THE GOALS OF AMPLITUDE

Amplitudes Meet Cosmology symposium
A field named by an observable, not theories/models (sugra/strings, susy, BSM..), nor particular systems (nuclear, cosmic ray..)

Reflection of the attitude that the answer is only the beginning

- What is it’s (often surprising) behavior telling us ?
- What are the interesting “angles” of view ?
- Does it show us an alternative path ?
- What are the other questions this is an answer to ?
A field named by an observable, not theories/models (sugra/strings, susy, BSM..), nor particular systems (nuclear, cosmic ray..)

Inviting with hints of hidden messages open to different “interpretations”

Ask and truth shall be revealed, punishes the “morally” incorrect
From the S-matrix to
Hints at simplicity beyond the classical space-time

\[ \mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^e F_{\mu\nu}^e \]
\[ = -\frac{1}{4} (\partial^\mu A^e_{\nu} - \partial^\nu A^e_{\mu} + g f^{abc}_{\mu
u} A^{a}_{\mu} A^{b}_{\nu}) (\partial_\mu A_\nu^e - \partial_\nu A_\mu^e + g f^{cde}_{\mu\nu} A_\mu^c A_\nu^d) \]
Hints at simplicity beyond the classical space-time

Why is it so simple?
• Amplitudes must factorize on to itself

• Amplitudes are Lorentz invariant, little group covariant functions.

These properties can be most efficiently utilized using "on-shell" variables

\[ A_n(\{p_\mu, e^I_\mu\}) \rightarrow A_n(\{\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}\}) \]

\[
p^\mu \rightarrow p^{\alpha\dot{\alpha}} = p_\mu (\sigma^\mu)^{\alpha\dot{\alpha}} = \begin{pmatrix}
\frac{p^0 + p^3}{2} & \frac{p^1 + ip^2}{2} \\
\frac{p^1 - ip^2}{2} & \frac{p^0 - p^3}{2}
\end{pmatrix}
\]

\[ p^\mu p_\mu = \text{Det}[p^{\alpha\dot{\alpha}}] = 0 \rightarrow p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \]

These new spinor variables transform under both Lorentz and little group

\[
\langle ij \rangle = \lambda^\alpha_i \lambda^\beta_j \epsilon_{\alpha\beta} \\
[ij] = \tilde{\lambda}_{\dot{\alpha}}^\alpha \tilde{\lambda}^{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}
\]

Lorentz inv \hspace{1cm} \text{Little group cov}

\[
\lambda^\alpha \rightarrow e^{-\frac{1}{2}i\theta} \lambda^\alpha, \quad \tilde{\lambda}^{\dot{\alpha}} \rightarrow e^{\frac{1}{2}i\theta} \tilde{\lambda}^{\dot{\alpha}}
\]
• Little group requires

\[ A_n(\{\lambda_i, \tilde{\lambda}_i\}, h_i) \bigg|_{\lambda_i \rightarrow e^{-\frac{1}{2}h_i}, \tilde{\lambda}_i \rightarrow e^{\frac{1}{2}h_i}} = \prod_{i=1}^{n} (e^{h_i \theta_i}) A_n(\{\lambda_i, \tilde{\lambda}_i\}, h_i) \]

• Only one set of factorization channels

\[ A_n(\cdots, i^{-1}, \cdots, j^{-1}, \cdots) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1n \rangle \langle n1 \rangle} \]
From the analytic behavior of the amplitude, factorization, soft behaviors

Amplitude bootstrap equations

- Deform $A_n \rightarrow A_n(z)$ where the poles in $A_n(z)$ are in the kinematic configuration where the properties are understood, exp factorization, soft behaviors
From the analytic behavior of the amplitude, factorization, soft behaviors

Amplitude bootstrap equations

- Deform $A_n \rightarrow A_n(z)$ where the poles in $A_n(z)$ are in the kinematic configuration where the properties are understood

- BCFW tree recursion (factorization) $A_n(z) = A_n(p_1 \rightarrow p_1 + zq, \ p_2 \rightarrow p_2 - zq)$

Britto, Cachazo, Feng, Witten
From the analytic behavior of the amplitude, factorization, soft behaviors

Amplitude bootstrap equations

- Deform \( A_n \rightarrow A_n(z) \), the poles in \( A_n(z) \) are at the kinematic configuration where the properties are understood.

