Equivariant Minimal Model Program with a View Towards Algebraic and Arithmetic Dynamics

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24th May 2019

Simons Symposium on Algebraic, Complex, and Arithmetic Dynamics
We work over an algebraically closed field $k$ of characteristic 0.

The focus of the talk will be on polarized or amplified endomorphisms of normal projective varieties.

This talk is based on the following joint works.

[**MZ18**] Meng and Zhang, Semi-group structure of all endomorphisms of a projective variety admitting a polarized endomorphism, arXiv:1806.05828

[**M17**] Meng, Building blocks of amplified endomorphisms of normal projective varieties, Mathematische Zeitschrift (to appear), arXiv:1712.08995

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Introduction

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Definition of Polarized, or Int-Amplified Endomorphisms

Let $X$ be a projective variety.

A $(\mathbb{R}$-Cartier) divisor on $X$ is **nef** if $\deg D|_C = D.C \geq 0$ for every curve $C$ on $X$.

A divisor $D$ on $X$ is called **big** if $D = H + E$ for an ample $\mathbb{Q}$-divisor $H$ and effective divisor $E$.

**Definition (Polarized, Amplified)**

Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety, $q \in \mathbb{Z}_{> 1}$.

1. $f$ is **numerically polarized** if $f^*L \equiv qL$ for ample Cartier divisor $L$.
2. $f$ is **numerically quasi-polarized** if $f^*L \equiv qL$ for big Cartier divisor $L$.
3. $f$ is **quasi-polarized** if $f^*L \sim qL$ for big Cartier divisor $L$.
4. $f$ is **q-polarized** if $f^*L \sim qL$ for ample Cartier divisor $L$.
5. $f$ is **int-amplified** if $f^*L - L = H$ for ample Cartier divisors $L$, $H$.
6. $f$ is **amplified** if $f^*L - L = H$ for (not necessarily ample) Cartier divisor $L$ and ample Cartier divisor $H$.

It turns out that the first 4 conditions above are equivalent.
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Theorem (MZ16, MZ18; Polarized Endomorphism: Equivalence)

Suppose $f : X \rightarrow X$ is separable (this is the case if char $k = 0$). Then:

(1) If $f^* L \equiv qL$ for some $\mathbb{R}$-Cartier big divisor $L$, then $f^* H \sim qH$ for some ample Cartier (integral) divisor $H$.

(2) The first 4 conditions in the definition above are equivalent.

Proof. Apply below to $V = \text{NS}_\mathbb{Q}(X)$, $\varphi = f^*$, and nef divisor cone and pseudo effective divisor cone of $X$.

Proposition (MZ16; Cone-Preserving Map)

Let $\varphi : V \rightarrow V$ be an invertible linear map of a positive dimensional real vector space equipped with a norm. Assume $\varphi(C) = C$ for a convex cone $C \subseteq V$ such that $C$ spans $V$ and its closure $\overline{C}$ contains no line. Let $q$ be a positive number. Then the (i) and (ii) below are equivalent.

(1) $\varphi(u) = qu$ for some $u \in C^\circ$ (the interior part of $C$).

(2) There exists a constant $N > 0$, such that $\frac{||\varphi^i||}{q^i} < N$ for all $i \in \mathbb{Z}$.

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Definition (Q-Abelian Variety)

A variety $X$ is \textbf{Q-abelian} if there is a quasi-étale (i.e., étale-in-codimension 1) finite surjective morphism $A \rightarrow X$ from an abelian variety $A$.

A result of Greb-Kebekus-Peternell [GKP16] (extending a result of Nakayama-Zhang [NZ10] to higher dimension) is used in the proof below.

Theorem (NZ10, GKP16, MZ16, M17; Non-Uniruled Case)

Let $f : X \rightarrow X$ be an int-amplified (resp. polarized) endomorphism of a normal projective variety $X$. Suppose \textbf{either one of} the following three conditions.

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(iii) the Kodaira dimension $\kappa(X) \geq 0$.

Then $X$ is Q-abelian, and $f : X \rightarrow X$ lifts to an int-amplified (resp. polarized) endomorphism $f_A : A \rightarrow A$ of a quasi-étale abelian variety cover $A$ of $X$. 
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Non-Uniruled Variety with an Int-Amplified Endomorphism

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Non-Uniruled Variety with an Int-Amplified Endomorphism

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Theorem (MZ16, M17; Equivariant MMP)

Let \( f : X \to X \) be an int-amplified endomorphism of a normal projective variety with at worst \( \mathbb{Q} \)-factorial Kawamata log terminal singularities. Then there exist a finite-index submonoid \( G \) of \( \text{SEnd}(X) \), a \( \mathbb{Q} \)-abelian variety \( Y \), and a \( G \)-equivariant relative MMP:

\[
X = X_0 \to \cdots \to X_t = Y
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over \( Y \) (i.e. every \( g \in G = G_0 \) descends to \( g_i \in G_i \) on each \( X_i \)), such that:

1. There is a quasi-étale Galois cover \( A \to Y \) from an abelian variety \( A \) such that \( G_t \) lifts to a submonoid \( G_A \) of \( \text{SEnd}(A) \).
2. The rational map \( X_{t-1} \to X_t = Y \) is a Fano contraction (a morphism).
3. For any subset \( H \subseteq G \) and its descending \( H_Y \subseteq \text{SEnd}(Y) \), \( H \) acts via pullback on \( \text{NS}_{\mathbb{Q}}(X) \) or \( \text{NS}_{\mathbb{C}}(X) \) as commutative diagonal matrices with respect to a suitable basis if and only if so does \( H_Y \).

