From Amplitudes to Correlators

Radu Roiban
Pennsylvania State University

an overview
Correlation functions: wealth of information

• In general field theories
  -- Correlation functions of fundamental fields \( \xrightarrow{\text{LSZ reduction}} \) scattering amplitudes
  -- Certain quantities/processes (typically) formulated in terms of correlation fcts
    e.g. anomalies, IR properties, factorization formulae, DIS, etc
  -- Certain inclusive properties of the final state (e.g. energy and charge flow)
  -- all \( n \)-point functions define the theory (Wightman)
  -- Effective action in background fields
    -- special properties and relations to other quantities of interest

• In conformal field theories:
  -- next more complicated quantities after dimensions of operators
  -- \( n \)-point functions may be constructed out of 3-point functions
    -- related to string theory effective action
    -- integral part of gauge/string dualities

• Cosmology:
  -- the only relation between theory and observations
Two types of correlation functions

1. Time-ordered/Euclidian correlation functions

\[ \langle 0 | T \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle = \int D\Phi \; \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{i S[\Phi]} \]

\[ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int D\Phi \; \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S_E[\Phi]} \]

2. in-in correlation functions

\[ \langle \mathcal{O}(t) \rangle = \langle \Omega(t_0) | U^{-1}_{\text{int}}(t, t_0) \mathcal{O}(t) U_{\text{int}}(t, t_0) | \Omega(t_0) \rangle \]

\[ U_{\text{int}}(t, t_0) = T \exp \left( -i \int_{t_0}^{t} H_{\text{int}}(t') dt' \right) \]

\[ \frac{1}{\sigma_{\text{tot}}} \frac{d\sigma_{\text{tot}}}{de} = \sum_{N} d\sigma_{X \to N} \delta(e - \hat{e}(N)) \]

Event shapes/final state correlations:

\[ \mathcal{E}(\vec{n})|k_1, \ldots, k_N \rangle = \sum_{i=1}^{N} \delta(\cos \theta - \cos \theta_i) \delta(\phi - \phi_i) k^0_i |k_1, \ldots, k_N \rangle \]

\[ \langle \mathcal{E}(\vec{n}_1) \cdots \mathcal{E}(\vec{n}_n) \rangle \equiv \frac{\ln \langle 0 | \mathcal{O}^\dagger U^{-1}_{(-\infty, -\infty)} \mathcal{E}(\vec{n}_1) \cdots \mathcal{E}(\vec{n}_n) U_{(+\infty, -\infty)} \mathcal{O} | 0 \rangle_{\text{in}}}{\ln \langle 0 | \mathcal{O}^\dagger \mathcal{O} | 0 \rangle_{\text{in}}} \]

\[ \mathcal{E}(\vec{n}) = \lim_{R \to \infty} R^2 \int_{-\infty}^{+\infty} dt \; n^i T^0_i(t, R\vec{n}^i) \]

Cosmology:

\[ \Delta(p) \propto \frac{1}{p^2 + m^2 - i\epsilon} \]

Basham, Brown, Ellis, Love; Belitsky, Koschemsky, Sterman, Collins, Soper, ...; Hofman, Maldacena

more recently in QCD by Dixon et al
Wightman vs. Euclidian/time-ordered correlation functions:
discussed by Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov

Correlation functions \( \langle O_1(x_1^4, \vec{x}_1) \ldots O_n(x_n^4, \vec{x}_n) \rangle \) in an Euclidian CFT can be analytically continued as \( x_k^4 = -\epsilon_k + it_k \) and are holomorphic in the domain \( 0 < \epsilon_1 < \cdots < \epsilon_n, \; t_k \in \mathbb{R}, \; (\forall) k = 1, \ldots, n \).

The formal limit \( \epsilon_k \to 0 \) defines a single-valued function of \( t_k \) which is the non-time-ordered correlation function \( \langle O_1(t_1, \vec{x}_1) \ldots O_n(t_n, \vec{x}_n) \rangle \) in Minkowski space.

E.g. \[ \langle O(x_1^\dagger) O(x_2) \rangle = \frac{1}{(x_{12}^2)\Delta} \quad \quad \quad G_W(t_1, \vec{x}_1; t_2, \vec{x}_2) = \frac{1}{(-(t_{12} - i\epsilon)^2 + \vec{x}_{12}^2)\Delta} \]

- 3-point functions: Same result as Schwinger-Keldysh technique

- Technically-involved beyond 3-point correlators
Wightman vs. Euclidian/time-ordered correlation functions:

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E.g.
\[
\langle \mathcal{O}^\dagger(x_1)\mathcal{O}(x_2) \rangle = \frac{1}{(x_{12}^2)^\Delta} \quad \rightarrow \quad G_W(t_1, \vec{x}_1; t_2, \vec{x}_2) = \frac{1}{(-(t_{12} - i\epsilon)^2 + \vec{x}_{12}^2)^\Delta}
\]

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Analytically continue through the inverse Mellin transform

\[
G_E^c(\{x_i\}) = \prod_{1 \leq i < j \leq n} \int_{-i\infty}^{+i\infty} \frac{d\delta_{ij}}{2\pi i} \frac{\Gamma(\delta_{ij})}{(x_{ij}^2)\delta_{ij}} \prod_{i=1}^{n} \delta(\Delta_i - \sum_{j \neq i=1}^{n} \delta_{ij}) M(\{\delta_{ij}\})
\]

T-ordered
\[
\begin{align*}
x_{ij}^2 & \rightarrow x_{ij}^2 - i\epsilon \\
\end{align*}
\]

non-T-ordered
\[
\begin{align*}
x_{ij}^2 & \rightarrow x_{ij}^2 - i\epsilon x_{ij}^0 \theta(j - i) \\
\end{align*}
\]
Goal of this talk

**Advances in scattering amplitude calculations**  
**Improved methods for (certain) correlation function calculations**

- Generalized unitarity

- Color-kinematics duality & the double copy

- Recursion relations of various types

- A peek towards curved space correlators (more in Paolo’s & Lionel’s talks)
Focus on general techniques; gloss over aspects of theories with special properties:

-- use information from integrability
  currently restricted to 1-loop; special op’s
  Okuyama, Tseng; RR, Volovich
  Alday, Narain;
  Gromov, Vieira & Escobedo & Sever

-- semiclassical approach at strong coupling
  currently restricted to special configurations of operators
  Extensive work on 3pf starting w/ Zarembo
  n-pf: Buchbinder, Tseytlin

-- recursion relations at strong coupling
  currently restricted to correlators of BPS operators to leading order
  S. Raju

-- symmetry restrictions, general considerations
  some aspects restricted to special classes of operators
  Penedones, Kaplan,
  Fitzpatrick, Raju, van Rees,
  Simmons-Duffin, Costa, Poland, Rychkov

-- hidden symmetries
  currently restricted to correlation functions of chiral stress tensor multiplets
  Eden, Heslop, Korchemsky, Sokatchev

-- etc.
T-ordered (or Euclidian) correlation functions vs. scattering amplitudes

Close relation between correlation functions and certain scattering amplitudes

\[
Z[J_1, \ldots, J] = \int [D\Phi] e^{-S_E[\Phi]} - \int d^d x \sum_i J_i(x) \mathcal{O}_i(x) - J(x) \Phi(x)
\]

Local sources; interpret as (infinitely many) fictitious non-dynamical fields

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \frac{(-)^n \delta^n}{\delta J_1(x_1) \ldots \delta J_n(x_n)} Z[J_1, \ldots, J] \bigg|_{J \rightarrow 0}
\]

-- formal similarity with fundamental field Green’s functions \[\rightarrow\] amplitudes

\[
\langle \Phi(y_1) \ldots \Phi(y_m) \rangle = \frac{(-)^m \delta^m}{\delta J(y_1) \ldots \delta J(y_m)} Z[J_1, \ldots, J] \bigg|_{J \rightarrow 0}
\]

-- same goes for single and multi-operator form factors \[\langle 0|\mathcal{O} \ldots |1 \ldots m \rangle\]

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \Phi(y_1) \ldots \Phi(y_m) \rangle = \frac{(-)^n \delta^n}{\delta J_1(x_1) \ldots \delta J_n(x_n)} \frac{(-)^m \delta^m}{\delta J(y_1) \ldots \delta J(y_m)} Z[J_1, \ldots, J] \bigg|_{J \rightarrow 0}
\]

correlation functions \& form factors \[\leftrightarrow\] scattering amplitudes of \[J\] with action

\[
S = S_E[\Phi] + \int d^d x \sum_i J_i(x) \mathcal{O}_i(x)
\]

Can also be interpreted as form factors of the 0-momentum interaction Lagrangian
(Momentum space) correlation functions

\[ \langle \tilde{O}_1(q_1) \ldots \tilde{O}_n(q_n) \rangle \quad \text{with} \quad \tilde{O}(q) = \int \frac{d^d q}{(2\pi)^d} e^{i q \cdot x} \mathcal{O}(x) \]

- can use same techniques as for scattering amplitudes
- Some comments:
  - momentum space is slightly inefficient
    e.g. L.O. momentum space correlators in free field theory are nontrivial

\[ \mathcal{J} \quad \langle O(x_1) \ldots O(x_n) \rangle \quad \mathcal{J} \]

Nontrivial integral

- (In general) not clean separation between planar and nonplanar contrib’s to correlation fct’s— some nonplanar parts of source-field amplitudes needed
(Momentum space) correlation functions

\[ \langle \tilde{\mathcal{O}}_1(q_1) \cdots \tilde{\mathcal{O}}_n(q_n) \rangle \quad \text{with} \quad \tilde{\mathcal{O}}(q) = \int \frac{d^d q}{(2\pi)^d} e^{i q \cdot x} \mathcal{O}(x) \]

- can use same techniques as for scattering amplitudes
  - Flat space
    - Generalized unitarity method
    - Color/kinematics duality and double copy
    - On-shell recursion relation(s)
    - Perturbative approach (leading order)
  - Position space as a step towards curved space
    - Generalized unitarity in position space
    - On-shell recursion relation(s)
    - Perturbative approach (leading order)
Generalized unitarity method – scattering amplitudes

Loop amplitude = $\int [dL] \text{ Loop integrand}$

Idea: reconstruct loop integrand from its poles and residues

Bern, Dixon, Dunbar, Kosower
1 loop improvements: Britto, Cachazo, Feng

complicated, transcendental
rational, with prescribed poles and residues
Generalized unitarity method – scattering amplitudes

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Idea: reconstruct loop integrand from its poles and residues

- Possible poles: each propagator in any Feynman graph consistent with given external legs and loop order when its corresponding momentum is on shell

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Generalized unitarity method – scattering amplitudes

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- Residues (generalized cuts): products of (tree-level) amplitudes

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Generalized unitarity method – scattering amplitudes

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- Possible poles: each propagator in any Feynman graph consistent with given external legs and loop order when its corresponding momentum is on shell

- Residues (generalized cuts): products of (tree-level) amplitudes

- Reconstruction algorithm:
  - Case-by-case simplifying features (symmetries, power counting, etc)
  - Cut merging
  - Method of maximal cuts/ansatz
  - Method of maximal cuts/Contact terms

- Checks of the result: special kinematic limits; cuts not used in the construction, etc
E.g. Method of maximal cuts/contact terms

\[ N^k\text{-contact} = N^k\text{-max cut} - (N^k\text{-max cut of approximation of amp.}) \]

- Each cut gives an independent contrib. to amplitude
- Freedom in choosing each of them
- Lots of cuts
- But a finite number!