- Soft tree recursion (soft-limits)

Cheung, Kampf, Novotny, Shen, Trnka

\[
A_n(z) = \frac{A_n(p_i \rightarrow (1 - a_i z)p_i)}{\prod_{i=1}^{n}(1 - a_i z)^{d_i}}
\]

\[
A_n|_{p_n=q\rightarrow 0} = S_{-1}A_{n-1} + S_0A_{n-1} + \mathcal{O}(q^1)
\]

\( A_{n-q} \otimes A_{q+2} \)
Analytic bootstrap at loop-level: **Generalized unitarity**

By ``cutting'' (multi-dimensional residues), the integrand must yield a product of tree amplitudes. The solution for the integrand is not unique, Feynman rules are just one solution:

\[
\int d^D \ell \frac{n}{\ell^2 (\ell + k_i)^2} \cdots
\]

Loop amplitudes are not rational, but integrands are rational.

By ``cutting'' (multi-dimensional residues), the integrand must yield a product of tree amplitudes. The solution for the integrand is not unique, Feynman rules are just one solution:

\[\text{Ansatz} \left[ \sum_{G=dias} \frac{n(G)c(G)}{D(G)} \right]_{\text{cuts}} = A_{\text{tree}} \otimes A_{\text{tree}} \otimes \cdots \otimes A_{\text{tree}}\]
Analytic bootstrap at loop-level: Generalized unitarity

By ``cutting'' (multi-dimensional residues), the integrand must yield a product of tree amplitudes. The solution for the integrand is not unique, Feynman rules are just one solution.

- At one-loop there are well established scalar integral basis, and one simply computes the coefficients from cuts.
- There are cut free parts of the amplitude, which can be captured utilizing Locality constraints, or simply D-dimensional cuts.
- Beyond one-loop there are no preferred basis, one has to build own’s ansatz (surprise awaits!)
The simplicity of the answer can be understood from the constraints imposed by unitarity, Lorentz invariance and locality.

In turn, new computation methods are built upon these constraints. Yielding new equivalent representations of the final answer.

In turn, reveals hidden structures that were invisible or unnoticed!
Perturbative gauge/gravity duality

From string theory world-sheet perspective, the open/closed relation is simply that of a KLT double copy (Kawai, Lewellen, Tye)

\[
A^{(M)}_{\text{open (tachyon)}} = \int_{-\infty}^{\infty} dx_1 \cdots dx_M \frac{|(x_a - x_b)(x_b - x_c)(x_c - x_d)|}{dx_a dx_b dx_c} \times \prod_{M \geq i > j \geq 1} |x_i - x_j|^{k_i \cdot k_j}.
\]

\[
A^{M}_{\text{closed (tachyon)}} = \pi \kappa^{M-2} \int d^2 \zeta_1 \cdots d^2 \zeta_M \frac{1}{\Delta} \times \prod_{M \geq i > j \geq 1} (\zeta_i - \zeta_j)^{k_i \cdot k_j} (\bar{\zeta}_i - \bar{\zeta}_j)^{k_i \cdot k_j}.
\]

\[
A^{(4)}_{\text{closed}} = -\pi \kappa^2 \sin \left( \frac{\pi}{2} \alpha^\prime k_2 \cdot k_3 \right) A^{(4)}_{\text{open}}(\alpha^\prime s, \alpha^\prime t) \tilde{A}^{(4)}_{\text{open}}(\alpha^\prime t, \alpha^\prime u).
\]

This seems to be applicable only to string theory states and their field theory limits, and only at tree-level.
Perturbative gauge/gravity duality

\[ A_{\text{closed}}^{(4)} = -\pi \kappa^2 \sin(\pi \frac{1}{2} \alpha' k_2 \cdot k_3) A_{\text{open}}^{(4)}(\alpha'^{1/4} s, \alpha'^{1/4} t) \tilde{A}_{\text{open}}^{(4)}(\alpha'^{1/4} t, \alpha'^{1/4} u) \]

This seems to be applicable only to string theory states and their field theory limits, and only at tree-level. Not true!