(Remark: “\( X \) is smooth rational connected" \( \Rightarrow \) \( Y = \text{pt.} \)"
Theorem (MZ16, M17; Equivariant MMP)

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Equivariant MMP Relative to the Monoid $\text{SEnd}(X)$

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Let $f : X \to X$ be an int-amplified endomorphism of a normal projective variety with at worst $\mathbb{Q}$-factorial Kawamata log terminal singularities. Then there exist a finite-index submonoid $G$ of $\text{SEnd}(X)$, a $\mathbb{Q}$-abelian variety $Y$, and a $G$-equivariant relative MMP:

$$X = X_0 \to \cdots \to X_t = Y$$

over $Y$ (i.e. every $g \in G = G_0$ descends to $g_i \in G_i$ on each $X_i$), such that:

1. There is a quasi-étale Galois cover $A \to Y$ from an abelian variety $A$ such that $G_t$ lifts to a submonoid $G_A$ of $\text{SEnd}(A)$.
2. The rational map $X_{t-1} \to X_t = Y$ is a Fano contraction (a morphism).
3. For any subset $H \subseteq G$ and its descending $H_Y \subseteq \text{SEnd}(Y)$, $H$ acts via pullback on $\text{NS}_\mathbb{Q}(X)$ or $\text{NS}_\mathbb{C}(X)$ as commutative diagonal matrices with respect to a suitable basis if and only if so does $H_Y$.

(Remark: “$X$ is smooth rational connected” $\Rightarrow$ $Y = \text{pt.}$)
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(Remark: “$X$ is smooth rational connected” $\Rightarrow$ $Y = \text{pt.}$)
Let $X$ be a rationally connected smooth projective variety admitting an int-amplified endomorphism $f$. Then there are a finite-index submonoid $G \leq \text{SEnd}(X)$ such that:

$$G^*|_{\text{NS}_\mathbb{Q}(X)}$$

is a commutative diagonal monoid with respect to a suitable $\mathbb{Q}$-basis $B$ of $\text{NS}_\mathbb{Q}(X)$.

Further, for every $g$ in $G$, the representation matrix $[g^*|_{\text{NS}_\mathbb{Q}(X)}]_B$ relative to $B$ satisfies:

$$[g^*|_{\text{NS}_\mathbb{Q}(X)}]_B = \text{diag}[q_1, q_2, \ldots]$$

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Applications of EMMP: Diagonalization

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However, we have the following result, which answers affirmatively (up to finite index) a question of Xinyi Yuan and Shou-Wu Zhang (2013) for rationally connected varieties.

**Theorem (MZ18; Product of Polarized or Int-Amplified Maps)**

Let $X$ be a rationally connected smooth projective variety. Then there is a constant $M \geq 1$ (depending only on $X$) such that if $f_i : X \to X$ ($i = 1, \ldots, s$) are polarized (resp. int-amplified) endomorphisms then the composition

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Applications of EMMP: $\text{Aut}(X)$

**Theorem (MZ16, MZ18; $\text{Aut}(X)$)**

Let $X$ be a rationally connected smooth projective variety. Suppose $X$ admits an int-amplified (or polarized) endomorphism. Then we have:

1. $\text{Aut}(X)/\text{Aut}_0(X)$ is a finite group. More precisely, $\text{Aut}(X)$ is a linear algebraic group (with only finitely many connected components).

2. Every amplified endomorphism of $X$ is int-amplified.

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Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety over $\overline{\mathbb{Q}}$. Let $h_X : X(\overline{\mathbb{Q}}) \rightarrow [0, \infty)$ be a Weil height on $X$ relative to an ample divisor, and $h_X^+ = \max\{1, h_X\}$. The Arithmetic degree of $f$ at a point $P \in X(\overline{\mathbb{Q}})$ is the quantity

$$\alpha_f(P) = \lim_{s \to \infty} h^+(f^s(P)))^{1/s}.$$ 

Here the existence of the above limit is proved by Kawaguchi-Silverman 2012.