- Effectively a tree-level calculation
- Ideal if cuts are organized in terms of cubic tree graphs
Generalized unitarity method – momentum space correlation functions

\[
\text{Loop amplitude of non-dynamical fields} = \int [dL] \text{ Loop integrand of non-dynamical fields}
\]
Generalized unitarity method – momentum space correlation functions

Correlation function = \int [dL] Correlator Integrand

complicated, transcendental
rational, with prescribed poles and residues

Idea: reconstruct loop integrand from its poles and residues

- Possible poles: each propagator in any Feynman graph consistent with deformed \( \mathcal{L} \), given external legs and loop order when its corresponding momentum is on shell

\[ \mathcal{J} \leftrightarrow \text{Tr}[\bar{\phi} \phi] \]

+ 2 more labeling

- Residues (generalized cuts): products of (tree-level) amplitudes and form factors

- Reconstruction algorithm:

- Checks of the result: cuts not used in the construction, OPE, symmetries, etc
Complete correlator vs. separated points: \( C(x_1, x_2, \ldots x_n) \equiv C(x_i - x_j; i \neq j = 1, \ldots, n) \)

If we are interested in correlation functions with all points separated, then generalized cuts with multi-operator form factors can be ignored.

\[
\int d^d q_1 d^d q_2 \delta^d \left( \sum q_i \right) e^{i(q_1 \cdot x_1 + q_2 \cdot x_2)} f_{ct}(q_1 + q_2, \ldots) = \int d^d (q_1 - q_2) e^{i/2(q_1 - q_2) \cdot (x_1 - x_2)} \int d^d (q_1 + q_2) \delta^d \left( \sum q_i \right) e^{i/2(q_1 + q_2) \cdot (x_1 + x_2)} f_{ct}(q_1 + q_2, \ldots) \propto \delta^d(x_1 - x_2)
\]

For separated points: any two operators must have at least one propagator between them

\( \rightarrow \) Fully captured by single-operator form factors; can ignore multi-operator form factors

\( \mathcal{J} \leftrightarrow \text{Tr}[\bar{\phi} \phi] \)
Correlation functions and form factors from generalized unitarity: brief summary

- Same steps as for scattering amplitudes

- Building blocks of generalized cuts are tree-level amplitudes and tree-level form factors*

- Most of the properties of tree-level amplitudes are inherited by cuts of form factors and correlation functions

- Generalized cuts of some higher-loop correlation function are the same as those of a leading order correlator of the same operators and some number of zero-momentum (complete) Lagrangians (more later)
Example: A 3-point correlator in dimensional reduction of YM theory

\[ \mathcal{O}_1 = \text{Tr}(\phi^I_1 \phi^J_1) \quad \mathcal{O}_2 = \text{Tr}(\phi^I_2 \phi^J_2) \quad \mathcal{O}_{3;x,S} = \text{Tr}(D^+_x \phi^I_3 D^S_x - x \phi^J_3) \]

“Good” operator:

\[ \mathcal{O}_3 = i^S (n \cdot D_1 + n \cdot D_2)^S P_S^{(0,0)} \left( \frac{n \cdot D_2 - n \cdot D_1}{n \cdot D_2 + n \cdot D_1} \right) \text{Tr}[\phi^I_3(\xi_2) \phi^J_3(\xi_1)] \bigg|_{\xi_1 = \xi_2} \]

Belitsky, Henn, Jarczak, Mueller, Sokatchev

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = c_1^x \delta\{J_1|I_3\} \delta_J \delta_I + c_2^x \delta\{J_1|I_3\} \delta_J \delta_I \]

• Choose \( I_3 = J_3 \) the two structures become the same

Leading order relevant form factors:

\[ \langle \tilde{\mathcal{O}}^{AB,CD}_{2,S,x} | \phi_{AB} \phi_{CD} \rangle = (p_a^{-})^x (p_b^{-})^{S-x} \]

• Leading order correlator:

\[ \tilde{c}_1^x(0) + \tilde{c}_2^x(0) = \int \prod_{i=1}^3 \frac{d^4 l_i}{(2\pi)^4} \delta^4(q_1 - l_2 - l_3) \delta^4(q_2 + l_3 - l_1) \delta^4(q_3 + l_1 + l_2) \frac{l_1^x l_2^{S-x} + l_3^{S-x} l_2^x}{l_1^2 l_2^2 l_3^2} \]

Symmetry is consequence of \( I_3 = J_3 \)
Example: A 3-point correlator in dimensional reduction of YM theory

\[ \mathcal{O}_1 = \text{Tr}(\phi^I_1 \phi^J_1) \quad \mathcal{O}_2 = \text{Tr}(\phi^I_2 \phi^J_2) \quad \mathcal{O}_{3;x,S} = \text{Tr}(D^x_+ \phi^I_3 D^{S-x}_+ \phi^J_3) \]

“Good” operator:

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = c_1^x \delta^{\{J_1 \delta^I_1\}} \delta^I_3 + c_2^x \delta^{\{J_1 \delta^I_1\}} \delta^I_3 \]

Choose \( I_3 = J_3 \) the two structures become the same

Leading order relevant form factors:

\[ \langle \tilde{\mathcal{O}}^{AB,CD}_{2,S,x} | \phi_{AB} \phi_{CD} \rangle = (p_a^-)^x (p_b^-)^{S-x} \]

Cuts relevant at next-to-leading order:

Next-to-leading order relevant form factors:

\[ \langle \tilde{\mathcal{O}}^{AB,CD}_{2,S,x} | \phi_{AB} \phi^+_{CD} \rangle = \frac{\langle ba \rangle}{\langle ai \rangle \langle ib \rangle \langle ba \rangle} \left( (p_a^-)^x (p_i^- + p_b^-)^{S-x} \frac{\langle a|\sigma^-|b \rangle}{2p_i^-} + (p_a^- + p_i^-)^x (p_b^-)^{S-x} \frac{\langle a|\sigma^-|p_i|b \rangle}{2p_i^-} \right) \]
\[ \tilde{c}_1^{x(1)} + \tilde{c}_2^{x(1)} = \left( (l_1^-)^{S-x} (l_2^-)^x + (l_2^-)^{S-x} (l_1^-)^x \right) \times \left( q_3^2 \frac{q_1}{l_2} + q_1^2 \frac{q_2}{l_2} + q_2^2 \frac{q_1}{l_2} \right) + \left( \frac{l_4^-}{l_2^-} \right) \left( (l_1^- + l_2^-)^{S-x} (l_3^-)^x + (l_1^- + l_2^-)^x (l_3^-)^{S-x} \right) \]