<table>
<thead>
<tr>
<th>Bi-Adjoint Scalar:</th>
<th>color (\times) color</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bern, de Freitas, Wong (’99); Bern, Denren, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O’Connell</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(S)YM (…(S)QCD…):</th>
<th>color (\times) spin-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>BCJ (’08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, (’08-’11); Johansson, Ochirov</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(S)Gr (…(S)Einstein-YM…):</th>
<th>spin-1 (\times) spin-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>KLT(’86); BCJ (’08); Chiodaroli, Guneydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kälk, Mogull</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NLSM / Chiral Lagrangian:</th>
<th>“color” (\times) even-spin-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen, Du ’13 Cachazo, He, Yuan ’14 Cheung, Shen ’16</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(S)Born-Infeld:</th>
<th>spin-1 (\times) even-spin-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cachazo, He, Yuan ’14</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Special Galileon:</th>
<th>even-spin-0 (\times) even-spin-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cachazo, He, Yuan ’14 Cheung, Shen ’16</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Open String:</th>
<th>(\alpha' \times) spin-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broedel, Schlotterer, Stieberger</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Closed String:</th>
<th>spin-1 (\times) (\alpha') corrected spin-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broedel, Schlotterer, Stieberger;</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Z-theory:</th>
<th>(\alpha' \times) “color”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer</td>
<td></td>
</tr>
</tbody>
</table>
Perturbative gauge/gravity duality

In solving unitarity cuts, one can utilize the KLT relation to simplify the tree-amplitudes in the cuts.

However, one can be even more clever. A more optimal ansatz is if the YM numerators satisfy the same algebraic relation as its color factors

\[
\begin{align*}
\mathbf{n}_1 &= (\mathbf{k}_4 \cdot \mathbf{k}_5)(\mathbf{k}_3 \cdot \mathbf{e}_1)(\mathbf{e}_2 \cdot \mathbf{e}_3)(\mathbf{e}_4 \cdot \mathbf{e}_5) + \cdots \\
\mathbf{c}_1 &= f^{34a}_f a5b f b12 \\
\mathbf{c}_2 &= f^{34a}_f a2b f b15 \\
\mathbf{c}_3 &= f^{34a}_f a1b f b25
\end{align*}
\]

\[\mathbf{c}_1 = \mathbf{c}_2 - \mathbf{c}_3 \iff \mathbf{n}_1 = \mathbf{n}_2 - \mathbf{n}_3\]

A new color/kinematic duality (BCJ)  Bern Carrasco Johansson
Perturbative gauge/gravity duality  
A new color/kinematic duality (BCJ)

\[ c_1 = c_2 - c_3 \quad \leftrightarrow \quad n_1 = n_2 - n_3 \]

Once a BCJ \( n_i \) is found, one obtains gravity:

\[
\text{YM: } A_n^{\text{tree}} = g^{n-2} \sum_i \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}
\]

\[
\text{Gravity: } M_n^{\text{tree}} = i \left( \frac{\kappa}{2} \right)^{n-2} \sum_i \frac{n_i \bar{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}
\]

Most importantly, the same for loops:

\[
\frac{(-i)^L}{g^{n-2+2L}} A_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{c_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2}
\]

\[
\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} M_n^L = \sum_j \int \prod_{\ell=1}^L \frac{d^D p_\ell}{(2\pi)^D} \frac{\bar{n}_j n_j}{S_j \prod_{\alpha_j} p_{\alpha_j}^2}
\]
Perturbative gauge/gravity duality

A new color/kinematic duality (BCJ)

The square of which directly yield three-loops N=8 sugra!
Going back to where it all started, instead of asking why it was so simple, we ask could there be a new formulation that directly gives this result in one setting?

