \supset O(m + 1, 1) = H. \]
Theorem 3.

1. The irreducible representation $\Pi_i$ is $H$-distinguished if $i \leq n - m$.

2. The outer tensor product representation $\Pi_i \boxtimes \pi_j$ has a nontrivial $H$-period if $0 \leq i - j \leq n - m$. 
Using the chain of subgroups

\[ G = O(n + 1, 1) \supset O(n, 1) \supset O(n - 1, 1) \supset \cdots \supset O(m + 1, 1) = H. \]

we also define a vector \( v \) in the minimal \( K \)-type of \( \Pi_i \) and prove
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**Theorem 4.**

Suppose that \( G = O(n + 1, 1) \) and \( \Pi_i \) is the irreducible representation with trivial infinitesimal character \( \rho \) defined as above. Then the value of the \( O(n + 1 - i, 1) \)-period on \( v \in \Pi_i \) is

\[
\frac{\pi^{\frac{1}{2}}(n-1)^i (n+1)!}{((n-i)!)^{i+1}(n+1-i)}. 
\]
Unitary Representations with nontrivial \((g, K)\)-cohomology are important in the cohomology of locally symmetric spaces and so we consider next \textbf{symmetry breaking for representations with nontrivial \((g, K)\)-cohomology.}

\textbf{Technical Remark:} Recall

The \((g, K)\)-cohomology groups are the right derived functor of

\[ \text{Hom}_{g, K}(\mathbb{C}, \ast) \]

from the category of \((g, K)\)-modules.
Theorem 5.

1. Let $\Pi$ be an irreducible unitary representation of $G = O(n + 1; 1)$ such that 

$$H^*(g; K, \Pi) \neq 0.$$ 

Then the smooth representation $\Pi^\infty$ is isomorphic to a representation $\Pi_{\ell, \delta}$ for some $0 \leq \ell \leq n + 1$ and $\delta \in \{+/ -1\}$

2. Suppose $0 \leq \ell \leq n + 1$, $j \in \mathbb{N}$, and $\delta \in \{\pm\}$. Then

$$H^j(g, K; \Pi_{\ell, \delta}) = \begin{cases} 
\mathbb{C} & \text{if } j = \ell \text{ and } \delta = (-1)^\ell, \\
\{0\} & \text{otherwise.}
\end{cases}$$

20
Zhu proved that for the pair $G = \text{Gl}(n, \mathbb{R})$, $H = \text{Gl}(n - 1, \mathbb{R})$ symmetry breaking introduces a nonvanishing bilinear form on the $(g; K)$-cohomologies of tempered representations. We generalize this to pairs representations with nontrivial $(g, K)$-cohomology of $O(n+1,1), O(n,1)$, which are not all tempered.
Theorem 6.

Let \((G, G') = (O(n + 1, 1), O(n, 1))\), \(0 \leq i \leq n\), and \(\delta \in \{\pm\}\).

1. The symmetry breaking operator \(T: \Pi_{i, \delta} \rightarrow \pi_{i, \delta}\) induces bilinear forms

\[
B_T: H^j(g, K; \Pi_{i, \delta}) \times H^{n-j}(g', K'; \pi_{n-i, (-1)^i n\delta}) \rightarrow \mathbb{C}
\]

for all \(j\).

2. The bilinear form \(B_T\) is nonzero if and only if \(j = i\) and \(\delta = (-1)^i\).
Main tools for the proof:

- the classification of symmetry breaking operators for principal series representations

- functional equations for the integral symmetry breaking operators

- for principal series reps with infinitesimal character $\rho$ we determined completely their kernels and images
Final remark:

We expect that most these theorem are also true if the infinitesimal characters of the representations of $G$ and $G'$ satisfy some "interlacing conditions"