- no pole at \( l_2^- = 0 \)

\[ + \frac{l_5^-}{l_2^-} \left( (l_1^-)^{S-x} (l_2^- + l_3^-)^x + (l_1^-)^x (l_2^- + l_3^-)^{S-x} \right) \]

\[ + \frac{l_2^- + l_3^-}{l_2^-} \left( (l_1^-)^{S-x} (l_2^- + l_3^-)^x - (l_1^- + l_2^-)^{S-x} (l_3^-)^x \right) \]

\[ + \frac{l_1^- + l_2^-}{l_2^-} \left( (l_3^-)^x (l_1^- + l_2^-)^{S-x} - (l_2^- + l_3^-)^x (l_1^-)^{S-x} \right) \]

\[ + \frac{l_1^- + l_2^-}{l_2^-} \left( (l_3^-)^{S-x} (l_1^- + l_2^-)^x - (l_2^- + l_3^-)^{S-x} (l_1^-)^x \right) \]

• structure required by conformal invariance emerges after reconstruction of \( O_3 \)

• last two integrals are responsible for the anomalous dimension term
Direct gravity amplitudes calculations

It is not easy to use Feynman graphs (though many have done it):

- 3 terms per propagator
- $O(100)$ terms per 3-point vertex
- Too many per higher-point vertex

Current method of choice for gravitational amplitude calculations:
relate them to simpler gauge theory ones

Two options:
1. Gravity generalized cuts $\leftrightarrow$ Gauge theory generalized cuts
2. Complete gravity integrands $\leftrightarrow$ Complete gauge theory integrands

Two technical tools:
1. The Kawai-Lewellen-Tye relations between tree amplitudes
2. The double-copy construction of Bern, Carraso, Johansson
Gravity correlators from gauge theory correlators

Several options:

1. Construct generalized cuts of gravity correlators mostly from gauge theory data
   - Amplitude factors from KLT or double copy
   - Form-factor factors: directly or from KLT/double copy, if possible

   Assemble cuts following same standard algorithm(s) as for amplitudes

2. Construct integrands of correlation functions through double copy

Caveats:

   a. Operators need to cooperate; must identify the gauge theory operators which correspond to desired gravity operator

   b. In the double copy approach it is not obvious how to discard all contact terms
A fairly general illustration:
A fairly general illustration:

Product of amplitudes: use KLT relations between gauge and gravity trees

\[ M_{3}^{tr}(1, 2, 3) = iA_{3}^{tr}(1, 2, 3)A_{3}^{tr}(1, 2, 3) \]
\[ M_{4}^{tr}(1, 2, 3, 4) = -is_{12}A_{4}^{tr}(1, 2, 3, 4)A_{4}^{tr}(1, 2, 4, 3) \]
\[ M_{5}^{tr}(1, 2, 3, 4, 5) = is_{12}s_{34}A_{5}^{tr}(1, 2, 3, 4, 5)A_{5}^{tr}(2, 1, 4, 3, 5) + (2 \leftrightarrow 3) \]
\[ M_{6}^{tr} = 12 \text{ terms of the type } s^{3}A_{6}A_{6} \]

Gravity states = (states of gauge theory 1) \times (states of gauge theory 2)

Sometimes further projection may be necessary!

Sum over states with momenta \( l_6 \) and \( l_5 \):

\[
\sum_{\text{grav states}(5,6)} M_{4}^{tr}(l_1, l_2, l_5, l_6)M_{4}^{tr}(l_3, l_4, -l_5, -l_6)
\]
\[
= -(l_1 + l_2)^2 \sum_{\text{gt}(5,6) \times \text{gt}'(5,6)} A_{4}^{tr}(l_1, l_2, l_5, l_6)A_{4}^{tr}(l_2, l_1, l_5, l_6) A_{4}^{tr}(l_3, l_4, -l_5, -l_6)A_{4}^{tr}(l_4, l_3, -l_5, -l_6)
\]
\[
= -(l_1 + l_2)^2 \sum_{\text{gt}(5,6)} A_{4}^{tr}(l_1, l_2, l_5, l_6)A_{4}^{tr}(l_3, l_4, -l_5, -l_6) \sum_{\text{gt}'(5,6)} A_{4}^{tr}(l_2, l_1, l_5, l_6)A_{4}^{tr}(l_4, l_3, -l_5, -l_6)
\]

\[ \rightarrow \text{ State projection can (sometimes) be implemented by correlating the state sums} \]
A fairly general illustration:

(Product of) form factors: \( \langle 0 | \mathcal{O}_G(q_i) | n \text{ gravity states} \rangle \)

- Direct evaluation of local part of form factors with e.g. Feynman rules or other graph-based methods. (nonlocal part has poles and is therefore captured by other cuts)

- Attempt to identify gauge theory operators such that

\[
\langle 0 | \mathcal{O}_G(q_i) | n \text{ gravity states} \rangle = \text{KLT} [ \langle 0 | \mathcal{O}_{GT1}(q_i) | n \text{ states of GT1} \rangle, \langle 0 | \mathcal{O}_{GT2}(q_i) | n \text{ states of GT2} \rangle ]
\]

Possible issue: - there exist local gauge theory operators which are gauge invariant
- there are no local gravity operators which are gauge invariant

(With small number of exceptions) any local deformation of the gravitational action couples directly to the metric (e.g. via \( \sqrt{-g} \)) while the double copy of gauge-inv. operators need not.

Possible way(s) out:
1. Local sources transform nonlinearly under diffeo \( \longrightarrow \) not visible on shell
2. Deformation is added to the gauge-fixed action; diffeo imposes no constraints
3. “Things work out”: deformations with constant sources can be related \( \text{Dixon, Broedel} \)
Color/kinematics and double-copy (or how to make sure KLT works)

An illustrative example:

- tree-level 4-gluon amplitude in a gauge theory:

\[
\mathcal{A}^{\text{tree}}_4(1, 2, 3, 4) = g^2 \left( \frac{c_s n_s(p, \epsilon)}{s} + \frac{c_t n_t(p, \epsilon)}{t} + \frac{c_u n_u(p, \epsilon)}{u} \right)
\]

\[
c_s = f^{A_1 A_2 B} f^{B A_3 A_4}
\]

\[
c_t = f^{A_2 A_3 B} f^{B A_1 A_4}
\]

\[
c_u = f^{A_3 A_1 B} f^{B A_2 A_4}
\]