\[
\begin{array}{c}
\langle 13 \rangle^4 \\
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle
\end{array}
\]
Going back to where it all started, instead of asking why it was so simple, we ask could there be a new formulation that directly gives this result in one setting?

\[
\frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}
\]

A hint can be found by parameterizing our on-shell data, in a different fashion

\[
\chi_i^a = \left( \begin{array}{c} 1 \\ \sigma_i \end{array} \right) t_i \quad \rightarrow \quad \prod_i \frac{1}{\langle ii+1 \rangle} \sim \frac{1}{\sigma_i - \sigma_{i+1}}
\]

The Parke-Taylor factor looks like a world sheet correlation function!

*Witten’s twistor string theory*: a worldsheet model for gluon tree amplitudes

*amps = string correlators* fixed by a map from $\mathbb{CP}^1$ to $\mathbb{CP}^{3|4}$ (twistor space) [Witten, 2003]

\[
j_A(z) j_B(z') = \frac{f^C_{AB} j_C}{z - z'} + \text{double poles} + \ldots
\]

Can we extend beyond four-dimensions and $N=4$?
The CHY formalism: (Cachazo, He, Yuan)

Lesson: $\lambda_i^\alpha \rightarrow (z_i, 1)$ kinematics determine coordinates of the sphere!

Scattering Equations: 

$$E_a = \sum_{i \neq a} \frac{k_i \cdot k_a}{\sigma_i - \sigma_a} = 0$$

Massless kinematics parameterizes the moduli space of $n$-punctured Riemann spheres

Factorization kinematics maps to degenerate limits of the moduli space.
The CHY formalism: \( \text{(Cachazo, He, Yuan)} \)

\[
M_n = \int \frac{d^n\sigma}{\text{vol \, SL}(2, \mathbb{C})} \prod_a \delta(E_a) \mathcal{I}(\{k, \epsilon, \sigma\}) = \sum_{\{\sigma\} \in \text{solns.}} \frac{\mathcal{I}(\{k, \epsilon, \sigma\})}{J(\{\sigma\})}
\]

- Amplitude as an integral over moduli space localized by scattering equations, or equivalently a sum over the solutions, with certain “CHY integrand” [CHY 2013]

- \( n-3 \) integrals with \( n-3 \) delta functions; \( J=\det'(\partial E/\partial \sigma) \) is the Jacobian

What are the integrands?
The CHY formalism: (Cachazo, He, Yuan)

- **Yang-Mills:**

\[
M_n^{YM}[\pi] = \int d\mu_n \text{PT}[\pi] \text{Pf}' \Psi, \quad \text{PT}[\pi] = \frac{1}{\prod_i (\sigma_{\pi_i} - \sigma_{\pi_{i+1}})}
\]

- **Gravity:**

\[
M_n^{h+B+\phi} = \int d\mu_n \text{Pf}' \Psi(\epsilon) \text{Pf}' \Psi(\epsilon')
\]

where Pf' [\psi] is manifest gauge invariant

\[
\text{Pf}' \Psi := \frac{\text{Pf}[\psi]_{i,j}^{i,j}}{\sigma_{i,j}} \quad A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases} \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}
\]

\[
\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \quad C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}
\]
The CHY formalism: (Cachazo, He, Yuan)
Recall our tree recursion formula, requires us to shift two legs

\[ A_n(z) = A_n(p_1 \rightarrow p_1 + zq, \ p_2 \rightarrow p_2 - zq) \]

Cauchy theorem tells us that different shifts are all equivalent. Let’s look at \( k=1 \), which \((k+2)\)-negative helicities

\[ p_1 \rightarrow \lambda_1 (\tilde{\lambda}_1 + z\tilde{\lambda}_n) \]

\[ A_6[1-2-3-4+5+6+] = \text{diagram A} + \text{diagram B} \]

\[ = ?? \]

\[ p_1 \rightarrow (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 \]

\[ A_6[1-2-3-4+5+6+] = \text{diagram A'} + \text{diagram B'} + \text{diagram C'} \]

