A SURFACE IN ODD CHARACTERISTIC WITH DISCRETE AND NON-FINITELY GENERATED AUTOMORPHISM GROUP

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1. EXTENDED ABSTRACT OF MY TALK

In my talk, I have presented the following result with fairly complete proof. This result is much inspired by recent two remarkable works, due to Lesieutre [Le17] in which a 6-dimensional example as in Theorem 1.1(2), also over characteristic 2, is constructed, and due to Dinh and me [DO19] in which a complex surface example as in Theorem 1.1(2) is finally constructed.

**Theorem 1.1.** Let $\mathbb{F}_p$ be the prime field of characteristic $p \geq 3$. Then:

1. Let $k_0$ be an algebraic closure of $\mathbb{F}_p$. Then for any smooth projective surface $Y$ birational to a K3 surface over $k_0$ and for any field extension $k_0 \subset L$, the automorphism group $\text{Aut}(Y_L/L)$ is (discrete and) finitely generated.

2. Let $k$ be an algebraically closed field such that $\mathbb{F}_p(t) \subset k$, where $t$ is transcendental over $\mathbb{F}_p$. Then there is a smooth projective surface $Y$ birational to some K3 surface, over $k$, such that $\text{Aut}(Y/k)$ is (discrete but) not finitely generated.

It is known that the automorphism group of a K3 surface, hence the automorphism group of any smooth projective surface birational to a K3 surface, is discrete. Indeed $H^0(S, T_S) = \{0\}$ for such surfaces.

**Corollary 1.2.** Let $k$ be as in Theorem 1.1 (2). Then, for any integer $d$ such that $d \geq 2$, there is a smooth projective variety $Y_d$ of dim $Y_d = d$, defined over $k$, such that $\text{Aut}(Y_d/k)$ is discrete but not finitely generated.

Let $S$ be a K3 surface defined over an algebraically closed field $K$. Sterk [St85] shows the finite generation of $\text{Aut}(S/K)$ when $K$ is of characteristic zero by using the Torelli theorem for complex K3 surfaces. Then Lieblich and Maulik [LM18] shows the finite generation of $\text{Aut}(S/K)$ when $K$ is of odd characteristic as Theorem 1.3 below. They reduce to characteristic zero when $S$ is not supersingular (Theorem 1.3 (2)), while they use the crystalline Torelli theorem, which is not yet settled in characteristic 2, when $S$ is supersingular.

**Theorem 1.3.** Let $S$ be a K3 surface defined over an algebraically closed field $K$ of odd characteristic. Then

1. $\text{Aut}(S/K)$ is finitely generated.

2. Assume in addition that $S$ is not supersingular. Then there are a discrete valuation ring $R$ with residue field $K$ and fraction field $Q(R)$ of characteristic 0 and a

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smooth projective morphism $\pi: X \to \text{Spec} R$ with special fiber $S$ such that the specialization map

$$\text{Aut}(\tilde{S}/\tilde{K}) \to \text{Aut}(S/K)$$

has finite kernel and cokernel. Here $\tilde{S}$ is the geometric generic fiber of $\pi$ and $\tilde{K}$ is an algebraic closure of the fractional field $Q(R)$, in particular, $\tilde{S}$ is a K3 surface defined over an algebraically closed field $\tilde{K}$ of characteristic zero.

So, K3 surfaces themselves can not be candidate surfaces with non-finitely generated automorphism groups. Our proof of Theorem 1.1 (2) is quite close to [DO19] and explicit. Indeed, we show that some K3 surface birational to the Kummer surface $Km(E \times F)$ such that the elliptic curves $E$ and $F$ are not isogeneous over $k$ and $E$ is defined by $y^2 = x(x - 1)(x - t)$, where $t \in k$ is transcendental over $\mathbb{F}_p$, satisfies the requirement of Theorem 1.1 (2).

We should emphasize that it remains completely open if there is a smooth projective rational surface $S$ over a field $k$ such that $\text{Aut}(S/k)$ is discrete but not finitely generated. This question is completely open even for $k = \mathbb{C}$.

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References