Linearized gauge transformation: \( \epsilon_1 \rightarrow p_1 \)

\[
\delta \mathcal{A}^{\text{tree}}_4 = (c_s + c_t + c_u) \alpha(p, \epsilon) = 0 \quad \text{(color Jacobi relation)}
\]

The surprise:

\[
n_s + n_t + n_u = 0 \quad \text{“kinematics parallels color”}
\]

A/The consequence:

\[
M_4 \propto \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u}
\]

is annihilated by \( \epsilon_1^\mu \tilde{\epsilon}_1^\nu \rightarrow p_1^\mu \tilde{\xi}_1^\nu \pm \xi_1^\mu p_1^\nu \)

\( \implies \) Belongs in a gravity theory

With suitable normalization: \( M_4 = -i s_{12} \mathcal{A}^{\text{tree}}_4(1, 2, 3, 4) \tilde{\mathcal{A}}^{\text{tree}}_4(1, 2, 4, 3) \)
The general picture/conjecture: a duality between color and kinematics for scattering amplitudes

\[ \mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{l \in G_3} \int \prod_{l=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{1}{S_i} \prod_{\alpha} n_i C_i \]  
\[ n_i = n_i(p_\alpha \cdot p_\beta, \epsilon \cdot p_\alpha, \ldots) \]

(s)YM theories in any dimension with certain additional matter and/or with higher-dimension operators is said to obey color kinematics duality if the kinematic factors have the same algebraic properties as the color factors must have for the amplitude to be gauge invariant.

- Proved at tree level in (s)YM theories; evidence for QCD, Coulomb branch, \( \phi^3 \), NLSM, Z-theory, BLG, ABJM, \( (DF)^2 \), \( F^3 \),

- Loop-level evidence in various theories and to various loop orders

E.g. Jacobi relations of the gauge group
The general picture/conjecture: color-kinematics duality and double copy for scattering amplitudes

Bern, Carrasco, Johansson

Organize amplitudes in terms of cubic graphs:

\[
\mathcal{A}_{m}^{L-\text{loop}} = i^{L} g^{m-2+2L} \sum_{i \in G_3} \int \prod_{l=1}^{L} \frac{d^{D}p_{l}}{(2\pi)^D} \frac{1}{S_{i}} \frac{n_{i}C_{i}}{\prod_{\alpha_{i}} p_{\alpha_{i}}^{2}} \quad n_{i} = n_{i}(p_{\alpha} \cdot p_{\beta}, \epsilon \cdot p_{\alpha}, \ldots)
\]

With manifest color-kinematics duality \(\longrightarrow\) Double-copy to gravitational amplitudes

\[
\mathcal{M}_{m}^{L-\text{loop}} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i \in G_3} \int \prod_{l=1}^{L} \frac{d^{D}p_{l}}{(2\pi)^D} \frac{1}{S_{i}} \frac{n_{i}\tilde{n}_{i}}{\prod_{\alpha_{i}} p_{\alpha_{i}}^{2}}
\]

- Property of many pure & YM/Maxwell-Einstein SGs w/ further matter, gauged SG, open strings, self-dual gravity, \(R + R^3\), EYM+SSB, Conformal SG, ... [many references] more in talk by Johansson

- Expected to hold to all loop orders; partial arguments available
- Thoroughly tested (e.g. through 5 loops in \(\mathcal{N}=8\) supergravity)
- Workaround if manifest color-kinematics gauge amplitudes are unavailable [many references]

- As in the tree-level example, gauge invariance (linearized diffeo invariance) singles out this structure; it also guarantees that all cuts are gauge invariant
What about correlation functions and form factors?

Form factors: require (at least) the kinematic relations dual to the color identities required by gauge invariance

Other definitions:

- For $\text{Tr}[F^3]$ operator with constant source: impose Jacobi relations on all edges
  - tree-level amplitudes of various multiplicities
  - Not all operators are compatible with $c/k$: $D^2 F^4 \oplus F^5$ but not separately or $\text{Tr}[F^4]$ D, B

- Demand Jacobi relations on all edges not attached directly to the operator
  - Successfully used through 5 loops in $\mathcal{N}=4$ sYM for 20 of $SO(6)$ B, K, T, Y

Correlation functions: require (at least) the kinematic relations dual to the color identities necessary for all generalized cuts to be gauge invariant.

Other definitions:

- Demand Jacobi relations on all edges not attached directly to the operators
  - e.g. correlator of two 20 of $SO(6)$ and one twist-2 spin-$S$ E, RR

- Operators with constant sources and suitable color factor: impose Jacobi relations on all edges
What about correlation functions and form factors?

Form factors: require (at least) the kinematic relations dual to the color identities required by gauge invariance

Other definitions:
- For $\text{Tr}[F^3]$ operator with constant source: impose Jacobi relations on all edges
- Impose Jacobi relations on all edges not attached directly to the operator

Correlation functions: require (at least) the kinematic relations dual to the color identities necessary for all generalized cuts to be gauge invariant.

Other definitions:
- Impose Jacobi relations on all edges not attached directly to the operators
- Operators with constant sources and suitable color factor: impose Jacobi relations on all edges

Double copy

Zero-momentum form factors: $\int d^4 x \text{Tr}[F^3] \leftrightarrow \int d^4 x \sqrt{-g} (\phi R^2 \oplus R^3)$

$q$-dependent $c/k$-dual form factors: Proper interpretation is an open question
Classical double copy

Gauge theory classical solutions \(\leftrightarrow\) gauge theory tree-level amplitudes

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \quad h_{\mu\nu} = -\frac{\kappa}{2} \phi k_\mu k_\nu \]

\[
\bar{g}_{\mu\nu} k^\mu k^\nu = 0 \quad (k \cdot D) k = 0
\]

Lots of examples:

Schwarzschild & Kerr black holes
Taub-NUT spaces
Maximally-symmetric space-times
Plane waves
Gravitational radiation
Perturbative space-times

E.g. \(A^\mu = g \frac{1}{4\pi r} k^\mu \rightarrow h^{\mu\nu} = -\frac{M\kappa}{2} \frac{1}{4\pi r} k^\mu k^\nu\)