\( \text{anti-MHV x NMHV} \)

\( \text{MHV x MHV} \)

\( \text{MHV x MHV} \)
Recall our tree recursion formula, requires us to shift two legs

$$A_n(z) = A_n(p_1 \rightarrow p_1 + zq, \ p_2 \rightarrow p_2 - zq)$$

Why are these two expression equivalent?

$$\left( \frac{[4|5 + 6|1]^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3 + 4|5]\ S_{234}} \right) \delta\left(\sum_{i=1}^{6} p_i\right)$$

$$+ \ \frac{[6|1 + 2|3]^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3 + 4|5]\ S_{612}}$$

$$\left( \frac{(S_{123})^3}{[12][23]\langle 45\rangle\langle 56\rangle[1|2 + 3|4][3|4 + 5|6]} \right) \delta\left(\sum_{i=1}^{6} p_i\right)$$

$$+ \ \frac{\langle 12\rangle^3[45]^3}{\langle 16\rangle [34][3|4 + 5|6][5|6 + 1|2]\ S_{612}}$$

$$+ \ \frac{\langle 23\rangle^3[56]^3}{\langle 34\rangle[16][1|2 + 3|4][5|6 + 1|2]\ S_{234}}$$
You just have to view them in the right variables! Penrose

Twistors: $Z_\mu = \begin{pmatrix} \lambda^\alpha \\ \mu^{\hat{\alpha}} \end{pmatrix}$

$\mu^{\hat{\alpha}} = x^{\alpha\alpha} \lambda_\alpha$

---

You just have to view them in the right variables! Hodges

Twistors: $W_\mu = \begin{pmatrix} \lambda^\alpha \\ \mu^{\hat{\alpha}} \end{pmatrix}$ but now $\mu^{\hat{\alpha}} = y_i^{\hat{\alpha}\alpha} \lambda_\alpha$, $y_i - y_{i+1} = p_i$
Remarkably each term is simply:

\[
\left( \frac{(S_{123})^3}{[12][23][45][56][1][2 + 3][4][5 + 6]} \right) + \left( \frac{(12)^3[45]^3}{[16][34][3][4 + 5][6][5 + 6 + 1][2]S_{012}} \right) + \left( \frac{(23)^3[56]^3}{[34][16][1][2 + 3][4][5 + 6 + 1][2]S_{234}} \right) = [1,3,4,5,6]+[3,5,6,1,2]+[5,1,2,3,4]
\]

What are true entity of these blocks?

The area of a two-dimensional triangle:

\[
\text{Area} = \frac{1}{2} \left( \frac{(a, b, c)^2}{(0, b, c)(0, a, b)(0, c, a)} \right) \equiv [a, b, c]
\]

The volume of a four-dimensional simplex:

\[
\frac{\langle a, b, c, d, e \rangle^4}{\langle 0, a, b, c, d \rangle \langle 0, b, c, d, e \rangle \langle 0, c, d, e, a \rangle \langle 0, d, e, a, b \rangle \langle 0, e, a, b, c \rangle}
\]
The amplitude correspond to the volume of a **polytope**, and different BCFW representation are simply different **triangulations**!

The **facets** of the polytope are determined by the propagators
A polytope can be constructed by considering the convex hull of vectors:

$$\bar{A} = \frac{\sum_{i=1}^{5} m_i \bar{u}_i}{\sum_{i=1}^{5} m_i}$$

Write things projectively:

$$A' = \left( \begin{array}{c} 1 \\ \bar{A} \end{array} \right), \quad U'_i = \left( \begin{array}{c} 1 \\ \bar{u}_i \end{array} \right)$$

The convex hull is the inside of the polygon

$$A' = w_1 U'_1 + \cdots + w_n U'_n, \quad w_i > 0, \quad \sum_{i=1}^{n} w_i = 1$$

