Some results on the cohomology of finite flat group schemes

Peter Scholze
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Schloss Elmau

Finite flat := finite locally free for this talk

Thm. 1. For any commutative finite flat gp. scheme $G/R$,

$$\text{RT}^{aff}(R, G) = \text{RT}^{fppf}(R, G),$$

("flat descent").

Restricted to $R$ that are Heuselian along $\bullet$, $\text{RT}^{aff}(-, G)$ satisfies $\bullet$-complete flat descent; $\uparrow$ perfectoid $R$, satisfies $\uparrow$-complete arc descent.

Thm. 2. For $R, G$ as above, $Z \subseteq \text{Spec } R$ closed, $R \rightarrow S$ that is iso.

along $Z: 0 \rightarrow S \rightarrow \mathcal{O}_Z$, $\text{RF}_z(R, G) \rightarrow \text{RT}_z(S, G)$ ("excision").

If $Z \subset U$ and $\text{RT}_z(-, G)$ satisfies $\bullet$-complete flat descent, $\uparrow$-complete arc descent on perfectoids.

Thm. 3. For $(R, m)$ North, local complete intersection of dim $d$,

$G/R$ comm. finite flat, $H^i_m(R, G) = 0$ for $i < d$.

General strategy: reduce to similar statements in coherent coho. via


(1) Deformation thry.

Ex. If $I \subset R \rightarrow S$ is a square zero-thickening, $G/R$, $w_G := e^* LG/R$, then $\text{RHom}_R(w_G, I) \rightarrow \text{RT}(R, G) \rightarrow \text{RT}(S, G)$

(2) Dieudonné thry. Let $R$ be an integral perfectoid ring.

Thm. (Lau, $p \gg 3$), S., Anschnitt-Le Bras

$\mathcal{F}$-div. $\mathcal{G}$-gps. $/R \cong \mathcal{B}K\mathcal{F}(R) := \{ \text{finite proj. } A_{\text{inf}}(R) \text{- mod's } M \}

\phi^\mathcal{F}: M \rightarrow M \phi^\mathcal{F}$-linear

$V: M \rightarrow M \psi^{-1}$-linear

$\mathcal{E}N = d(d), VF = d$, with $d$ generator of $\text{ker}(A_{\text{inf}}(R) \rightarrow R)$

Ex. If $d$ is chosen right, then

$\mathbb{Q}_p/\mathbb{Z}_p \rightarrow (A_{\text{inf}}(R), F = d, V = \psi^{-1})$

$\mathcal{C}$-div. gp. schemes $/R \cong \mathcal{B}K\mathcal{F}(R)$

$G \rightarrow \text{M}(G)$

as before but $M$ for torsion?

Will later see that $\text{RT}(R, G) \cong [\text{M}(G) \stackrel{1-V}{{\longrightarrow}} \text{M}(G)]$. 

$\text{M}(G)$
Recall (for Thm 1) \( R \to S \) is \( p \)-completely flat if

\[
\exists \mathcal{L}_R, R/p = S/p \quad \text{and} \quad R/p \to S/p \quad \text{flat (faithfully)}
\]

\( R \to S \) is a \( p \)-complete arc cover if for all \( p \)-complete \( \mathcal{L} \)-valuation rings \( V \) and \( R \to V \), \( \exists \quad \mathcal{L}_R, R \to V \) faith fully flat

\[
S \to V
\]

\( p \)-complete \( \mathcal{L} \)-valuation ring

Thus (Blaschke-Newton) If \( R \) Henselian along \( p \) and \( G/\mathcal{L}_p \) of \( p \)-power order, then \( RT: (R[[p]], G) \to RT(R^+[[p]], G) \) and \( R \to RT(R[[p]], G) \) satisfies \( p \)-complete arc-descent.

Thus (Blaschke-S.) On perfectoid \( R \), \( R \to R \) satisfies \( p \)-complete arc cover.

Prop. \( R \) Henselian along \( p \) with \( R[[p^\infty]] \) killed by \( p^\infty \) (so derived and classical \( p \)-adic completions agree). Then

\[
RT_{\text{et}}(R, G) \to \text{Rlim}(RT(R/p^\infty, G)).
\]

In particular, \( RT(R, G) \simeq RT(R^+, G) \)

(\( \text{In general, for } R_{\text{et}} Z/p^\infty \to \hat{R} = \text{derived } p \text{-adic completion.} \)

Let \( R \to R^* \) be an ind-ff hypercover and \( R \to S^* \) levelwise \( p \)-adic Henselization.

Claim: \( \text{LHS for } R = \text{Rlim (LHS for } S^*) \)

\[
\text{RHS} \quad \text{— } \text{RHS}
\]

RHS: \( S^*/p^\infty = R^*/p^\infty \), so follows from ind-ff descent.
LHS: use functorial triangle

\[
RT(S^*/p^\infty, G) \to RT(S^*/p^\infty, G) \to \text{Rlim}(RT(S^*/p^\infty, G))
\]

ok by ind-ff descent

May assume \( R \) has no nonsplit \( \text{ff} \) covers.

\[
\Rightarrow H^{i, \text{ff}}(R, G) = 0 \quad \text{for } i > 0 \quad \text{(same for } R/p^\infty \text{ by using syntomic covers)}
\]
Remains: $H^0(R,G) \Rightarrow \text{Rl}_n H^0(R/p^n, G)$, which is ok by Elkik. □

Pf of Thm: 

\[
\begin{align*}
RT(R, G) & \rightarrow RT(R/p, G) = \text{Rl}_n RT(R/p^n, G) \\
RT(R[I], G) & \rightarrow RT(R[I]/p, G)
\end{align*}
\]

ok be away from $p$ 

⇒ reduce to $R/p^n$

Deformation thy ⇒ $R$ an $\mathbb{F}_p$-alg.

Extract $p$-power roots ⇒ reduce to semiperfect, then to perfect $R$,

then use Dieudonné thy. □

Cor. $R$ perfectoid, $G/R$, then $RT(R, G) = [M(G) \rightarrow M(G)]$.

If both sides satisfy $p$-complete and have no higher cohomology locally, □

Proof of Purity $H^i_n(R, G) = 0$ for $i < d$ w/ $R$ lax of dim $d$.

Reductions: $G$ of $p$-power order, $k = R/m$ alg. closed, $R$ complete.

Induct on $i$. Cohen: $W(k)[x_1, \ldots, x_n]/(f_1, \ldots, f_m) = R$

\[
\begin{align*}
&\text{id.-smth. completed} \\
&\text{perfectoid } S \\
&\text{perfectoid, so finish using Dieudonné thy.}
\end{align*}
\]