Kerr-Schild ansatz

Luna, Monteiro, O’Connell and White; Luna, Monteiro, Nicholsen, O’Connell and White; Ridgway and Wise; Carrillo González, Penco, Trodden; Adamo, Casali, Mason, Nekovar; Goldberger and Ridgway; Chen; Luna, Monteiro, Nicholson, Ochirov; Bjerrum-Bohr, Donoghue, Vanhove; O’Connell, Westerberg, White; Kosower, Maybee, O’Connell, etc

The hope: perturbation theory around double-copy space-times can be expressed in terms of gauge theory building blocks, perhaps around the single-copy solutions

Bahjat-Abbas, Luna, White; Casali, Mason, Nekovar

more in talks by O’Connell, Huang & Porto

more in talks by Benincasa and Mason
Color/kinematics and double copy:

Brief summary

- Color/kinematics and the double copy is a powerful framework for phrasing gravity calculations in terms of simpler gauge theory ones
- Certain zero-momentum form factors exhibit color/kinematics duality
- With a looser definition more zero-momentum form factors may exhibit the duality
- Minimal definition follows from gauge invariance on all cuts
  similar definition can be used for amplitudes: 2-loop example Bern, Davies, Nohle
- Certain momentum space correlators could exhibit color/kinematics duality (with the appropriate definition)
Scattering amplitudes and form factors at tree level

Several available efficient tree-level techniques for scattering amplitudes:

- MHV vertex expansion
  - Cachazo, Svrcek, Witten
- On shell (BCFW) recursion relations
  - Britto, Cachazo, Feng, Witten
- Off shell recursion relations
  - Berends, Giele
- Perturbiner formalism
  - Rosly, Selivanov
- Feynman diagrams
Idea: tree-level amplitudes are rational functions with poles at known positions and with known residues; reconstruct them from this data.

Tag poles through momentum flow

\[ p_i \rightarrow \hat{p}_i = p_i + zk \]
\[ p_j \rightarrow \hat{p}_j = p_j - zk \]

\[ k^2 = 0 \quad k \cdot p_i = 0 = k \cdot p_j \]

\[ A_{n}^\text{tree} \equiv A_{n}^\text{tree}(1, \ldots, i \ldots, j \ldots, n) \rightarrow A_{n}^\text{tree}(1, \ldots, \hat{i} \ldots, \hat{j} \ldots, n) \equiv A_{n}^\text{tree}(z) \]

Obtain the original amplitude by setting \( z = 0 \)

\[ A_{n}^\text{tree} = \frac{1}{2\pi i} \oint_{C_0} \frac{dz}{z} A_{n}^\text{tree}(z) \]
Scattering amplitudes and form factors at tree level

On shell (BCFW) recursion relations for amplitudes

Idea: tree-level amplitudes are rational functions with poles at known positions and with known residues; reconstruct them from this data.

Tag poles through momentum flow

\[
p_i \rightarrow \hat{p}_i = p_i + zk, \quad k^2 = 0, \quad k \cdot p_i = 0 = k \cdot p_j
\]

\[
A_{n}^{\text{tree}} \equiv A_{n}^{\text{tree}}(1, \ldots i \ldots j \ldots n) \rightarrow A_{n}^{\text{tree}}(1, \ldots \hat{i} \ldots \hat{j} \ldots n) \equiv A_{n}^{\text{tree}}(z)
\]

Obtain the original amplitude by setting \( z = 0 \) or by picking up all the other poles

\[
A_{n}^{\text{tree}} = \frac{1}{2\pi i} \oint_{C_0} \frac{dz}{z} A_{n}^{\text{tree}}(z) = -\frac{1}{2\pi i} \oint_{C_0^\infty} \frac{dz}{z} A_{n}^{\text{tree}}(z) \quad (p_i + A)^2 \rightarrow (p_i + A)^2 + 2z k \cdot A
\]

Pole at infinity related to powercounting; may need special treatment

More elaborate momentum shifts may improve with the large-\( z \) scaling

If no pole at infinity:

\[
A_{n}^{\text{tree}} = \sum_{l,h} A_L(\hat{i}, \ldots, l, \hat{P}^h) \frac{i}{P^2} A_R(-\hat{P}^{-h}, l + 1, \ldots, \hat{j}) \bigg|_{z=z(\hat{P}^2=0)}
\]

Can be extended to curved space: in anti-de-Sitter space
On-shell recursion relation for form factors

Same reasoning and same general structure for the result

\[ A_n^{\text{tree}} = \sum_{l,h} A_L(i, \ldots, l, h) \frac{i}{P^2} A_R(-h, l + 1, \ldots, j) \bigg|_{z=z(\hat{P}^2=0)} \]

- One of the two amplitude factors carries the source so it is a form factor

\[ \langle 0|O_1|1 \ldots n \rangle \longrightarrow \langle 0|O_1|1, \ldots, m < n - 1, P \rangle \frac{1}{P^2} A_{n-m+1}^{\text{tree}}(-P, m + 1, \ldots, n) \]

- Generalized form factors: either one or both factors are form factors

\[ \langle 0|O_1O_2|1 \ldots n \rangle \longrightarrow \langle 0|O_1O_2|1, \ldots, m < n - 1, P \rangle \frac{1}{P^2} A_{n-m+1}^{\text{tree}}(-P, m + 1, \ldots, n) \]

\[ \oplus \langle 0|O_1|1, \ldots, m < n - 1, Q \rangle \frac{1}{Q^2} \langle 0|O_2| - Q, m + 1, \ldots, n \rangle \]

Some known results obtained through this or related methods: \( \mathcal{N} = 4 \) sYM theory

All tree-level form factors from BCFW

All tree-level form factors from twistor space

All tree-level form factors from Feynman rules + LSZ reduction:

May be extended to curved space along the anti-de-Sitter discussion of S. Raju
Perturbiner approach for tree-level amplitudes and (generalized) form factors

Idea: tree-level scattering amplitudes encode all solutions of the classical field equations

The perturbiner: - ansatz for such a solution of classical field equations written as an infinite expansion in plane waves, which can be thought of as generating function(al) for tree-level amplitudes
- generating function for fundamental field Green’s functions with one field/leg off shell (Berends-Giele currents)