The inside is determined by $Det[A, U_i, U_j] > 0$

In general complicated to compute
Our amplitudes live in the convex hull of $Z_i$:

$$Y = \sum_i c_i Z_i, \quad c_i > 0$$

With the co-dimension one boundaries (factorisation), given by

$$(Z_i, Z_{i+1}, Z_j, Z_{j+1})$$

This is a special positive geometry: if

$$\det(Z_{i_1}, Z_{i_2}, Z_{i_3}, \ldots) > 0 \quad \forall i_1 < i_2 < i_3 < \cdots$$

Then the convex hull of the $Z$s are a cyclic polytope, where

- All $Z$s are vertices
- For even $Z$, the boundaries are $(1, i, i+1, j, j+1 \ldots)$ and $(\ldots, i, i+1, j, j+1, n)$
- For odd $Z$, the boundaries are $(i, i+1, j, j+1 \ldots)$

Thus the amplitude lives in the space

$$Y = \sum_i c_i Z_i, \quad \text{Gr}_+(1,n) \quad \text{Gr}_+(5,n)$$

The amplitude is then a canonical form with dlog form on the boundary of this space.
For \( k+2 \) negative helicity amplitudes, the geometry generalizes to \( \text{Gr}(k,4+k) \) and \( \text{Gr}+(4+k,n) \) and \( \text{Gr}+(k,n) \).

The amplitude is then a canonical form with dlog singularity on the boundary of this space: the tree Amplitudehedron (Arkani-hamed, Trnka).

A new general picture of what amplitudes are:

Identical picture with bi-adjoint \( \phi^3 \) (Arkani-hamed, Bai, He, Yan)
Many more things to say: loop-level bootstrap using analytic properties of the function, new differential approaches to loop integration, new understanding for structures appearing in field theory limits of string theory, the interplay between the different new formulations .......

But let me spend some time on extension of these ideas into other arenas

- Can these novel structure be extended to other observables?
- If so are they useful? what do they tell us?
- Is there similar “better variables” that can expose the simplicity of the answer?
Double copy beyond flat space scattering:

There has been a growing number of work, on relating classical solutions in GR and Yang-Mills.

Special class of (stationary) Kerr-Schild solutions in gravity, yields single copy for abelian Yang-Mills theory Monteiro, O’Connell, White see Donal’s talk

\[ g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi \rightarrow A^a_\mu = c^a k_\mu \phi \]

As well as time-dependent plane wave solutions, Adamo, Casali, Mason, Nekovar see Mason’s talk

As well as perturbative solutions (Goldberger, Ridgeway, Luna, Monteiro, Nicholson, Ochirov, O’Connell, Westerberg, White)
Double copy beyond flat space scattering:

For classical gravitation potential

\[ V(v, r) \sim -\frac{Gm^2}{r} \left( 1 + v^2 + \frac{Gm}{r} + \cdots \right) \]

\[ V(p, q) \sim -\frac{4\pi Gm^2}{q^2} \left( 1 + \frac{p^2}{m^2} + Gmq + \cdots \right) \]

At long distances BH’s can be treated as point particles, and the conservative part of the potential can be obtained from the rational part of the q-expansion.
Double copy beyond flat space scattering:

The double copy of flat-space amplitudes, can be directly imported!

Use generalized unitarity and the double-copy construction inside the cuts to construct the integrand -> State of the art 3PM computation

Bern, Cheung, Roiban, Shen, Solon, Zeng
Better variables for classical potentials

Consider an amplitude for massive states.

\[ \langle \text{in } t \to +\infty | \text{out } t \to -\infty \rangle \rightarrow M_n \]

I = 1, 2 are doublets of SU(2) Little group.