**Conjecture (KSC, Kawaguchi-Silverman 2012)**

Let $P \in X(\overline{\mathbb{Q}})$. Suppose that the (forward) orbit $\{f^s(P) \mid s \geq 1\}$ is Zariski dense in $X$. Then

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The Kawaguchi-Silverman Conjecture = KSC

Let \( f : X \to X \) be a surjective endomorphism of a projective variety over \( \overline{\mathbb{Q}} \). Let

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Theorem (M17, MZ18; KSC when $X$ is non-uniruled)

Let $X$ be a normal projective variety with an int-amplified endomorphism. Suppose either one of the following three conditions.

(i) $X$ has at worst Kawamata log terminal singularities and the canonical divisor $K_X$ is pseudo-effective (i.e., a limit of effective divisors).
(ii) $X$ is non-uniruled, i.e., $X$ is not covered by rational curves.
(iii) the Kodaira dimension $\kappa(X) \geq 0$.

Then every surjective endomorphism $g : X \to X$ satisfies the KSC.

Indeed, $X$ is a $\mathbb{Q}$-abelian variety and $g$ lifts to an endomorphism $g_A : A \to A$ of a quasi-étale abelian variety cover $A$ of $X$. 
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Applications of EMMP: Kawaguchi-Silverman Conjecture, Non-Uniruled Case

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Applications of EMMP: Kawaguchi-Silverman Conjecture, Rationally Connected Threefolds Case

Theorem (Meng-Zhang, forthcoming; KSC for Rat Conn Variety)

Suppose $X$ is a smooth projective threefold over $\overline{\mathbb{Q}}$ such that

1. $X$ is rationally connected (e.g. birational to a Fano variety), i.e., every two points are connected by a rational curve, and
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Then every surjective endomorphism $g$ of $X$ (which is not necessarily amplified) satisfies the Kawaguchi-Silverman conjecture. Namely,

$$\alpha_g(P) = d_1(g)$$

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De-Qi Zhang (NUS) Equivariant Minimal Model Program with a View Towards Algebraic and Arithmetic Dynamics 24th May 2019 12/16
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(2) $X$ admits an int-amplified endomorphism.

Then every surjective endomorphism $g$ of $X$ (which is not necessarily amplified) satisfies the Kawaguchi-Silverman conjecture. Namely,

$$\alpha_g(P) = d_1(g)$$

where $P$ is any point in $X$ with Zariski-dense orbit, $\alpha_g(P)$ is the arithmetic degree of $g$ at $P$, and $d_1(g)$ is the first dynamical degree of $g$, i.e., the spectral radius of $g^*|_{\text{NS}_{\mathbb{C}}(X)}$. 
Theorem (Meng-Z. forthcoming)

*KSC holds for any surjective morphism on any projective surface X.*

Remark

The above result was known in the following cases:

1. $X$ is automorphism [Kawaguchi 2005]
2. $X$ is smooth surface [Matsuzawa-Sano-Shibata 2017]
3. $X$ has an int-amplified endomorphism, and at worst Kawamata log terminal singularities when $K_X \equiv 0$ [Matsuzawa-Yoshikawa 2019]

Matsuzawa (2019) proved KSC for connected linear algebraic groups, (weak) Mori Dream Spaces, as well as (KLT) rationally connected varieties admitting an int-amplified endomorphism by assuming the anti-Kodaira dimension $\kappa(X, -K_X) \geq 1$, lower dimensional KSC and the **Flip Termination Conjecture (FTC).**

**FTC** is a very hard conjecture in birational geometry, would imply the existence of minimal models for non-uniruled varieties, and is still open in dimension $> 3$. 
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Our Idea of Proving KSC for Rat Conn Threefolds

Let $X$ be a rationally connected smooth projective variety admitting an int-amplified endomorphism $f$.

We run MMP on $X$ which is equivariant relative to $\text{SEnd}(X)$ up to finite index. Since $X$ is smooth and rationally connected, it (even with a co-dimension-two subset removed) is still simply connected. We can then show that the EMMP ends with $Y$ a point.

Next, the adjunction formula

$$K_X = f^*K_X + R_f$$

implies that $(f^* - \text{id})(-K_X) = R_f$ which is effective. Using the property of $f$ being int-amplified and the analysis on the cone of effective divisors, we can show that the anti-Kodaira dimension $\kappa(X, -K_X) \geq 1$ (cf. [M17]).

This condition was assumed in Matsuzawa’s approach. Since KSC holds for surfaces and using our characterisation of toric varieties, we can prove KSC for rationally connected threefolds.
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Question (Dynamical Degree vs Entropy)

Let

\[ f : X \rightarrow X \]

be a dominant rational map with first dynamical degree \( d_1(f) > 1 \).

Can we find a birational model \( X' \) of \( X \) such that the induced map \( f|_{X'} : X' \rightarrow X' \) has the topological entropy \( h(f|_{X'}) > 0 \)?

The converse \( h(f|_{X'}) > 0 \Rightarrow d_1(f) > 1 \) is by Dinh-Sibony; they proved

\[
\max_i \log d_i(f) \geq h(f).
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Case 1. \( f \) is a morphism. Yes (Gromov, Yomdin)

Case 2. \( f \) is birational map.

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