Various results available:

SD perturbiners for Yang-Mills theory and gravity
- MHV amplitudes, dressed with gravitons of the same helicity
Perturbiners in $\mathcal{N}=3$ and $\mathcal{N}=4$ sYM theory & sinh-Gordon model

Better perturbiner for (s)YM theory & its dimensional reduction

Nonlinear sigma model, Born-Infeld, Z theory

YM theory with $F^3$ and $F^4$ deformations

For our purpose: - Generalize to (generalized) form factors of local composite operators
- Recursive; cannot be separated from amplitudes’ perturbiner
Simple example amplitudes perturbiner for an unordered theory

Lagrangian and equation of motion:

\[ \mathcal{L} = -\frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - \frac{g}{3} \varphi^{3} \quad \Box \varphi = g \varphi^{2} \]

Perturbiner ansatz:

\[ \varphi(x) := \sum_{\mathcal{P}} \varphi_{\mathcal{P}} e^{k_{\mathcal{P}} \cdot x} \]

\[ = \sum_{i} \varphi_{i} e^{k_{i} \cdot x} + \sum_{i<j} \varphi_{ij} e^{k_{ij} \cdot x} + \sum_{i<j<l} \varphi_{ijl} e^{k_{ijl} \cdot x} + \ldots \]

Replace and organize:

\[ \sum_{i} \sum_{p} = \sum_{i<p} + \sum_{p<i}, \quad \sum_{i<j} \sum_{p} = \sum_{i<j<p} + \sum_{i<p<j} + \sum_{p<i<j} \]

Separate Fourier modes:

\[ k_{r}^{2} \varphi_{r} = 0, \quad k_{rs}^{2} \varphi_{rs} = g(\varphi_{r} \varphi_{s} + \varphi_{s} \varphi_{r}), \]

\[ k_{rst}^{2} \varphi_{rst} = g(\varphi_{rs} \varphi_{t} + \varphi_{rt} \varphi_{s} + \varphi_{st} \varphi_{r} + \varphi_{r} \varphi_{st} + \varphi_{s} \varphi_{rt} + \varphi_{t} \varphi_{rs}) \]

Solve w/ initial condition \( \varphi_{i} = 1 \):

\[ \varphi_{12} = \frac{g}{s_{12}}, \quad \varphi_{123} = \frac{g^{2}}{s_{123}} \left( \frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}} \right) \ldots \]

Amplitudes:

\[ A_{n}^{\varphi^{3}} = \lim_{k_{n}^{2} \to 0} s_{12} \ldots n-1 \varphi_{12} \ldots n-1 \varphi_{n} \]
Simple example amplitudes & form factors perturbiner for an unordered theory

- **Lagrangian and equation of motion:**
  \[ \mathcal{L} = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{g}{3} \varphi^3 - \frac{1}{2} J \varphi^2 \quad \Box \varphi = g \varphi^2 + J \varphi \]

- **Perturbiner ansatz:**
  \[ \varphi(x) = \sum_i \varphi_i e^{k_i \cdot x} + \sum_{i<j} \varphi_{ij} e^{k_{ij} \cdot x} + \sum_{i<j<l} \varphi_{ijl} e^{k_{ijl} \cdot x} + \ldots \quad J = \sum_j J_j e^{q_j \cdot x} \]

- **Replace and organize:**
  \[ \Box \varphi = g \left( \sum_i \sum_p \varphi_i \varphi_p e^{k_{ip} \cdot x} + \sum_{i<j} \sum_p \varphi_{ij} \varphi_p e^{k_{ijp} \cdot x} + \sum_i \sum_{p<q} \varphi_i \varphi_{pq} e^{k_{ipq} \cdot x} + \ldots \right) \]
  \[ + \sum_i \sum_p J_i \varphi_p e^{\tilde{k}_{ip} \cdot x} + \sum_i \sum_{p<q} J_i \varphi_{pq} e^{\tilde{k}_{ipq} \cdot x} + \ldots \]

- **Separate Fourier modes:**
  \[ k_r^2 \varphi_r = 0, \quad k_{rs}^2 \varphi_{rs} = \varphi_r \varphi_s + \varphi_s \varphi_r + \varphi_r J_s + J_s \varphi_r \]
  \[ k_{rst}^2 \varphi_{rst} = \varphi_{rs} \varphi_t + \varphi_{rt} \varphi_s + \varphi_{st} \varphi_r + \varphi_r \varphi_{st} + \varphi_s \varphi_{rt} + \varphi_t \varphi_{rs} \]
  \[ + \varphi_{rs} J_t + J_t \varphi_{rs} \]

- **Solve w/ initial condition \( \varphi_i = 1 \):**
  \[ \varphi_{12}^J = \varphi_{12} + \frac{J_1 + J_2}{s_{12}} \]
  \[ \varphi_{123}^J = \varphi_{123} + \frac{1}{s_{123}} \left( \frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}} \right) (J_1 + J_2 + J_3) \]
  \[ + \frac{1}{s_{123}} \left( \frac{J_3 (J_1 + J_2)}{s_{12}} + \frac{J_2 (J_1 + J_3)}{s_{13}} + \frac{J_1 (J_2 + J_3)}{s_{23}} \right) \]
Simple example amplitudes & form factor perturbiner for an unordered theory

- **Lagrangian and equation of motion:**
  \[
  \mathcal{L} = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{g}{3} \varphi^3 - \frac{1}{2} J \varphi^2 \quad \square \varphi = g \varphi^2 + J \varphi
  \]

- **Perturbiner ansatz:**
  \[
  \varphi(x) = \sum_i \varphi_i e^{k_i \cdot x} + \sum_{i<j} \varphi_{ij} e^{k_{ij} \cdot x} + \sum_{i<j<l} \varphi_{ijl} e^{k_{ijl} \cdot x} + \ldots \quad J = \sum_j J_j e^{q_j \cdot x}
  \]