We introduce

\[ p^{\alpha \dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^\dot{\alpha} \]

In doing so we simply have:

\[ M_n \{ l_1 \alpha \ldots l_{2s_1} \}, \ldots = \lambda_1^{l_1 \alpha_1} \lambda_1^{l_2 \alpha_2} \ldots \lambda_1^{l_{2s_1} \alpha_{2s_1}} M_n, \{ \alpha_1 \alpha_2 \ldots \alpha_{2s_1} \} \ldots \]

Using this the general three-point amplitude simplifies,

\[ M^h_{\{ \alpha_1 \alpha_2 \ldots \alpha_{2s_1} \}, \{ \beta_1 \beta_2 \ldots \beta_{2s_2} \}} \]

we need two vectors to span the space

\[ (v_1 \alpha, v_2 \alpha) = (\lambda_3 \alpha, p_2 \alpha \tilde{\lambda}^\alpha_3) \]

\[ m_1 = m_2 \quad \Rightarrow \quad 2p_2 \cdot p_3 = \langle 3 | p_2 | 3 \rangle = 0 \]

\[ x \lambda_3^\alpha = \frac{p_2^{\alpha \dot{\alpha}} \tilde{\lambda}_3^{\dot{\alpha}}}{m} \]

Three point amplitude is constructed from \((x, \lambda, \varepsilon)\)
Consider the three point amplitude with one massless and two equal mass

\[ s = \frac{1}{2} : \quad x \left( \epsilon_{\alpha\beta} + g_1 x \frac{\lambda_\alpha \lambda_\beta}{m} \right) \]
\[ s = 1 : \quad x \left( \epsilon_{\alpha_1\beta_1} \epsilon_{\alpha_2\beta_2} + g_1 \epsilon_{\alpha_2\beta_2} x \frac{\lambda_{\alpha_1} \lambda_{\beta_1}}{m} + g_2 x^2 \frac{\lambda_{\alpha_1} \lambda_{\beta_1} \lambda_{\alpha_2} \lambda_{\beta_2}}{m^2} \right) \]
\[ \vdots \]
\[ s = 2 : \quad x \left( \epsilon^4 + g_1 \epsilon^3 x \frac{\lambda^2}{m} + g_2 \epsilon^2 x^2 \frac{\lambda^4}{m^2} + g_3 \epsilon x^3 \frac{\lambda^6}{m^3} + g_4 \frac{\lambda^8}{m^4} \right) \]
\[ \vdots \]

Finite size effect

Since BH's are viewed as point particles, set \( g_{-i} = 0 \) to compute spin-dependent terms.
Positive geometry beyond flat space amplitudes:

• Do we encounter convex hull problems physics?
• Do we find cyclic polytopes in such scenario?
Do we encounter convex hull problems physics?

The space of consistent CFT amounts to the space of consistent 4-pt functions (encodes spectrum, OPE couplings, e.t.c.)

\[ \langle \phi(0)\phi(z)\phi(1)\phi(\infty) \rangle \equiv F(z) \]

Unitarity and OPE tells us that it is positively expandable on the conformal blocks

\[ F(z) = \sum_{\Delta} p_{\Delta} C_{\Delta}(z), \quad C_{\Delta}(z) = z^{\Delta} \, _2F_1(\Delta, \Delta, 2\Delta, z) \]

We can consider the polynomial expansion around \( z=1/2 \)

\[ F\left(\frac{1}{2} + y\right) = \sum_{q=0}^{\infty} f_q y^q \]
CFT

We can consider the polynomial expansion around $z=1/2$

$$ F \left( \frac{1}{2} + y \right) = \sum_{q=0}^{\infty} f_q y^q $$

We can also expand the conformal blocks around $z=1/2$

$$ C_\Delta \left( \frac{1}{2} + y \right) = \left( \frac{1}{2} + y \right)^\Delta \, _2F_1(\Delta, \Delta, 2\Delta, \frac{1}{2} + y) = \sum_{q=0}^{\infty} c_{\Delta,q} y^q $$

Unitarity again tells us that the four-point function live in the convex hull of Block vectors!

$$ F = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{L-1} \end{pmatrix} \leq \sum_{\Delta} p_\Delta \begin{pmatrix} c_{\Delta,0} \\ c_{\Delta,1} \\ \vdots \\ c_{\Delta,L-1} \end{pmatrix} \quad p_\Delta > 0 $$
CFT

