K3 dynamics: tropical and Berkovich versions

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Introduction

By definition a K3 surface has trivial canonical bundle and is simply connected. As a consequence, it has a nowhere vanishing 2-form $\Omega$, so it can be viewed as a symplectic surface.

Among all complex or algebraic surfaces, K3s are the only ones (with the exception of tori) that admit volume-preserving automorphisms. Furthermore, the existence of a canonical volume form leads, by Yau’s solution of the Calabi conjecture, to an abundance of Ricci-flat Kähler metrics.

There are many classes of K3 surfaces, but two are particularly helpful to keep in mind. The first are Kummer K3s, which are desingularizations of $A/\pm 1$, where $A$ is a complex 2-torus and $\pm 1$ is its involution when viewed as an abelian group. Holomorphic (or algebraic) automorphisms of $A$ preserving the origin will descend to the Kummer K3. The second class is given by surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by equations of bidegree $(2,2,2)$. These admit 3 dynamically interesting involutions, each given by swapping the two points that have two of the coordinates equal.

Some results

Let now $X$ be a K3 surface and $f: X \to X$ an automorphism. Suppose that $f_*$ acting on the real Neron–Severi group has an eigenvector $[\eta_+]$ of eigenvalue $\lambda > 1$. Then it will have another eigenvector $[\eta_-]$ of eigenvalue $\frac{1}{\lambda}$.

Suppose now that $X$ is defined over the complex numbers. Then theorems of Gromov and Yomdin imply that the topological entropy of $f$ is $\log \lambda$.

**Theorem (Cantat):** There exist closed, positive currents $\eta_\pm$ on $X$ such that $f_*\eta_\pm = \lambda^{\pm 1} \eta_\pm$, and such that $\mu := \eta_- \wedge \eta_+$ is the unique measure of maximal entropy.

In fact, Cantat proved many further properties of the currents and measure.

Suppose now that $X$ is defined over $K = \mathbb{C}((t))$. Let $X^K$ and $f^K$ denote the Berkovich analytifications.

**Theorem:** There exist closed, positive currents $\eta_\pm$ on $X^K$ such that $f^K_*\eta_\pm = \lambda^{\pm 1} \eta_\pm$, and such that $\mu := \eta_- \wedge \eta_+$ is an $f^K$-invariant measure.
The currents $\eta_{\pm}$ and the measure are understood in the sense of Chamber-Loir and Ducros, who developed a formalism of differential forms on Berkovich spaces.

The skeleton

Associated to a K3 surface $X$ over a non-archimedean field $K$ is the skeleton $Sk(X) \subset X^{an}$, defined by Kontsevich and Soibelman. In situations of interest, the skeleton is homeomorphic to a 2-sphere.

By choosing judicious charts in projective realizations of $X$, one can tropicalize the K3 surface and observe the skeleton as a piecewise-linear (PL) polyhedron in Euclidean space. The automorphisms of $X$ can also be tropicalized, leading to explicit PL maps of Euclidean spaces.

The PL automorphisms, as well as their Berkovich versions, preserve a Lebesgue-class volume form on the skeleton. As shown by Boucksom–Jonsson, when $K = \mathbb{C}((t))$ and the K3 comes from a degenerating family of complex K3s, the volume form on the skeleton is an appropriately normalized limit of the volume forms on the elements of the family.