- **Separate Fourier modes:**
  \[
  k_r^2 \varphi_r = 0, \quad k_{rs}^2 \varphi_{rs} = \varphi_r \varphi_s + \varphi_s \varphi_r + \varphi_r J_s + J_s \varphi_r
  \]
  \[
  k_{rst}^2 \varphi_{rst} = \varphi_r \varphi_t + \varphi_r \varphi_s + \varphi_s \varphi_r + \varphi_r \varphi_{st} + \varphi_s \varphi_{rt} + \varphi_{rt} \varphi_{rs} + \varphi_{rs} J_t + J_t \varphi_{rs}
  \]

- **Interpret:**
  \[
  \varphi_{12}^J = \varphi_{12} + \frac{J_1 + J_2}{s_{12}}
  \]
  \[
  \varphi_{123}^J = \varphi_{123} + \frac{1}{s_{123}} \left( \frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}} \right) (J_1 + J_2 + J_3)
  \]
  \[
  + \frac{1}{s_{123}} \left( \frac{J_3(J_1 + J_2)}{s_{12}} + \frac{J_2(J_1 + J_3)}{s_{13}} + \frac{J_1(J_2 + J_3)}{s_{23}} \right)
  \]

- **Amplitudes & form factors:**
  \[
  A_n^{\varphi^3 + J \varphi^2} = \lim_{k_n^2 \to 0} s_{12}\ldots n-1 \varphi_{12}\ldots n-1 \varphi_n
  \]
An alternative way to evaluate in-in correlation functions

- Recall expression for final-state correlations:

\[ G(\vec{n}_1, \ldots, \vec{n}_n, \mathcal{O}) = \langle \mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n) \rangle \equiv \frac{\langle 0|\mathcal{O}^\dagger U^{-1}(+\infty,-\infty)\mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)U(+\infty,-\infty)\mathcal{O}|0\rangle_{\text{in}}}{\langle 0|\mathcal{O}^\dagger \mathcal{O}|0\rangle_{\text{in}}} \]

- Insert complete set of out states which are eigenstates of \( \mathcal{E} \)

\[
\langle 0|\mathcal{O}^\dagger U^{-1}(+\infty,-\infty)\mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)U(+\infty,-\infty)\mathcal{O}|0\rangle_{\text{in}} = \sum_{N,N'} \langle 0|\mathcal{O}^\dagger|N'\rangle_{\text{out}} \langle N'|\mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)|N\rangle_{\text{out}} \langle N|\mathcal{O}|0\rangle_{\text{in}} \\
= \sum_{N} \langle N|\mathcal{E}(\vec{n}_1) \ldots \mathcal{E}(\vec{n}_n)|N\rangle_{\text{out}} |\langle N|\mathcal{O}|0\rangle_{\text{in}}|^2 \\
\langle 0|\mathcal{O}^\dagger \mathcal{O}|0\rangle_{\text{in}} = \sum_{N} |\langle N|\mathcal{O}|0\rangle_{\text{in}}|^2
\]

- Express in-in correlators in terms of time-ordered form factors
  How to deal with final states at finite time? 
  (possible solution Collins 1904.10923)

- Possible finer grading: - restrict to fixed final state multiplicity
  - related to loop expansion of form factors
Position space generalized unitarity-like approach

Goals: - avoid loop order miscount in momentum space
- connect with curved space, where momentum space is not natural

- Builds on the idea that loop expansion of amplitudes can be organized in terms of zero-momentum form factors of the interaction Lagrangian (Lagrangian-insertion formalism)

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int d^d y_1 \ldots d^d y_m \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \mathcal{L}_{\text{int}}(y_1) \ldots \mathcal{L}_{\text{int}}(y_m) \rangle_0
\]

- Complete permutation symmetry in \( y_i \)
  - Fixed scaling

\( \mathcal{N}=4 \) sYM: Complete permutation symmetry and conformal symmetry
- correlation function of 4 stress tensor multiplets through 7 loops
  ( \( \mathcal{L} \) is part of this multiplet \( \longrightarrow \) larger permutation symmetry)

Ambrosio, Eden, Goddard, Heslop, Korchemsky, Taylor, Sokatchev

Higher-point correlators of stress tensor multiplets

Chicherin, Doobary, Eden, Heslop, Korchemsky, Sokatchev
Position space generalized unitarity-like approach

Goals:  
- avoid loop order miscount in momentum space  
- connect with curved space, where momentum space is not natural  
- Builds on the idea that loop expansion of amplitudes can be organized in terms of zero-momentum form factors of the interaction Lagrangian (Lagrangian-insertion formalism)

\[
\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int d^d y_1 \cdots d^d y_m \ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \mathcal{L}_{\text{int}}(y_1) \cdots \mathcal{L}_{\text{int}}(y_m) \rangle_0
\]

Rational
Complete permutation symmetry in \( y_i \)
Fixed scaling
Poles at prescribed positions
Residues with physical meaning

Poles: \( \Delta(x_i - x_j) \propto \frac{1}{|x_i - x_j|^2} \) \( \longrightarrow \) Poles at null-separated points

Residues: sequence of propagators connecting collinear null-separated points becomes an open null Wilson line \( \text{Eden, Heslop, Korchemsky, Sokatchev; Engelund, RR} \)

\( \longrightarrow \) Expectation values of: (products of) null Wilson lines (products of) null Wilson lines and other operators

Assembly: Strategy similar to momentum space amplitudes \( \text{examples: Engelund} \)
Summary & Outlook

- Correlation functions – off shell yet close enough to scattering amplitudes
  relation between the time-ordered and non-time-ordered correlators

- Described extension of some general momentum-space on shell methods
  generalized unitarity; color/kinematics and double copy;
  recursion relations; perturbiners

  left out others: e.g. inverse soft limits, $Q$-cuts, more rec. rel., etc
  maybe more in talks by Bourjaily & Carrasco

- One may formulate some of them in position space; stepping stone towards curved space

- Important to figure out how to carry out curved space computations efficiently
  more in talks by Benincasa & Mason

- A prime candidate: the double copy as it provides a bridge between (certain) curved space(s)
  and nontrivial gauge theory backgrounds.
  more in talks by O’Connell

Many powerful on shell techniques with applications beyond their original design.
We should look forward to the talks and discussions this week for a common way forward!