Unitarity again tells us that the four-point function live in the convex hull of Block vectors

\[
F = \begin{pmatrix}
    f_0 \\
    f_1 \\
    \vdots \\
    f_{L-1}
\end{pmatrix} \in \sum_\Delta p_\Delta \begin{pmatrix}
    c_{\Delta,0} \\
    c_{\Delta,1} \\
    \vdots \\
    c_{\Delta,L-1}
\end{pmatrix}, \quad p_\Delta > 0
\]

Crossing symmetry tells us

\[
z^{-2\Delta_\phi} F(z) = (1 - z)^{-2\Delta_\phi} F(1 - z) \rightarrow F(z) = \left( \frac{z}{1 - z} \right)^{2\Delta_\phi} F(1 - z)
\]
When the block vectors are ordered in conformal dimension, we get positive determinants!

The convex hull of block vectors yield a cyclic polytope

\[
C^I_\Delta = \begin{pmatrix}
1 \\
2\Delta\alpha \\
4\Delta(\Delta - 1) + 2\Delta\alpha \\
\frac{8}{3}\Delta\alpha(\Delta^2 - \Delta + 1) \\
\frac{5}{3}\Delta(\Delta - 1)(\Delta^2 - \Delta + 3) + 4\Delta\alpha \\
\frac{15}{19} [\Delta\alpha(\Delta^2 - \Delta + 1)(\Delta^2 - \Delta + 6) - 2\Delta^2(\Delta - 1)^2] \\
\vdots
\end{pmatrix},
\]

where

\[
\alpha \equiv \frac{2\mathrm{F}_1(\Delta, \Delta + 1, 2\Delta, \frac{1}{2})}{\mathrm{F}_1(\Delta, \Delta, 2\Delta, \frac{1}{2})}.
\]

We know the complete boundaries and the space of intersection is given by simple combinatorial rules.
EFT

- Do we encounter convex hull problems physics?

Consider the coefficients of irrelevant operators

\[ M^{IR}(s, t) \equiv M(s, t)|_{s, t \to 0} = \{\text{massless poles}\} + \sum_{k, q} g_{k, q} s^{k-q} t^q \]

The well controlled nature of the near forward (below threshold) limit discontinuities allows us to write

\[ g_{k, q} = \frac{1}{q!} \frac{d^q}{dt^q} \left( \sum_a \frac{\text{Res}_{s=m_a^2} M(s, t)}{(m_a^2)^{k-q}} \right) + \int \frac{ds'}{s^{k-q+1}} \text{Dis}_{s \geq 4m_a^2} M(s, t) \bigg|_{t=0} + \{u\} \]

The discontinuity is a positive function

\[ \text{Res}_{s=m_a^2} M(s, t) = \sum_a p_a G_{\ell_a}^\alpha (\cos \theta), \quad p_a > 0 \]

\[ \text{Dis}_{s \geq 4m^2} M(s, t) = \sum_{\ell} p_\ell (s) G_{\ell}^\alpha (\cos \theta), \quad p_\ell (s) > 0 \]
EFT

• Do we encounter convex hull problems physics?

Consider the coefficients of irrelevant operators

\[ M^{IR}(s,t) \equiv M(s,t)|_{s,t>0} = \{ \text{massless poles} \} + \sum_{k,q} g_{k,q} s^{k-q} t^q \]

\[ \text{Res}_{s=m^2} M(s,t) = \sum_a p_a G_{\ell_a}^\alpha (\cos \theta), \quad p_a > 0 \]

\[ \text{Dis}_{s>4m^2} M(s,t) = \sum_{\ell} p_{\ell}(s) G_{\ell}^\alpha (\cos \theta), \quad p_{\ell}(s) > 0. \]

\[ G_{\ell}^\alpha (1 + \delta) = \sum_q v_{\ell,q}^\alpha \delta^q. \]

\[ g_{k,q} = \sum_a p_a \left( \frac{2^q u_{\ell_a,k,q}}{(m_a^2)^{k+1}} \right) \quad p_a > 0 \]

The couplings must sit inside the convex hull of cyclic polytope!
• What is the goal?